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Deformable Permanent Ferroelectric or Ferromagnetic Media

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Abstract

In the framework of continuum mechanics, one of possible consistent definitions of deformable permanent magnets is introduced and explored. Similar model can be used for ferroelectric substances. Based on the suggested definition, we establish the key kinematic relationship for the deformable permanent magnet and suggest the simplest master system, allowing to analyze behavior of such substances.

Keywords

Ferroelectric and Ferromagnetic Media, Thermodynamic Consistency, The Aleph Tensor, Boundary Value Problems of Electro- and Magnetostatics and Quasi-Statics

1. Introduction and Notation

Magnetic phenomena have found thousands of applications in science and engineering. Of the multitude of authoritative textbooks, dealing with continuum description of magnetic phenomena we recommend the comprehensive treatise [1] [2] [3] [4], in which the readers will be able to get information about various theoretical models of magnetic phenomena and the essential difficulties still existing in the fundamentals of electromagnetism. In order to avoid some of those difficulties, we suggested [5] [6] the models based on the concept of the Aleph tensors. The Aleph tensors have common features with the energy-momentum tensors of the classical field theories.

Important classes of the models of magnets are known as the permanent magnets. Several monographs [7] [8] [9] [10] are devoted to the permanent magnets, with which all of us are familiar from childhood. The concept of permanent magnet can be easily introduced for the rigid (*i.e.*, non-deformable) solids. Needless to say, that the permanent magnet is an idealization, which ignores many features of real magnetic substances. As always, the idealization implies some advantages and disadvantages. It ignores several key features. On the other hand, it allows to investigate more deeply the remaining features. In this paper, we try to generalize the intuitive understanding of permanent magnet with the important property of deformability. It is intuitively clear that when rigid permanent magnet is rotated, the magnetic field and magnetic polarization rotate with the rigid body. But how to define the permanent magnet when dealing with deformable magnetized solid? Presumably, this question does not have a unique answer. One of possible answers is suggested below.

Notation of this paper is presented in full in monographs [11] [12]; for the readers' convenience, we give in the Section "**Appendix**" brief sketch of this notation.

Consider the immobile spatial coordinate system, referred to the coordinates z^i (the reference indexes from the middle of the Latin alphabet i, j, k run the values 1, 2, 3) and assume, for simplicity, that our space is Euclidean. In this space, we consider a material body *B*, referred to the material coordinates x^a (the material indexes, taken from the beginning of the Latin alphabet a, b, c, run the values 1, 2, 3 as well). We accept the standard concepts of the covariant and contravariant indexes and accept the standard agreement regarding summation over the repeat covariant and contravariant indexes of the same type (*i.e.*, of the reference or of the material type).

In addition to the two different sets of coordinate variables, we distinguish also between two different configurations of the body: the initial and the current configurations of the body. Let the functions $z^i = z^i\left(x^a, t\right)$ be the Eulerian coordinates z^i in the current configuration of the material point with the material coordinates x^a at the moment of time *t*. Technically, it is important to show explicitly the arguments of different functions, although quite often this practice makes the relationships rather cumbersome. To make the relationships more compact, we often skip indexes in the arguments of different functions.

We use the notation $x^a = x^a\left(z^i, t\right)$ for the inverse of the function $z^i\left(x^a, t\right)$. Let us use the notation $Z_{ij}\left(z\right)$ for the deformation-independent metrics of the reference spatial system, and the notation $X_{ab}\left(x, t\right)$ for the deformation-dependent metrics of the actual material configurations. These two metrics tensors are interconnected by the relationships

$$X_{ab}\left(x,t\right) = Z_{ij} z^{i\cdot}_{\cdot a} z^{i\cdot}_{\cdot b},\; Z_{ij} = X_{ab} x^{a\cdot}_{\cdot i} x^{b\cdot}_{\cdot j} \tag{1.1}$$

where the mixed shift-tensors $z^{i\cdot}_{\cdot a}$ and $x^{a\cdot}_{\cdot i}$ are defined as

$$z^{i\cdot}_{\cdot a} \equiv \frac{\partial z^i\left(x,t\right)}{\partial x^a},\; x^{a\cdot}_{\cdot i} \equiv \frac{\partial x^a\left(z,t\right)}{\partial z^i} \tag{1.2}$$

The reference and the coordinate configurations are characterized by the current covariant bases $\boldsymbol{Z}_i\left(z\right)$ and contravariant bases and $\boldsymbol{X}_a\left(x,t\right)$, respectively.

We use the standard notation ∇_i and ∇_a for the reference and material contravariant differentiation in the metrics of the actual configuration (see the relationships ((A.1)-(A.10)) of the section "**Appendix**").

Magnetization is a vector quantity. A distributed magnetization field is characterized by the density per unit mass $\boldsymbol{M}$ or per unit volume $\rho\boldsymbol{M}$, where ρ is the mass density. Vector field $\boldsymbol{M}$ can be decomposed with respect to the material basis $\boldsymbol{M} = M^a \boldsymbol{X}_a$ or the spatial basis $\boldsymbol{M} = M^i \boldsymbol{Z}_i$. By definition, in vacuum, the vector $\boldsymbol{M}$ of vacuum is equal to zero.

