Collatz Iterative Trajectories of All Odd Numbers Attain Bounded Values

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Abstract

The aim of this paper is to study the $3x + 1$ problem based on the Collatz iterative formula. It can be seen from the iterative formula that the necessary condition for the Collatz iteration convergence is that its slope being less than 1. An odd number N that satisfies the condition of a slope less than 1 after n^{th} Collatz iterations is defined as an n-step odd number. Through statistical analysis, it is found that after n^{th} Collatz iterations, the iterative value of any n-step odd number N that is greater than 1 is less than N, which proves that the slope less than 1 is a sufficient and necessary condition for Collatz iteration convergence.

1. Introduction

The $3x + 1$ problem, also known as the "Collatz conjecture" or the "Syracuse problem", is an interesting topic. It is easy to understand, but not yet proved. The $3x + 1$ problem has attracted many mathematic lovers including famous scholars [1]-[7]. The problem can be described as follows: Take any odd number, multiply it by 3 and add 1 to make it an even number, divide it by the power of 2 to make it an odd number, and repeat the arithmetic operation, the final result must be 1. The $3x + 1$ problem can also be written as: Take any odd number greater than 1, multiply it by 3 and add 1 to make it an even number, divide it by the power of 2 to make it an odd number, and repeat the arithmetic operation, the final result must be less than the original odd number. Documents [1] and [2]

have proved that the necessary condition for $C_n(N) < N$ is $m = \text{int}\left(n\dfrac{\log 3}{\log 2} + 1\right)$

and also proved that when $m = \text{int}\left(n\dfrac{\log 3}{\log 2} + 1\right)$, there are many odd numbers

that are also convergent, that is, $C_n(N) < N$, but it has not been proven that all odd numbers are convergent. This paper is based on the Collatz iterative formula $C_n(N) = \dfrac{3^n}{2^m}N + S_n(N)$, through the change patterns of $\log\left[\dfrac{2^m S_n(N)}{\left(2^m - 3^n\right)N}\right]$

towards $\log\left(\dfrac{2^m - 3^n}{3^n}N\right)$, to prove Collatz iteration convergence, that is, the

necessary and sufficient condition for $C_n(N) < N$ is $m = \text{int}\left(n\dfrac{\log 3}{\log 2} + 1\right)$.

2. Collatz Iterative Formula

For any odd number N, multiply by 3 and add 1 to make it an even number is called an odd transformation, and divide by the power of 2 to make it an odd number is called an even transformation. This process can be called an odd-even transformation.

The Collatz algorithm of an odd-even transformation can be written as:

$$C_1(N) = \frac{3N+1}{2^{m_1}} = \frac{3}{2^{m_1}}N + S_1(N) \tag{1.1}$$

$$S_1(N) = \frac{1}{2^{m_1}} \tag{1.2}$$

The Collatz algorithm of 2nd odd-even transformations can be written as:

$$C_2(N) = \frac{3C_1(N)+1}{2^{m_2}} = \frac{3^2}{2^{m_2+m_1}}N + S_2(N) \tag{1.3}$$

$$S_2(N) = \frac{3S_1(N)+1}{2^{m_2}} = \frac{3^{2-1} + 2^{m_1}}{2^{m_2+m_1}} \tag{1.4}$$

The Collatz algorithm of 3rd odd-even transformations can be written as:

$$C_3(N) = \frac{3C_2(N)+1}{2^{m_3}} = \frac{3^3}{2^{m_3+m_2+m_1}}N + S_3(N) \tag{1.5}$$

$$S_3(N) = \frac{3S_2(N)+1}{2^{m_3}} = \frac{3^{3-1} + 3^{3-2}2^{m_1} + 2^{m_2+m_1}}{2^{m_3+m_2+m_1}} \tag{1.6}$$

Thus, the Collatz algorithm of nth odd-even transformations can be written as:

$$C_n(N) = \frac{3^n}{2^{m_n+m_{n-1}+\cdots+m_2+m_1}}N + S_n(N) \tag{1.7}$$

$$S_n(N) = \frac{3^{n-1} + 3^{n-2}2^{m_1} + 3^{n-3}2^{m_2+m_1} + \cdots + 3^1 2^{m_{n-2}+\cdots+m_2+m_1} + 2^{m_{n-1}+\cdots+m_2+m_1}}{2^{m_n+m_{n-1}+\cdots+m_2+m_1}} \tag{1.8}$$

Adopt the summation notation [3]:

$$m = \sum\nolimits_{i=1}^{n} m_i \tag{1.9}$$

$$m_{[1,k]} = \sum_{i=1}^{k} m_i \quad (1 < k < n) \tag{1.10}$$

m_i is the number of times of even transformations with regards to the ith odd transformations, m is the accumulative number of times of even transformations.