For isothermal systems, the bulk free energy density ψ per unit mass is given as a function of the actual material metrics X_{ab}, the Lagrangean components M^a of the magnetization vector per unit mass, and fixed material constants or tensors, which we do not mention explicitly:

$$\psi = \psi\left(X_{ab}, M^a\right) \tag{1.3}$$

The magnetoelastic Aleph tensor $\aleph^{ij}$ is defined as follows

$$\aleph^{ij} \equiv 2\rho \frac{\partial \psi}{\partial X_{(cd)}} z^{i\cdot}_{.c} z^{j\cdot}_{.d} - \frac{1}{8\pi} H_k H^k Z^{ij} + \frac{1}{4\pi} H^i H^j \tag{1.4}$$

where H^i are the Eulerian component of the magnetic field.

The bulk dynamics equations are postulated to be of the form

$$\rho\left(\frac{\partial V^i}{\partial t} + V^j \nabla_j V^i\right) = \nabla_j \aleph^{ij} \tag{1.5}$$

where V^i are the Eulerian components of the velocity field and ρ is the mass density.

The velocity field V^i is defined as

$$V^i\left(x,t\right) \equiv \frac{\partial z^i\left(x,t\right)}{\partial t} \tag{1.6}$$

We can also consider the velocity components as functions of the Eulerian coordinates z^i: we will use the notation $V^i\left(z,t\right)$ for this function. The functions $V^i\left(x,t\right)$ and $V^i\left(z,t\right)$ are different functions. This should not create any confusion even when we do not show the arguments explicitly, which of the two functions is meant should be clear from the context; for instance, in the equations (1.5) we mean $V^i = V^i\left(z,t\right)$.

The momentum balance condition at the boundary with vacuum reads

$$\begin{aligned} &\left(2\rho \frac{\partial \psi}{\partial X_{(cd)}} z^{i\cdot}_{.c} z^{j\cdot}_{.d} + \frac{1}{4\pi} H^i H^j - \frac{1}{8\pi} H_k H^k Z^{ij}\right)_{sub} N_j \\ &= \left(\frac{1}{4\pi} H^i H^j - \frac{1}{8\pi} H_k H^k Z^{ij}\right)_{vac} N_j, \end{aligned} \tag{1.7}$$

where N_j are the components of the unit normal to the boundary.

The relationships (1.3)-(1.6) should be amended with the magnetostatics bulk equations and boundary conditions. The bulk equations read

$$H_i = -\frac{\partial \varphi\left(z,t\right)}{\partial z^i} \tag{1.8}$$

$$\nabla_i B^i = 0 \tag{1.9}$$

with the magnetic induction B^i defined as

$$B^i \equiv H^i + 4\pi\rho M^i \tag{1.10}$$

The Equation (1.8) reflects the fact that in the absence of macroscopic electric current the magnetic field is irrotational. The Equation (1.9) reflects the fact that the magnetic induction is always divergence-free.

At the interfaces, the fields $\varphi(z,t)$, $H_i(z,t)$, and $B^i(z,t)$ and/or their derivatives experience finite jumps. Those jumps are not arbitrary but satisfy the boundary constraints of magnetostatics:

$$[\varphi]_-^+ = 0 \tag{1.11}$$

and

$$\left[B^i\right]_-^+ N_i = 0 \tag{1.12}$$

The bulk Equations (1.3)-(1.6), (1.8)-(1.10) should be amended with the following thermodynamics relationship:

$$H_a = \rho \frac{\partial \psi\left(X_{ab}, M^a\right)}{\partial M^a} \tag{1.13}$$

In order to get the mathematically closed master system, the relationships (1.1)-(1.13) should be amended with the initial conditions and conditions at infinity.

In the case of liquid substances, the free energy should be chosen in the form:

$$\psi\left(X_{ab}, M^a\right) = \Psi\left(\rho, M^a\right)\Big|_{\rho=\rho(X_{ab})} \tag{1.14}$$

Using the relationship (1.14), we get instead of (1.4) the following relationship of the Aleph tensor:

$$\aleph^{ij} \equiv -\rho^2 \frac{\partial \Psi(\rho)}{\partial \rho} Z^{ij} - \frac{1}{8\pi} H_k H^k Z^{ij} + \frac{1}{4\pi} H^i H^j \tag{1.15}$$

2. Definition of Permanent Deformable Magnet

How to define rigorously the notion of a permanent magnet on the macroscopic level? It is easy and straightforward to suggest such rigorous definition when dealing with the model of rigid solid. In words, we just say that the vector of magnetization "moves with the magnet". Or, in other words, we can say that the components of the magnetization vector with respect to the accompanying basis remain unchanged. Of course, the components with respect basis, fixed in the space, can change. Formally, let $\boldsymbol{G}_a = \{\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_3\}$ be the accompanying covariant triangle of the rigid body; this basis does not necessarily consist of the normalized orthogonal vectors. Let $\boldsymbol{G}^a = \left\{\boldsymbol{G}^1, \boldsymbol{G}^2, \boldsymbol{G}^3\right\}$ be the accompanying contra-variant triangle; this triangle is defined by the identities

$$\boldsymbol{G}_a \cdot \boldsymbol{G}^b = \delta_a^b \tag{2.1}$$

Needless to say, that both bases, $\boldsymbol{G}_a$ and $\boldsymbol{G}^b$, are time-dependent although they, in general, depend upon the Lagrangean (material) coordinates.

In terms of the accompanying bases we can define the permanent magnets by the relationships

$$\frac{\partial M^a(x,t)}{\partial t}=0 \quad \text{or} \quad \frac{\partial M_a(x,t)}{\partial t}=0 \tag{2.2}$$

where $M^a(x,t)$ and $M_a(x,t)$ are the components of the magnetization vector $\boldsymbol{M}(x,t)$ with respect to the accompanying bases $\boldsymbol{G}_a$ and $\boldsymbol{G}^b$, respectively.

What if we want to consider deformable magnets instead of rigid magnets? The simplest generalization will be the following. First, we replace the bases $\boldsymbol{G}_a$ and $\boldsymbol{G}^a$ with the Lagrangean (material) bases $\boldsymbol{X}_a(x,t)$, $\boldsymbol{X}^a(x,t)$, respectively. Then, we can define deformable permanent magnet by one of the relationships (2.2); for instance,

$$\frac{\partial M^a(x,t)}{\partial t}=\frac{\partial}{\partial t}\left(\boldsymbol{M}(x,t)\bullet\boldsymbol{X}^a(x,t)\right)=0, \tag{2.3}$$

where "$\bullet$" is the symbol of scalar product of corresponding vectors. In words, this definition means that the contravariant components of the magnetization vector with respect to the accompanying (material) basis remain fixed. To some extent this definition reminds the behavior of the vorticity vector in ideal liquid.

Also, conceptually it is logically possible to define the permanent magnet with the relationship

$$\frac{\partial M_a(x,t)}{\partial t}=\frac{\partial}{\partial t}\left(\boldsymbol{M}(x,t)\bullet\boldsymbol{X}_a(x,t)\right)=0 \tag{2.4}$$

The relationships (2.3), (2.4) are not mathematically equivalent and lead to different physical conclusions. Per Tamm [4], the ultimate choice should be made on the basis of comparison with experiment. In the absence of experimental evidence, we can make our choice on the basis of convenience of aesthetics. Below, we dwell on the choice based on the relationship (2.3).

3. The Master Equation for the Model of Permanent Deformable Magnet in the Eulerian Description

The relationships (2.3) provide the quite straightforward and simple definition of the model of permanent deformable magnet when using the Lagrangean (material) description of continuum media. However, the material description is quite cumbersome in various situations. In particular, when dealing with hydrodynamics researchers obviously prefer the Eulerian description. Below, we try to establish the Eulerian analogy of the definition of the permanent deformable magnet. We will establish the following key relationship:

$$\frac{\partial M^j(z,t)}{\partial t}+V^i(z,t)\nabla_i M^j(z,t)-M^i(z,t)\nabla_i V^j(z,t)=0 \tag{3.1}$$

where $M^j(z,t)$ and $V^j(z,t)$ are the Eulerian components of the magnetization and velocity fields, which expressed as functions of the variables z^i and *t*, and use the covariant differentiation ∇_i, based on the spatial metrics $Z_{ij}(z)$.

First, we rewrite the definition (2.2) of permanent deformable magnet as follows:

$$\frac{\partial}{\partial t}\left(\boldsymbol{M}(z,t)\big|_{z=z(x,t)}\cdot\boldsymbol{X}^a(x,t)\right)=0 \tag{3.2}$$

We remined that the function $\boldsymbol{M}(z,t)$ differs essentially from the function $\boldsymbol{M}(x,t)$; in fact, the following relationship hold

$$\boldsymbol{M}(x^a,t)\equiv\boldsymbol{M}(z^i,t)\Big|_{z^i=z^i(x^a,t)} \tag{3.3}$$

The notation (3.3) indicates that the variable z^i in the right-hand side of Equation (3.3) should be replaced with the function $z^i(x^a,t)$.

We, then, get, using the distributivity of the scalar product:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\boldsymbol{M}(z,t)\big|_{z=z(x,t)}\cdot\boldsymbol{X}^a(x,t)\right)\\&=\boldsymbol{X}^a(x,t)\cdot\frac{\partial}{\partial t}\left(\boldsymbol{M}(z,t)\big|_{z=z(x,t)}\right)+\boldsymbol{M}(z,t)\big|_{z=z(x,t)}\cdot\frac{\partial}{\partial t}\boldsymbol{X}^a(x,t)=0\end{aligned} \tag{3.4}$$

Let us differentiate the two terms in (3.4) separately.

First, we get the relationship

$$\frac{\partial}{\partial t}\left(\boldsymbol{M}(z,t)\big|_{z=z(x,t)}\right)=\left.\frac{\partial\boldsymbol{M}(z,t)}{\partial t}\right|_{z=z(x,t)}+\nabla_i\boldsymbol{M}(z,t)\big|_{z=z(x,t)}V^i(z,t)\big|_{z=z(x,t)} \tag{3.5}$$

as implied by the following chain:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\boldsymbol{M}(z,t)\big|_{z=z(x,t)}\right)&=\left.\frac{\partial\boldsymbol{M}(z,t)}{\partial t}\right|_{z=z(x,t)}+\left.\frac{\partial\boldsymbol{M}(z,t)}{\partial z^i}\right|_{z=z(x,t)}\frac{\partial z^i(x,t)}{\partial t}\\&=\left.\frac{\partial\boldsymbol{M}(z,t)}{\partial t}\right|_{z=z(x,t)}+\nabla_i\boldsymbol{M}(z,t)\big|_{z=z(x,t)}V^i(z,t)\big|_{z=z(x,t)}\end{aligned}$$

We, then, get the following identity

$$\boldsymbol{X}^a(x,t)=\boldsymbol{Z}^j(z)\big|_{z=z(x,t)}x^{a\cdot}_{\cdot j}(z,t)\big|_{z=z(x,t)} \tag{3.6}$$