So (1.7) can be written as:

$$C_n(N) = \frac{3^n}{2^m} N + S_n(N) \tag{1.11}$$

$\frac{3^n}{2^m}$ is the slope of the Collatz iterative formula, $S_n(N)$ is the intercept of the Collatz iterative formula.

And (1.8) can be written as:

$$S_n(N) = \frac{3^{n-1} + 3^{n-2} 2^{m_1} + 3^{n-3} 2^{m_{[1,2]}} + \cdots + 3^1 2^{m_{[1,n-2]}} + 2^{m_{[1,n-1]}}}{2^m} \tag{1.12}$$

3. Necessary Conditions for Collatz Iteration Convergence

3.1. Necessary Conditions and the Definition of *n*-Step Odd Number

Define $D_n(N)$ as follows:

$$D_n(N) = N - C_n(N) \tag{2.1}$$

Substitute Equation (1.11) into Equation (2.1) we can get:

$$D_n(N) = \frac{2^m - 3^n}{2^m} N - S_n(N) \tag{2.2}$$

Since $S_n(N) > 0$, then the necessary condition for $D_n(N) \geq 0$ is [2]:

$$2^m - 3^n > 0 \tag{2.3}$$

which is, the slope $\frac{3^n}{2^m}$ is less than 1.

From Equation (2.3) it can be obtained that:

$$m > n \frac{\log 3}{\log 2} \tag{2.4}$$

Mark the smallest m that satisfies (2.4) as:

$$m = \text{int}\left(n \frac{\log 3}{\log 2} + 1 \right) \tag{2.5}$$

Then it can be found that when $m < \text{int}\left(n \frac{\log 3}{\log 2} + 1 \right)$, the odd number N will not fall into a cycle. If an odd number does not fall into a cycle during the odd and even transformation process, then after n times of odd transformations, m times of even transformations will eventually occur, making Equation (2.5) true [4].

The odd number N that satisfies Equation (2.5) is called *n*-step odd number.

Define ε as follows:

$$\varepsilon = m - n\frac{\log 3}{\log 2} = \mathrm{int}\left(n\frac{\log 3}{\log 2} + 1\right) - n\frac{\log 3}{\log 2} \qquad (2.6)$$

ε is a function of n, and its value is greater than 0 and less than 1. The relationship between m, ε and n is shown in Table 1.

Table 1. The relationships among $m, \varepsilon, S_n(N)_{\min}, S_n(N)_{\max}$ and n.

n	m	ε	$n/2^{\varepsilon}$	$S_n(N)_{\max}$	$S_n(N)_{\min}$	$S_n(N)_{\max}/S_n(N)_{\min}$
1	2	0.4150	0.7500	0.2500	0.2500	1.0000
2	4	0.8301	0.5625	0.3125	0.3125	1.0000
3	5	0.2451	0.8438	0.7188	0.5938	1.2105
4	7	0.6601	0.6328	0.6641	0.5078	1.3077
5	8	0.0752	0.9492	1.2461	0.8242	1.5118
6	10	0.4902	0.7119	1.0596	0.6494	1.6316
7	12	0.9053	0.5339	0.9197	0.5027	1.8295
8	13	0.3203	0.8009	1.6295	0.7697	2.1172
9	15	0.7353	0.6007	1.3471	0.5851	2.3026
10	16	0.1504	0.9010	2.2707	0.8854	2.5646
20	32	0.3007	0.8118	4.0259	0.8116	4.9605
30	48	0.4511	0.7315	5.3488	0.7315	7.3124
40	64	0.6015	0.6591	6.4221	0.6591	9.7442
50	80	0.7519	0.5938	7.1836	0.5938	12.0971
60	96	0.9022	0.5351	7.7760	0.5351	14.5332
70	111	0.0526	0.9642	16.4415	0.9642	17.0523
80	127	0.2030	0.8687	16.9346	0.8687	19.4933
90	143	0.3534	0.7828	17.1048	0.7828	21.8522
100	159	0.5037	0.7053	17.0145	0.7053	24.1247
200	317	0.0075	0.9948	48.0471	0.9948	48.2976
300	476	0.5112	0.7016	50.6659	0.7016	72.2133
400	634	0.0150	0.9897	95.3488	0.9897	96.3453
500	793	0.5187	0.6980	84.0264	0.6980	120.3857
600	951	0.0225	0.9845	142.2404	0.9845	144.4761
700	1110	0.5262	0.6944	116.9818	0.6944	168.4748
800	1268	0.0300	0.9794	188.7254	0.9794	192.6908
900	1427	0.5337	0.6908	149.6505	0.6908	216.6470
1000	1585	0.0375	0.9743	234.4806	0.9743	240.6552
2000	3170	0.0750	0.9493	456.8498	0.9493	481.2273
3000	4755	0.1125	0.9250	667.5753	0.9250	721.7146
4000	6340	0.1500	0.9013	867.2626	0.9013	962.2862
5000	7925	0.1875	0.8781	1056.1896	0.8781	1202.7737

3.2. Collatz Iterative Trajectory Vector

In the Collatz iterative trajectory sequence $\{C_1(N), C_2(N), \cdots, C_{n-1}(N), C_n(N)\}$ of n-step odd number N, $C_1(N), C_2(N), \cdots, C_{n-1}(N)$ are all odd numbers, and $C_n(N)$ may be an odd number or an even number.

$(m_1, m_2, \cdots, m_n)$ which correspond to the Collatz iterative trajectory sequence is called the Collatz iterative trajectory vector of n-step odd number N, where $m_1, m_2, \cdots, m_{n-1} \geq 1$ and $m_n \geq 2$.

It can be seen that:

The number of types of 1-step odd number less than 2^2 is 1, and the Collatz iterative trajectory vector is:

(2)

The number of types of 2-step odd number less than 2^4 is 1, and the Collatz iterative trajectory vector is:

(1, 3)

The number of types of 3-step odd numbers less than 2^5 is 2, and the Collatz iterative trajectory vectors are:

(1, 1, 3)
(1, 2, 2)

The number of types of 4-step odd numbers less than 2^7 is 3, and the Collatz iterative trajectory vectors are:

(1, 1, 1, 4)
(1, 1, 2, 3)
(1, 2, 1, 3)

The number of types of 5-step odd numbers less than 2^8 is 7, and the Collatz iterative trajectory vectors are:

(1, 1, 1, 1, 4)
(1, 1, 1, 2, 3)
(1, 1, 1, 3, 2)
(1, 1, 2, 1, 3)
(1, 1, 2, 2, 2)
(1, 2, 1, 1, 3)
(1, 2, 1, 2, 2)

The number of types of 6-step odd numbers less than 2^{10} is 12, and the Collatz iterative trajectory vectors are:

(1, 1, 1, 1, 1, 5)
(1, 1, 1, 1, 2, 4)
(1, 1, 1, 1, 3, 3)
(1, 1, 1, 2, 1, 4)
(1, 1, 1, 2, 2, 3)
(1, 1, 1, 3, 1, 3)
(1, 1, 2, 1, 1, 4)
(1, 1, 2, 1, 2, 3)
(1, 1, 2, 2, 1, 3)

(1, 2, 1, 1, 1, 4)

(1, 2, 1, 1, 2, 3)

(1, 2, 1, 2, 1, 3)

By analogy, all odd-numbered Collatz iterative trajectory vectors of any number of steps can be constructed.

Assume that the number of n-step odd numbers less than $2^{\text{int}\left(n\frac{\log 3}{\log 2}+1\right)}$ is q, and its vector is: $\left(m_{j1}, m_{j2}, \cdots, m_{jn}\right)$ $j = 1, 2, \cdots, q$, then the number of $n + 1$-step odd numbers less than $2^{\text{int}\left((n+1)\frac{\log 3}{\log 2}+1\right)}$ is $\sum_{j=1}^{q} m_{jn} - q$.

3.3. The Maximum and Minimum Values of $S_n(N)$ and Their Relationships to n

By substituting the Collatz iterative trajectory vector components of n-step odd numbers into Equation (1.12), the $S_n(N)$ values of n-step odd numbers can be calculated.

1-step odd numbers can be represented by $2^2 x + 1$, and their $S_n(N)$ value is:

$$S_1\left(2^2 x + 1\right) = \frac{3^{1-1}}{2^2} = 0.25 \tag{2.7}$$

Note x is a non-negative integer.