Combining Equations (3.5), (3.6), we get

$$\begin{aligned}&\boldsymbol{X}^a(x,t)\cdot\frac{\partial}{\partial t}\left(\boldsymbol{M}(z,t)\big|_{z=z(x,t)}\right)\\&=x^{a\cdot}_{\cdot j}(z,t)\big|_{z=z(x,t)}\left.\left(\frac{\partial M^j(z,t)}{\partial t}+V^i(z,t)\nabla_i M^j(z,t)\right)\right|_{z=z(x,t)}\end{aligned} \tag{3.7}$$

as implied by the following chain:

$$\begin{aligned}&\boldsymbol{Z}^j(z)\big|_{z=z(x,t)}x^{a\cdot}_{\cdot j}(z,t)\big|_{z=z(x,t)}\cdot\left.\frac{\partial\boldsymbol{M}(z,t)}{\partial t}\right|_{z=z(x,t)}\\&+\boldsymbol{Z}^j(z)\big|_{z=z(x,t)}x^{a\cdot}_{\cdot j}(z,t)\big|_{z=z(x,t)}\cdot\nabla_i\boldsymbol{M}(z,t)\big|_{z=z(x,t)}V^i(z,t)\big|_{z=z(x,t)}\\&=x^{a\cdot}_{\cdot j}(z,t)\big|_{z=z(x,t)}\left.\frac{\partial M^j(z,t)}{\partial t}\right|_{z=z(x,t)}+x^{a\cdot}_{\cdot j}(z,t)\big|_{z=z(x,t)}\nabla_i M^j(z,t)\big|_{z=z(x,t)}V^i(z,t)\big|_{z=z(x,t)}\end{aligned}$$

$$= x^{a\cdot}_{\cdot j}(z,t)\Big|_{z=z(x,t)}\left(\frac{\partial M^j(z,t)}{\partial t} + V^i(z,t)\nabla_i M^j(z,t)\right)\Bigg|_{z=z(x,t)}$$

We turn now to the second term in (3.4). First, we get

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{X}^a(x,t) &= \frac{\partial}{\partial t}\left(X^{ab}(x,t)\boldsymbol{X}_b(x,t)\right) \\ &= \frac{\partial X^{ab}(x,t)}{\partial t}\boldsymbol{X}_b(x,t) + X^{ab}(x,t)\frac{\partial \boldsymbol{X}_b(x,t)}{\partial t}\end{aligned} \tag{3.8}$$

For any symmetric mutually inverse tensor fields $R_{ab}(x,t)$ and $S^{cd}(x,t)$ we get the identity

$$\frac{\partial S^{ab}(x,t)}{\partial t} = -S^{ac}S^{bd}\frac{\partial R_{cd}(x,t)}{\partial t},$$

which can be applied to the metric tensors $X_{ab}(x,t)$ and $X^{cd}(x,t)$:

$$\frac{\partial X^{ab}(x,t)}{\partial t} = -X^{ac}X^{bd}\frac{\partial X_{cd}(x,t)}{\partial t} \tag{3.9}$$

We also get the following relationship

$$\frac{\partial X_{cd}(x,t)}{\partial t} = \nabla_c V_d + \nabla_d V_c \tag{3.10}$$

Indeed, we get, using the definition of the actual material basis, and the Leibnitz rule of differentiation of products

$$\begin{aligned}\frac{\partial X_{cd}(x,t)}{\partial t} &= \frac{\partial}{\partial t}\left(\boldsymbol{X}_c(x,t)\cdot\boldsymbol{X}_d(x,t)\right) \\ &= \frac{\partial \boldsymbol{X}_c(x,t)}{\partial t}\cdot\boldsymbol{X}_d(x,t) + \boldsymbol{X}_c(x,t)\cdot\frac{\partial \boldsymbol{X}_d(x,t)}{\partial t}\end{aligned} \tag{3.11}$$

We proceed, using the relationship

$$\frac{\partial \boldsymbol{X}_c(x,t)}{\partial t} = \boldsymbol{X}_b(x,t)\nabla_c V^b(x,t), \tag{3.12}$$

as implied by the following chain

$$\begin{aligned}\frac{\partial \boldsymbol{X}_c(x,t)}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial \boldsymbol{X}(x,t)}{\partial x^c} = \frac{\partial}{\partial x^c}\frac{\partial \boldsymbol{X}(x,t)}{\partial t} = \frac{\partial \boldsymbol{V}(x,t)}{\partial x^c} = \nabla_c\boldsymbol{V}(x,t) \\ &= \nabla_c\left(\boldsymbol{X}_c(x,t)V^c(x,t)\right) = \boldsymbol{X}_b(x,t)\nabla_c V^b(x,t)\end{aligned}$$

Combining (3.11), (3.12), we get the relationship (3.10); indeed, we get the following chain:

$$\begin{aligned}\frac{\partial X_{cd}(x,t)}{\partial t} &= \frac{\partial \boldsymbol{X}_c(x,t)}{\partial t}\cdot\boldsymbol{X}_d(x,t) + \boldsymbol{X}_c(x,t)\cdot\frac{\partial \boldsymbol{X}_d(x,t)}{\partial t} \\ &= \boldsymbol{X}_c(x,t)\nabla_e V^c(x,t)\cdot\boldsymbol{X}_d + \boldsymbol{X}_c\cdot\boldsymbol{X}_e(x,t)\nabla_d V^e(x,t) \\ &= X_{cd}\nabla_c V^c + X_{ce}\nabla_d V^c = \nabla_c V_d + \nabla_d V_c\end{aligned}$$