2-step odd numbers can be represented by $2^4 x + 3$, and their $S_n(N)$ value is:

$$S_2\left(2^4 x + 3\right) = \frac{3^{2-1} + 2^1}{2^4} = 0.3125 \tag{2.8}$$

3-step odd numbers can be represented by $2^5 x + 23$ or $2^5 x + 11$, and their $S_n(N)$ values are:

$$S_3\left(2^5 x + 23\right) = \frac{3^{3-1} + 3^{3-2} 2^1 + 2^2}{2^5} = 0.59375 \tag{2.9}$$

$$S_3\left(2^5 x + 11\right) = \frac{3^{3-1} + 3^{3-2} 2^1 + 2^3}{2^5} = 0.71875 \tag{2.10}$$

4-step odd numbers can be represented by $2^7 x + 15$, $2^7 x + 7$, or $2^7 x + 59$, and their $S_n(N)$ values are:

$$S_4\left(2^7 x + 15\right) = \frac{3^{4-1} + 3^{4-2} 2^1 + 3^{4-3} 2^2 + 2^3}{2^7} = 0.50781 \tag{2.11}$$

$$S_4\left(2^7 x + 7\right) = \frac{3^{4-1} + 3^{4-2} 2^1 + 3^{4-3} 2^2 + 2^4}{2^7} = 0.57031 \tag{2.12}$$

$$S_4\left(2^7 x + 59\right) = \frac{3^{4-1} + 3^{4-2} 2^1 + 3^{4-3} 2^3 + 2^4}{2^7} = 0.66406 \tag{2.13}$$

5-step odd numbers can be represented by $2^8 x + 95$, $2^8 x + 175$, $2^8 x + 39$, $2^8 x + 79$, $2^8 x + 199$, $2^8 x + 219$ or $2^8 x + 123$, and their $S_n(N)$ values are:

$$S_5\left(2^8 x+95\right)=\frac{3^{5-1}+3^{5-2}2^1+3^{5-3}2^2+3^{5-4}2^3+2^4}{2^8}=0.82422 \tag{2.14}$$

$$S_5\left(2^8 x+175\right)=\frac{3^{5-1}+3^{5-2}2^1+3^{5-3}2^2+3^{5-4}2^3+2^5}{2^8}=0.88672 \tag{2.15}$$

$$S_5\left(2^8 x+39\right)=\frac{3^{5-1}+3^{5-2}2^1+3^{5-3}2^2+3^{5-4}2^4+2^5}{2^8}=0.98047 \tag{2.16}$$

$$S_5\left(2^8 x+79\right)=\frac{3^{5-1}+3^{5-2}2^1+3^{5-3}2^2+3^{5-4}2^3+2^6}{2^8}=1.01172 \tag{2.17}$$

$$S_5\left(2^8 x+199\right)=\frac{3^{5-1}+3^{5-2}2^1+3^{5-3}2^2+3^{5-4}2^4+2^6}{2^8}=1.11547 \tag{2.18}$$

$$S_5\left(2^8 x+219\right)=\frac{3^{5-1}+3^{5-2}2^1+3^{5-3}2^3+3^{5-4}2^4+2^5}{2^8}=1.12109 \tag{2.19}$$

$$S_5\left(2^8 x+123\right)=\frac{3^{5-1}+3^{5-2}2^1+3^{5-3}2^3+3^{5-4}2^4+2^6}{2^8}=1.24609 \tag{2.20}$$

It can be seen that there are multiple odd numbers that are equal to or greater than 3-step odd number. For the same step but different types of odd number, their $S_n(N)$ values are different, with $S_n(N)$ minimum value and maximum value exist.

From Equation (1.12), we can see that when the first $n-1$ components of the Collatz iterative trajectory vector are 1, and the last component is $m-n+1$, that is, the trajectory vector of the n-step odd number is $(1,1,\cdots,1,m-n+1)$ then $S_n(N)$ has the minimum value. Substitute the trajectory vector components into Equation (1.12), we can get:

$$\begin{aligned} S_n(N)_{\min} &=\frac{3^{n-1}+3^{n-2}2^1+3^{n-3}2^2+\cdots+3^1 2^{n-2}+2^{n-1}}{2^m} \\ &=\frac{\left(\frac{3}{2}\right)^{n-1}+\left(\frac{3}{2}\right)^{n-2}+\cdots+\frac{3}{2}+1}{2^{m-n+1}}=\frac{3^n-2^n}{2^m} \end{aligned} \tag{2.21}$$

Based on the value of n, Equation (2.5), and Equation (2.21), $S_n(N)_{\min}$ of n-step odd numbers can be calculated. Refer to Table 1.

Based on Equation (2.6), Equation (2.21) can be written as:

$$S_n(N)_{\min}=\frac{1}{2^\varepsilon}\left[1-\left(\frac{2}{3}\right)^n\right]<\frac{1}{2^\varepsilon}<1 \tag{2.22}$$

When n is large enough, then

$$S_n(N)_{\min}\approx\frac{1}{2^\varepsilon} \tag{2.23}$$

Based on Equation (1.12), the trajectory vector components of the maximum $S_n(N)$ should meet the following conditions:

$$m_{[1,k]}=\sum_{i=1}^{k}m_i=\text{int}\left(k\frac{\log 3}{\log 2}\right) \quad (1\le k<n, m_i=1 \text{ or } 2) \tag{2.24}$$

$$m = \sum_{i=1}^{n} m_i = \operatorname{int}\left(n\frac{\log 3}{\log 2} + 1\right) \quad (m_n = 2 \text{ or } 3) \tag{2.25}$$

Substitute the Collatz iterative trajectory vector components that satisfy Equation (2.24) and Equation (2.25) into Equation (1.12), the $S_n(N)_{\max}$ can be calculated. Refer to **Table 1**.

According to statistics, $S_n(N)_{\max}$ has a good linear relationship with $\frac{n}{2^\varepsilon}$.

Figure 1 shows their relationship when n is in the range of 1 - 50, and **Figure 2** shows their relationship when n in the range of 1 - 500. The larger the n is, the better the linear relationship is. If the value of n is small, an intercept can be added to the linear regression.
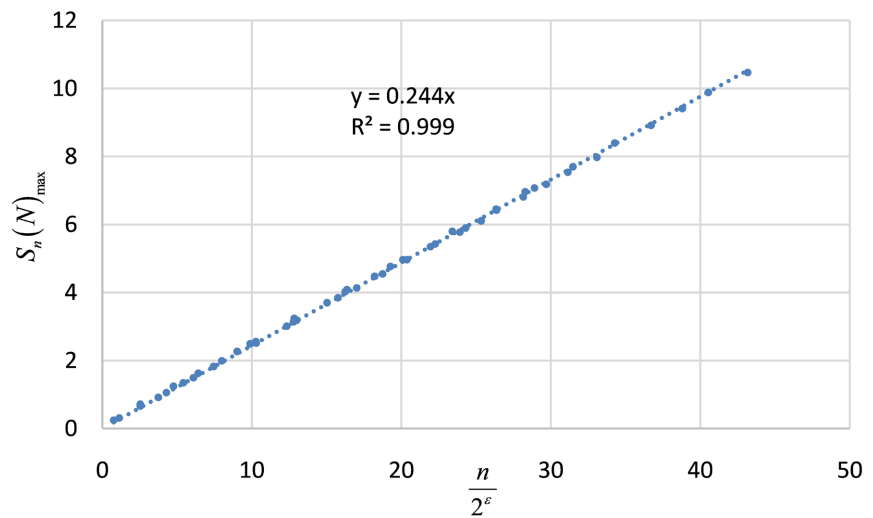
It can be seen from **Figure 2**.

Figure 1. The relationship between $S_n(N)_{\max}$ and $\frac{n}{2^\varepsilon}$ (n is in the range of 1 to 50).
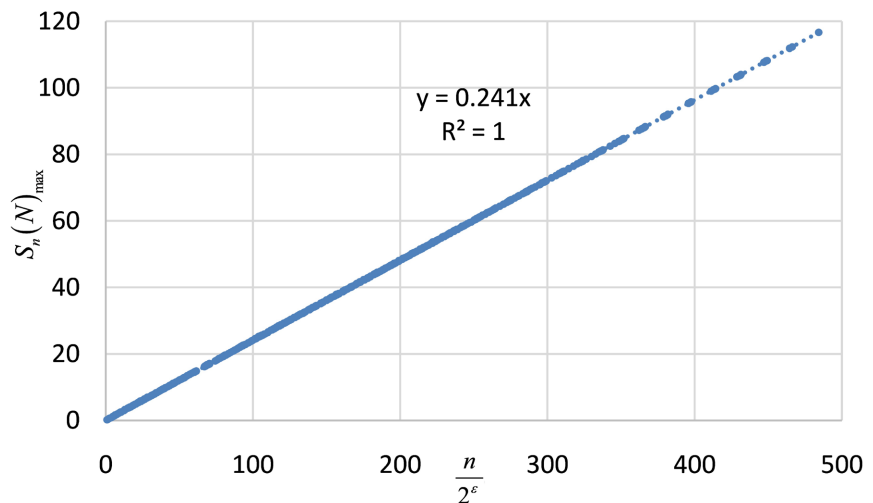