Combining (3.9), (3.10), we get

$$\frac{\partial X^{ab}(x,t)}{\partial t} = -\nabla^a V^b - \nabla^b V^a \tag{3.13}$$

Combining (3.12), (3.13), we get

$$\frac{\partial \boldsymbol{X}^b(x,t)}{\partial t} = -\boldsymbol{X}_c(x,t)\nabla^c V^b \tag{3.14}$$

as it follows from the chain:

$$\begin{aligned}\frac{\partial \boldsymbol{X}^b(x,t)}{\partial t} &= \frac{\partial\left(X^{bc}(x,t)\boldsymbol{X}_c(x,t)\right)}{\partial t} = \frac{\partial X^{bc}(x,t)}{\partial t}\boldsymbol{X}_c(x,t) + X^{bc}(x,t)\frac{\partial \boldsymbol{X}_c(x,t)}{\partial t}\\ &= -\left(\nabla^c V^b + \nabla^b V^c\right)\boldsymbol{X}_c(x,t) + X^{bc}(x,t)\boldsymbol{X}_d(x,t)\nabla_c V^d(x,t)\\ &= -\left(\nabla^c V^b + \nabla^b V^c\right)\boldsymbol{X}_c(x,t) + \boldsymbol{X}_c(x,t)\nabla^b V^c(x,t) = -\boldsymbol{X}_c(x,t)\nabla^c V^b\end{aligned}$$

Let us rewrite (3.14) as

$$\frac{\partial \boldsymbol{X}^a(x,t)}{\partial t} = -\boldsymbol{Z}_k x^a_{.i}\nabla^k V^i, \tag{3.15}$$

as implied by the following chain:

$$\frac{\partial \boldsymbol{X}^a(x,t)}{\partial t} = -\boldsymbol{X}_c(x,t)\nabla^c V^a = -\boldsymbol{Z}_k z^k_{.c}\nabla^c\left(V^i x^a_{.i}\right) = -\boldsymbol{Z}_k z^k_{.c}\nabla^c\left(V^i x^a_{.i}\right) = -\boldsymbol{Z}_k x^a_{.i}\nabla^k V^i$$

Combining (3.2), (3.5), (3.15), we get

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\boldsymbol{M}(z,t)\big|_{z=z(x,t)}\cdot \boldsymbol{X}^a(x,t)\right)\\ &= x^{a.}_{.j}(z,t)\big|_{z=z(x,t)}\left\{\frac{\partial M^j(z,t)}{\partial t} + V^i(z,t)\nabla_i M^j(z,t) - M^i\nabla_i V^j\right\}\Bigg|_{z=z(x,t)}\end{aligned} \tag{3.16}$$

as it is implied by the following chain:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\boldsymbol{M}(z,t)\big|_{z=z(x,t)}\cdot \boldsymbol{X}^a(x,t)\right)\\ &= x^{a.}_{.j}(z,t)\big|_{z=z(x,t)}\left\{\frac{\partial M^j(z,t)}{\partial t} + V^i(z,t)\nabla_i M^j(z,t)\right\}\Bigg|_{z=z(x,t)} - \boldsymbol{M}(z,t)\big|_{z=z(x,t)}\cdot \boldsymbol{Z}_k x^a_{.i}\nabla^k V^i\\ &= x^{a.}_{.j}(z,t)\big|_{z=z(x,t)}\left\{\frac{\partial M^j(z,t)}{\partial t} + V^i(z,t)\nabla_i M^j(z,t)\right\}\Bigg|_{z=z(x,t)} - x^a_{.j} M_k(z,t)\big|_{z=z(x,t)}\nabla^k V^j\\ &= x^{a.}_{.j}(z,t)\big|_{z=z(x,t)}\left\{\frac{\partial M^j(z,t)}{\partial t} + V^i(z,t)\nabla_i M^j(z,t) - M^i\nabla_i V^j\right\}\Bigg|_{z=z(x,t)}\end{aligned}$$

Combining the relationships (3.2), (3.16), we arrive at the required relationship (3.1).

4. The Full Master System for the Model of Permanent Deformable Magnet in the Eulerian Description

Analysis of the model of permanent deformable magnet requires some corrections in the master system (1.1)-(1.13). First of all, the free energy density ψ cannot depend upon the density of the magnetization vector M^a since this vector remains constant; thus, the free energy density reads

$$\psi = \psi\left(X_{ab}, T\right), \tag{4.1}$$

where T is the absolute temperature. For the sake of brevity, we will still consider only isothermal case and ignore the variable T.