Figure 2. The relationship between $S_n(N)_{\max}$ and $\frac{n}{2^\varepsilon}$ (n is in the range of 1 to 500).

$$S_n(N)_{\max} \approx \frac{0.241n}{2^\varepsilon} \tag{2.26}$$

Based on (2.23), Equation (2.26) can be written as:

$$\frac{S_n(N)_{\max}}{S_n(N)_{\min}} \approx 0.241n \tag{2.27}$$

That is, the ratio of $S_n(N)_{\max}/S_n(N)_{\min}$ is approximately proportional to n.

4. Necessary and Sufficient Conditions for Collatz Iteration Convergence

4.1. n-Step Odd Number Converges When $S_n(N) = S_n(N)_{\min}$

From Equation (2.2) we can get:

$$N = \frac{2^m D_n(N) + 2^m S_n(N)_{\min}}{2^m - 3^n} \tag{3.1}$$

Substitute Equation (2.21) into Equation (3.1) we can get:

$$N = \frac{2^m D_n(N) + 3^n - 2^n}{2^m - 3^n} \tag{3.2}$$

If the iteration is divergent, that is, $D_n(N) < 0$, because $D_n(N)$ is an integer, that is, $D_n(N) \le -1$, substitute it into Equation (3.2) we can get:

$$N \le \frac{-2^m + 3^n - 2^n}{2^m - 3^n} = -1 - \frac{2^n}{2^m - 3^n} < 0 \tag{3.3}$$

It can be seen that $D_n(N)$ cannot be less than 0.

If the iteration is cyclic, that is, $D_n(N) = 0$, substitute it into Equation (3.2) we can get:

$$\left(2^m - 3^n\right)N = 3^n - 2^n \tag{3.4}$$

For $S_n(N) = S_n(N)_{\min}$, the n-step odd number N can be expressed as:

$$N = 2^n x - 1 \tag{3.5}$$

Substitute Equation (3.5) into (3.4) and arrange it, we can get:

$$\left(2^m - 3^n\right)x = 2^{m-n} - 1 \tag{3.6}$$

It can be seen from the literature [5] that there is no other solution except $x = 1, n = 1, m = 2$ (that is, $N = 1$). This proves that when $S_n(N) = S_n(N)_{\min}$, n-step odd number $N(>1)$ neither diverges nor cycles. That is, for this special n-step odd number, Equation (2.5) is the necessary and sufficient condition for convergence.

4.2. n-Step Odd Number $N_{n\min}$ Converges

Literature [1] has proven that n-step odd numbers N satisfy Equation (2.5), then there do exist n-step odd numbers $2^m x + N$ such that $C_n\left(2^m x + N\right) < 2^m x + N$. Now we first prove that for the minimum odd number $N_{n\min}$ of n-step odd

numbers, the relationship $C_n(N_{n\min}) < N_{n\min}$ is established.

From Equation (2.2) we can get:

$$D_n(N) = \frac{2^m - 3^n}{2^m} N \left[1 - \frac{2^m S_n(N)}{(2^m - 3^n) N} \right] \tag{3.7}$$

Substitute $N = N_{n\min}$ into the above formula to get:

$$D_n(N_{n\min}) = \frac{2^m - 3^n}{2^m} N_{n\min} \left[1 - \frac{2^m S_n(N_{n\min})}{(2^m - 3^n) N_{n\min}} \right]$$
$$= \frac{2^\varepsilon - 1}{2^\varepsilon} N_{n\min} \left[1 - \frac{2^\varepsilon S_n(N_{n\min})}{(2^\varepsilon - 1) N_{n\min}} \right] \tag{3.8}$$

According to Equation (2.5), the minimum odd number $N_{n\min}$ of n-step odd numbers N ($1 < N < 2^{30}$) can be obtained. **Figure 3** shows the relationship between $\log\left[(2^\varepsilon - 1) N_{n\min}\right]$ and $\log(N_{n\min})$. It can be seen from **Figure 3** that $\log\left[(2^\varepsilon - 1) N_{n\min}\right]$ increases spirally as $\log(N_{n\min})$ goes up.

Figure 4 shows the relationship between $\log\left[\dfrac{2^\varepsilon S_n(N_{n\min})}{(2^\varepsilon - 1) N_{n\min}}\right]$ and $\log\left[(2^\varepsilon - 1) N_{n\min}\right]$. It can be seen from **Figure 4** that $\dfrac{2^\varepsilon S_n(N_{n\min})}{(2^\varepsilon - 1) N_{n\min}}$ is always less than 1 and gets close to 0 as $\log\left[(2^\varepsilon - 1) N_{n\min}\right]$ increases spirally. Therefore, $D_n(N_{n\min})$ is neither less than 0 nor equal to 0, but is always greater than 0. That is, the relationship $C_n(N_{n\min}) < N_{n\min}$ is established.

4.3. All Odd Numbers Greater Than 1 Converge

It is proved below that all odd numbers N less than 2^m satisfy $C_n(N) < N$. From Equation (3.7) we can get:
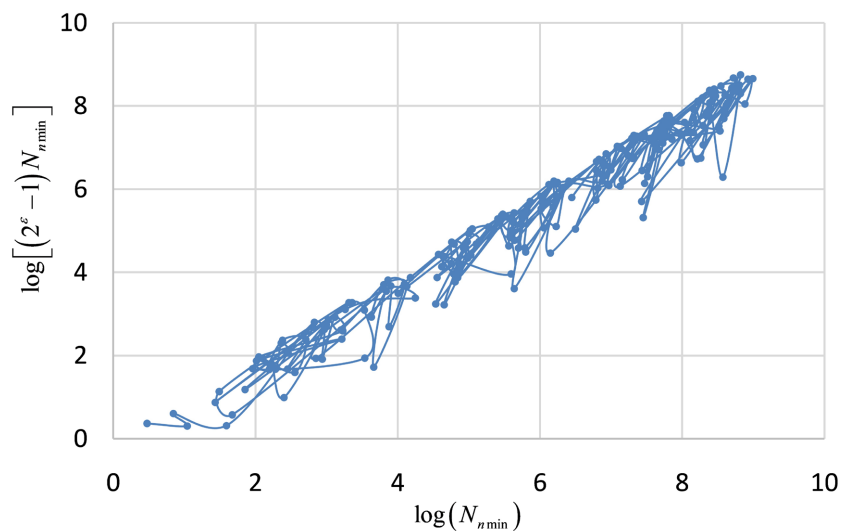

Figure 3. The relationship between $\log\left[(2^\varepsilon - 1) N_{n\min}\right]$ and $\log(N_{n\min})$ ($1 < N < 2^{30}$).