The appearance of the Lagrangean actual metrics X_{ab} looks like the inconsistency with the goal of providing the Eulerian description of the continuum media. Indeed, the definition of the metrics X_{ab} explicitly uses the Lagrangean (material) description. Therefore, some comments are required here. In fact, the Eulerian description basically means that the ultimate independent variables are actually the Eulerian coordinates z^k and the time t. That does not mean though, that the objects with the Lagrangean (material) indexes cannot appear in the intermediate relationships. For instance, the shift-tensor $x_{.i}^{a\cdot}\left(z^k,t\right)$ is one of the most widespread objects in the Eulerian description. Another widespread object $z_{.a}^{k\cdot}$ is the inverse of $x_{.i}^{a\cdot}\left(z^k,t\right)$. The inverse tensor can be presented as

$$z_{.a}^{k\cdot}=\frac{1}{A}\frac{\partial A}{\partial x_{.k}^{a\cdot}} \text{ where } A\equiv\det\left\|x_{.k}^{a\cdot}\left(z,t\right)\right\| \tag{4.2}$$

with this definition (4.2) the tensor field $z_{.a}^{k\cdot}$ appears to be the tensor-function of the independent variables $\left(z^k,t\right)$, as it should be when using the Eulerian description. There is also another definition of the tensor $z_{.a}^{k\cdot}$; it reads $z_{.a}^{k\cdot}=\partial z^k\left(x^c,t\right)/\partial x^a$. With the last definition, the same tensor $z_{.a}^{k\cdot}$ appears as the function of x^c,t. Thus, it cannot be the ultimate element of the Eulerian description, but it can be the intermediate object.

Another example is the definition of the permanent deformable magnet (2.3). This definition is relevant for the Lagrangean description, whereas the relationship (3.1) is relevant for the Eulerian description. The two definitions are mathematically and physically equivalent each other, although this equivalence is not obvious at all. But, technically, these definitions lead to quite different equations and may have essential advantages with respect to each other, depending on different circumstances (like, for instance, initial and boundary conditions, the equations of state, etc.)

It is essential that the key thermodynamic identity (1.13) at fixed M^a becomes meaningless and should be excluded from the master system for permanent deformable magnet. Instead of the identity (1.13), the relationship (3.1) must be included into the master system.

The bulk master system for the permanent magnetic liquid substance in the Eulerian description

A detailed discussion of magnetic liquids, including their applications and fundamentals, can be found in the monograph [13]. The full master system for the permanent deformable magnetic liquids consists of the following equations:

1) the momentum equation

$$\rho\left(\frac{\partial V^i}{\partial t}+V^j\nabla_jV^i\right)=\nabla_j\left(\frac{\partial\psi}{\partial X_{ab}}z_{.a}^{i\cdot}z_{.b}^{j\cdot}-\frac{1}{8\pi}H_kH^kZ^{ij}+\frac{1}{4\pi}H^iH^j\right) \tag{4.2}$$

2) the mass conservation equation

$$\frac{\partial\rho}{\partial t}+\nabla_i\left(\rho V^i\right)=0 \tag{4.3}$$

3) the thermodynamics relationship

$$\psi\left(X_{ab}, M^{a}\right) \equiv \Phi(\rho), \text{ where } \rho=\rho\left(X_{ab}\right) \tag{4.4}$$

4) the Eulerian kinematic definition of the permanent magnetic substance

$$\frac{\partial M^{j}(z,t)}{\partial t}+V^{i}(z,t) \nabla_{i} M^{j}(z,t)-M^{i} \nabla_{i} V^{j}=0 \tag{4.5}$$

5) the identity of the magnetic induction

$$\nabla_{i}\left(H^{i}+4 \pi \rho M^{i}\right)=0 \tag{4.6}$$

6) the identity of the magnetic field H^{i} and the potential φ (in the absence of electric currents)

$$H^{i} \equiv-\nabla^{i} \varphi \tag{4.7}$$

The natural arguments of the magnetic fields H^{i}, B^{i}, and φ are the Eulerian independent variables z^{i} and *t*. The only potential obstacle for the fully Eulerian description is the term $z_{.a}^{i\cdot} z_{.b}^{j\cdot} \partial \psi / \partial X_{ab}$. However, for the equation of state (4.4) the term $z_{.a}^{i\cdot} z_{.b}^{j\cdot} \partial \psi / \partial X_{ab}$ can be rewritten as

$$\frac{\partial \psi}{\partial X_{ab}} z_{.a}^{i\cdot} z_{.b}^{j\cdot}=-p(\rho) Z^{ij}, \tag{4.8}$$

where the "pressure" $p(\rho)$ is defined as $p(\rho) \equiv \rho^{2} \partial \Psi(\rho)$.

Using Equation (4.8), we can rewrite the momentum Equation (4.2) in the purely Eulerian form

$$\rho\left(\frac{\partial V^{i}}{\partial t}+V^{j} \nabla_{j} V^{i}\right)=\nabla_{j}\left(-p(\rho) Z^{ij}-\frac{1}{8 \pi} H_{k} H^{k} Z^{ij}+\frac{1}{4 \pi} H^{i} H^{j}\right) \tag{4.9}$$

In this form, all the functions obviously depend only on the Eulerian coordinates z^{i} and *t*.

5. The Bulk Master System for the Permanent Magnetic Liquid Incompressible Substance

When dealing with incompressible permanent magnet liquid substance, the free energy density appears to be the function of the absolute temperature. When consider the isothermal process, the free energy density appears to be just a fixed parameter. The pressure *p* cannot be expressed in terms of the free energy density, and the Equations (4.2), (4.3) should be replaced with the following pair:

$$\rho\left(\partial_{t} V^{i}+V^{j} \nabla_{j} V^{i}\right)=\nabla_{j}\left(-p Z^{ij}-(8 \pi)^{-1} H_{k} H^{k} Z^{ij}+(4 \pi)^{-1} H^{i} H^{j}\right) \tag{5.1}$$

$$\nabla_{i} V^{i}=0 \tag{5.2}$$

The Equations (4.5)-(4.7) remain unchanged.

6. Discussion and Conclusions

We introduced the notion of a *deformable* continuum with permanent magnetization. This logically rigorous definition generalizes the widespread intuitive notion of permanent magnet. This is our first main result. The intuitive definition is logically rigorous when using the model of rigid (*i.e.*, non-deformable) solid. However, the intuitive definition is unclear when considering deformable substances.

In the simplest form (2.2), our definition of permanent deformable magnet requires the Lagrangean description of continuum media. It claims that the contravariant material components $M^a\left(x^c,t\right)$ of the magnetization vectors remain unchanged at each material point. This definition appears to be totally consistent with the traditional intuitive understanding of the permanent magnet. It remains meaningful when considering deformable substances.

Despite its simplicity, the Lagrangean definition is often inconvenient because of some other reasons. For instance, traditionally the systems of hydrodynamics and electromagnetism make use of the Eulerian variables and description (see, for instance, [2] [4] [7]). Therefore, it is desirable to establish the description of deformable continuum with permanent magnetization based on the Eulerian variables and description. This problem is solved in the section "The master equation for the model of permanent deformable magnet in the Eulerian description"; see the keynote relationship (3.1). This relationship is our second main result.

In the section "The full master system for the model of permanent deformable magnet in the Eulerian description", we established the full master systems for permanent deformable magnetizable substances.

Needless to say, that several key features of real magnets are deliberately ignored in our models. This concerns, in particular, the equations of state for the permanent magnets. They will be taken into accounts in our future studies.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Stratton, J.A. (1941) Electromagnetic Theory. McGraw-Hill, New York.

[2] Landau, L.D. and Lifshitz, E.M. (1960) Electrodynamics of Continuous Media. Pergamon Press, Oxford.

[3] Truesdell, C.A. and Toupin, R.A. (1960) The Classical Field Theories. Springer, Berlin. https://doi.org/10.1007/978-3-642-45943-6_2

[4] Tamm, I.E. (1979) Fundamentals of the Theory of Electricity. Mir Publishers, Moscow.

[5] Grinfeld, M.A. and Grinfeld, P.M. (2020) Magneto-Solid Mechanics and the Aleph Tensor. *Applied Mathematics and Physics*, **8**, 8-13. https://doi.org/10.12691/amp-8-1-2

[6] Grinfeld, M.A. and Grinfeld, P.M. (2015) A Variational Approach to Electrostatics of Polarizable Heterogeneous Substances. *Advances in Mathematical Physics*, **2015**, Article ID 659127. https://doi.org/10.1155/2015/659127

[7] Moon, F.C. (1984) Magneto-Solid Mechanics. John Wiley & Sons, Milton Park,

Abingdon-on-Thames.

[8] Furlani, E.P. (2001) Permanent Magnet and Electromechanical Devices: Materials, Analysis, and Applications. 207-333. https://doi.org/10.1016/B978-012269951-1/50005-X

[9] Kraus, R.H. (1993) Overview, History of Permanent Magnet Devices in Accel Tech, LANL.

[10] Skomski, R. and Coey, M.D. (1999) Permanent Magnetism, IOP.

[11] Grinfeld, M. (1991) Thermodynamic Methods in the Theory of Heterogeneous Systems, Interaction of Mechanics and Mathematics Series. Longman Scientific & Technical, Harlow.

[12] Grinfeld, P. (2014) Introduction to Tensor Analysis and the Calculus of Moving Surfaces. Springer, New York, NY. https://doi.org/10.1007/978-1-4614-7867-6

[13] Rosensweig, R.E. (1997) Ferrohydrodynamics. Dover, New York.

Appendix

When describing electrostatics and anisotropy simultaneously we must use both the Lagrangean coordinates X^a and the Eulerian coordinates z^i.

Let the spatial radius-vector be the time-dependent vector-function of the Eulerian coordinates $\boldsymbol{Z}=\boldsymbol{Z}(z,t)$. Let $\boldsymbol{Z}_i(z,t)=\partial\boldsymbol{Z}(z,t)/\partial z^i$, $Z_{ij}(z,t)\equiv\boldsymbol{Z}_i(z,t)\cdot\boldsymbol{Z}_j(z,t)$ be the covariant basis and metrics of the reference (Eulerian) coordinate system. We use the notation ∇_i for the (time-dependent) spatial covariant differentiation based on the metrics $Z_{ij}(z,t)$. Let $\Gamma^i_{jk}(z,t)$ be the Christoffel associated with the metrics $Z_{ij}(z,t)$.

All other operations with tensors, having the indexes of the type "i,j,k", are introduced in a standard way. In this sketch we will not dwell on those.

Let the spatial radius-vector be the time-dependent vector-function of the Lagrangean coordinates $\boldsymbol{X}=\boldsymbol{X}(x,t)$ in the actual (current) configuration Let $\boldsymbol{X}_a(x,t)=\partial\boldsymbol{X}(x,t)/\partial x^a$, $X_{ab}(x,t)\equiv\boldsymbol{X}_a(x,t)\cdot\boldsymbol{X}_b(x,t)$ be the covariant basis and metrics of the actual (current) coordinate system. We use the notation ∇_a for the time-dependent covariant differentiation, based on the metrics $X_{ab}(x,t)$. Let $\Gamma^a_{bc}(x,t)$ be the Christoffels associated with the metrics $X_{ab}(x,t)$. All other operations with tensors, having the indexes of the type "$a,b,c,\cdots$" are introduced in a standard way. In this draft we will not dwell on those.