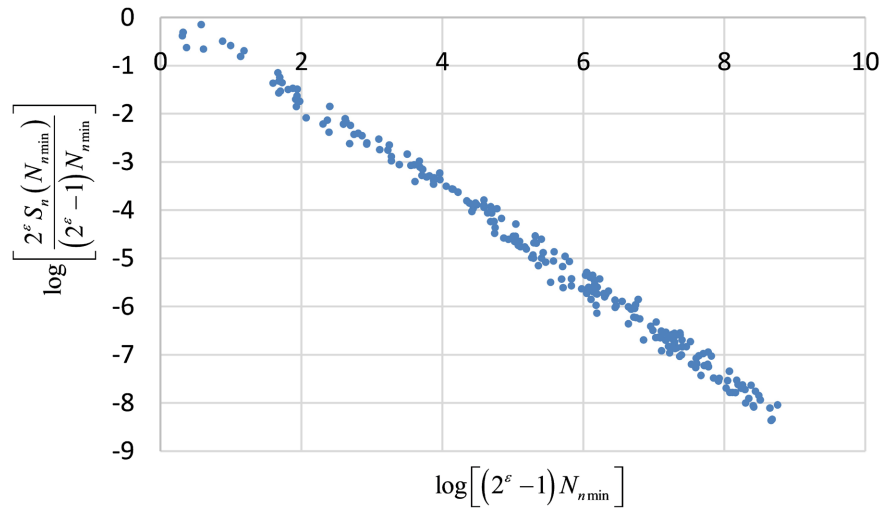**Figure 4.** The relationship between $\log\left[\dfrac{2^{\varepsilon} S_n\left(N_{n\min}\right)}{\left(2^{\varepsilon}-1\right)N_{n\min}}\right]$ and $\log\left[\left(2^{\varepsilon}-1\right)N_{n\min}\right]$ $\left(1 < N < 2^{30}\right)$.
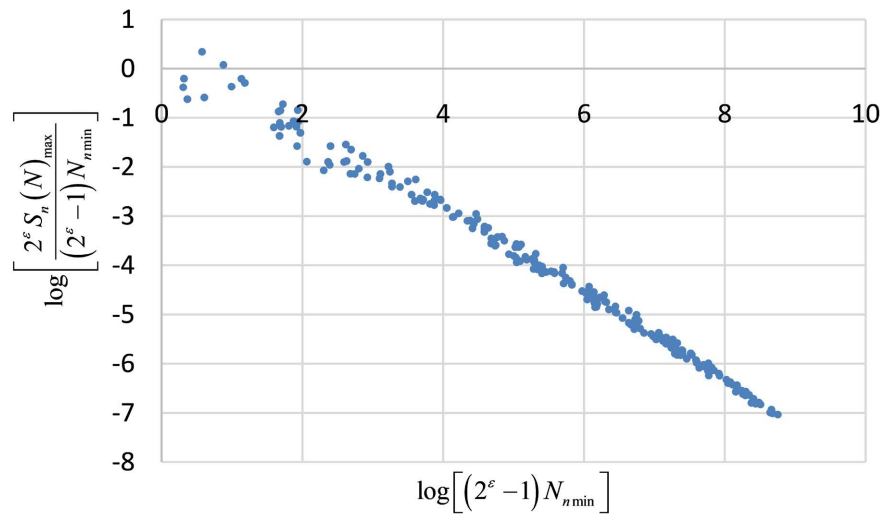
Figure 5. The relationship between $\log\left[\dfrac{2^{\varepsilon} S_n\left(N\right)_{\max}}{\left(2^{\varepsilon}-1\right)N_{n\min}}\right]$ and $\log\left[\left(2^{\varepsilon}-1\right)N_{n\min}\right]$ $\left(1 < N < 2^{30}\right)$.

$$D_n\left(N\right) = \frac{2^m - 3^n}{2^m}N\left[1 - \frac{2^{\varepsilon}S_n\left(N\right)}{\left(2^{\varepsilon}-1\right)N}\right] \geq \frac{2^m - 3^n}{2^m}N\left[1 - \frac{2^{\varepsilon}S_n\left(N\right)_{\max}}{\left(2^{\varepsilon}-1\right)N_{n\min}}\right] \quad (3.9)$$

According to Equation (2.24), Equation (2.25) and Equation (2.5), the maximum value $S_n\left(N\right)_{\max}$ of n-step odd number can be calculated. **Figure 5** shows the relationship between $\log\left[\dfrac{2^{\varepsilon} S_n\left(N\right)_{\max}}{\left(2^{\varepsilon}-1\right)N_{n\min}}\right]$ and $\log\left[\left(2^{\varepsilon}-1\right)N_{n\min}\right]$.

As depicted in **Figure 5**, only when $n = 34, N_{34\min} = 47$ and

$n = 37, N_{37\min} = 27$, $\log\left[\dfrac{2^{\varepsilon} S_n(N)_{\max}}{(2^{\varepsilon}-1) N_{n\min}}\right] > 0$, that is, $\dfrac{2^{\varepsilon} S_n(N)_{\max}}{(2^{\varepsilon}-1) N_{n\min}} > 1$; in other

cases, $\dfrac{2^{\varepsilon} S_n(N)_{\max}}{(2^{\varepsilon}-1) N_{n\min}}$ is always less than 1, and tends to be close to 0 as

$\log\left[(2^{\varepsilon}-1) N_{n\min}\right]$ spirals up. From Section 4.2, we can see that $N_{34\min} = 47$ and $N_{37\min} = 27$ are convergent, which proves that $D_n(N) = N - C_n(N) > 0$ for all odd numbers. Therefore, Collatz conjecture is established, and n-step odd numbers can also be called n-step convergence odd numbers.

5. Conclusions

1) For any odd number N, it will not fall into a cycle when $m < \text{int}\left(n\dfrac{\log 3}{\log 2} + 1\right)$, then after n times of odd transformations, m times of even transformations will eventually occur, making equation $m = \text{int}\left(n\dfrac{\log 3}{\log 2} + 1\right)$ true.

2) For any odd number N that is greater than 1, $m = \text{int}\left(n\dfrac{\log 3}{\log 2} + 1\right)$ is the necessary and sufficient condition for Collatz iteration convergence, that is, $C_n(N) < N$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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