So far, there was no difference between the Lagrangean and Eulerian coordinates. *The key difference between the two in various applications is the following*. The evolution of the Eulerian coordinate system (basically the radius-vector $\boldsymbol{Z}(z,t)$ field) is assumed known up-front: it is not part of solving problem under consideration. On the contrary, the radius-vector $\boldsymbol{X}(x,t)$ of the actual coordinate system is the central un-known field of the problem under study. Typically, we learn the motion of the actual radius-vector with respect to the reference coordinate system. The problems are usually formulated as finding of the motion-functions

$$z^i=z^i\left(x^a,t\right),\ x^a=x^a\left(z^i,t\right),\ z^i\equiv z^i\left(x^a\left(z^j,t\right),t\right) \tag{A.1}$$

Since, we know everything about the Eulerian (reference) coordinate system the ultimate independent unknown are $\left(z^i,t\right)$; This means, in particular, that the main unknown function is $x^a\left(z^i,t\right)$, not $z^i\left(x^a,t\right)$ (although the two are closely connected).

All said so far, can be expressed as the identities

$$\boldsymbol{Z}\left(z^i,t\right)\equiv\boldsymbol{X}\left(x^a\left(z^i,t\right),t\right),\ \boldsymbol{X}\left(x^a,t\right)\equiv\boldsymbol{Z}\left(z^i\left(x^a,t\right),t\right) \tag{A.2}$$

These identities imply the following ones:

$$z^{i\cdot}_{\cdot a}\left(x(z,t),t\right)x^{a\cdot}_{\cdot j}(z,t)=\delta^i_j\ ,\ z^{i\cdot}_{\cdot a}\left(x(z,t),t\right)x^{b\cdot}_{\cdot i}(z,t)=\delta^b_a \tag{A.3}$$

The two-point tensors $z^{i\cdot}_{\cdot a}(x,t)$ and $x^{a\cdot}_{\cdot j}(z,t)$ are called distortions. According to what was said above, even the distortion $z^{i\cdot}_{\cdot a}(x,t)$ should be ultimately treated as the tensor function of $\left(z^i,t\right)$.

The following identities are valid for the Christoffel symbols

$$\Gamma^{a}_{bc} = \Gamma^{i}_{jk} x^{a\cdot}_{\cdot i} z^{j\cdot}_{\cdot b} z^{k\cdot}_{\cdot c} + \frac{\partial^2 z^i}{\partial x^b \partial x^c} x^{a\cdot}_{\cdot i},$$
$$\frac{\partial^2 z^i}{\partial x^b \partial x^c} = \Gamma^{a}_{bc} z^{i\cdot}_{\cdot a} - \Gamma^{i}_{jk} z^{j\cdot}_{\cdot b} z^{k\cdot}_{\cdot c}, \ \frac{\partial^2 x^a}{\partial z^j \partial z^k} = \Gamma^{i}_{jk} x^{a\cdot}_{\cdot i} - \Gamma^{a}_{bc} x^{b\cdot}_{\cdot j} x^{c\cdot}_{\cdot k} \tag{A.4}$$

For the mixed tensors of the type $T^{i\cdot a\cdot}_{\cdot j\cdot b}(z,t)$ the actual covariant differentiation is defined as follows

$$\nabla_k T^{i\cdot a\cdot}_{\cdot j\cdot b}(z,t) \equiv \frac{\partial T^{i\cdot a\cdot}_{\cdot j\cdot b}(z,t)}{\partial z^k} + \Gamma^{i}_{kl} T^{i\cdot a\cdot}_{\cdot j\cdot b} - \Gamma^{l}_{kj} T^{i\cdot a\cdot}_{\cdot j\cdot b} + x^{d\cdot}_{\cdot k} \Gamma^{a}_{dc} T^{i\cdot c\cdot}_{\cdot j\cdot b} - x^{d\cdot}_{\cdot k} \Gamma^{c}_{db} T^{i\cdot a\cdot}_{\cdot j\cdot c} \tag{A.5}$$

The Lagrangean covariant differentiation ∇_k is defined according to the identity

$$\nabla_c = z^{k\cdot}_{\cdot c} \nabla_k, \nabla_k = x^{c\cdot}_{\cdot k} \nabla_c \tag{A.6}$$

The following relationships for the bases and distortions are of key importance

$$\nabla_k \boldsymbol{Z}_i = 0, \nabla_k \boldsymbol{X}_a = 0, \nabla_c \boldsymbol{Z}_i = 0, \nabla_c \boldsymbol{X}_a = 0 \tag{A.7}$$

and

$$\nabla_k z^{i\cdot}_{\cdot a}(z,t) = 0, \nabla_k x^{a\cdot}_{\cdot i}(z,t) = 0, \nabla_c z^{i\cdot}_{\cdot a}(z,t) = 0, \nabla_c x^{a\cdot}_{\cdot i}(z,t) = 0 \tag{A.8}$$

The following relationships for the metrics read

$$\nabla_k Z_{ij} = \nabla_c Z_{ij} = \nabla_k Z^{ij} = \nabla_c Z^{ij} = \nabla_c Z^{ij} = \nabla_k \delta^i_j = \nabla_c \delta^i_j = 0,$$
$$\nabla_k X_{ab} = \nabla_c X_{ab} = \nabla_k X^{ab} = \nabla_c X^{ab} = \nabla_c X^{ab} = \nabla_k \delta^a_b = \nabla_c \delta^a_b = 0 \tag{A.9}$$