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Lipschitz Regularity of Viscosity Solutions to the Infinity Laplace Equation

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Abstract

In this paper, we study the viscosity solutions of the Neumann problem

$$\begin{cases} \Delta_\infty^N u + \beta|Du| + \xi(x)\cdot Du + \eta(x)u = g(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

in a bounded C^2 domain Ω, where Δ_∞^N is called the normalized infinity Laplacian. The normalized infinity Laplacian was first studied by Peres, Shramm, Sheffield and Wilson from the point of randomized theory named tug-of-war, which has wide applications in optimal mass transportation, financial option price problems, digital image processing, physical engineering, etc. We give the Lipschitz regularity of the viscosity solutions of the Neumann problem. The method we adopt is to choose suitable auxiliary functions as barrier functions and combine the perturbation method and viscosity solutions theory.

Keywords

Normalized Infinity Laplacian, Viscosity Solution, Lipschitz Regularity

1. Introduction

In this paper, we study the Lipschitz regularity of the viscosity solutions of the Neumann problem

$$\begin{cases} \Delta_\infty^N u + \beta|Du| + \xi(x)\cdot Du + \eta(x)u = g(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded C^2 domain in $\mathbb{R}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$ denotes a set of $\mathbb{N}$-dimensional Euclidean space, $\beta \in \mathbb{R}$, $\partial\Omega$ denotes the boundary of Ω, $\vec{n}(x)$ is the unit exterior normal to the domain Ω at $x \in \partial\Omega$, $\xi(x):\Omega \to \mathbb{R}^{\mathbb{N}}$, $\eta(x):\Omega \to \mathbb{R}$

are continuous in $\bar{\Omega}$ (the closure of Ω), $g(x)$ is a bounded function in $\bar{\Omega}$ and

$$\Delta_{\infty}^{N} u=\left\langle D^{2} u \frac{D u}{|D u|}, \frac{D u}{|D u|}\right\rangle, \tag{2}$$

is called the normalized infinity Laplacian.

The infinity Laplace equation

$$\Delta_{\infty} u=\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial u}{\partial x_{i}} \frac{\partial^{2} u}{\partial x_{i} \partial x_{j}} \frac{\partial u}{\partial x_{j}}=0,$$

is the Euler-Lagrange equation associated with L^{∞}-variational problem related to the absolutely minimizing Lipschitz extensions, which was first studied by Aronsson [1] [2] [3] [4]. The infinity Laplacian has attracted more and more attention because it is highly degenerate and has no variational structure. It has been widely used in the Monge-Kantorovich mass transfer problem in [5], digital image processing in [6] [7] and financial mathematics in [8].

$\Delta_{\infty}^{N} u+\xi(x) \cdot D u$ is called the infinity Laplacian with a transport term related to tug-of-war. López, Navarro and Rossi [9] gave an explanation from the point of tug-of-war game. Let us briefly recall the game: let F be the final payoff function defined in a narrow strip around the boundary $\partial \Omega$. The tug-of-war game with a transport term is played with two stages. First the players toss an unfair coin, which has head probability $0<C(\varepsilon)<1$ and tail probability $1-C(\varepsilon)$. If the players have obtained a head, then they toss a new (fair) coin and the winner moves the token to any new position $x^{1} \in \bar{B}_{\varepsilon}\left(x^{0}\right)$. But if in the first (unfair) coin toss, the players obtain a tail, the token is moved to $x^{0}+\xi\left(x^{0}\right) \varepsilon$. Note that there is no strategies of the players involved if they get a tail in the first coin toss. The game continues until the first time the token arrives to $x^{\varsigma} \in \mathbb{R}^{\mathbb{N}} \backslash \Omega$ and then Player I earns $F\left(x^{\varsigma}\right)$, Player II earns $-F\left(x^{\varsigma}\right)$, where F is the extension of f from $\partial \Omega$ to a small strip $\Gamma_{\varepsilon}=\left\{x \in \mathbb{R}^{\mathbb{N}} \backslash \Omega: \operatorname{dist}(x, \partial \Omega)<\varepsilon\|\xi\|_{\infty}\right\}$ and gives the final payoff of the game. López, Navarro and Rossi found a viscosity solution to

$$\begin{cases}-\Delta_{\infty}^{N} u-\langle D u, \xi\rangle=0 & \text { in } \Omega, \\ u=f & \text { on } \partial \Omega,\end{cases} \tag{3}$$

where f is a Lipschitz continuous function. They obtained the existence and uniqueness of a viscosity solution by a L^p-approximation procedure when ξ is a continous gradient vector field. They also proved the stability of the unique solution with respect to ξ. In addition, when ξ is Lipschitz continuous but not necessarily a gradient, they proved that the problem (3) has a viscosity solution. Some kinds of modified tug-of-war have received a lot of attention, such as [10]-[19].

$\Delta_{\infty}^{N} u+\beta|D u|$ is called the β-biased infinity Laplacian, which was first introduced by Peres, Pete and Somersille when modelling the stochastic game named biased tug-of-war in [17]. They investigated the random game with a final payoff function and a running payoff function. It's a zero sum game with two players in which the earnings of one of them are the losses of the other. Armstrong, Smart and Somersille [20] studied the mixed Dirichlet-Neumann boundary value problem

$$\begin{cases} -\Delta_\infty^N u - \beta|Du| = g & \text{in } \Omega, \\ u = f & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \Gamma_N, \end{cases}$$

where $\Gamma_D \cup \Gamma_N = \partial\Omega$ is a partition of $\partial\Omega$ with Γ_D nonempty and closed. They obtained existence, uniqueness and stability results for the boundary-value problem. Liu and Yang [21] established the existence of the principal Dirichlet eigenvalue based on the comparison principle. They also established the Harnack inequality and the Lipschitz regularity of a nonnegative viscosity supersolution to the β-biased equation

$$\Delta_\infty^N u + \beta|Du| + \lambda a(x)u = g(x),$$

where $\lambda \in \mathbb{R}$ and $\lambda \geq 0$, the weight function $a(x)$ is positive in $\bar{\Omega}$ and $a(x) \in C(\Omega) \cap L^\infty(\Omega)$. The key of their method is to choose suitable exponential cones as barrier functions.

For the case $\beta = 0$, Lu and Wang [22] [23] studied the inhomogeneous infinity Laplace equation

$$\Delta_\infty^N u = g \quad \text{in } \Omega.$$

They showed existence and uniqueness of the viscosity solutions of the Dirichlet problem under the intrinsic condition that g does not change its sign from the PDE's methods. Patrizi [24] studied the following Neumann problem

$$\begin{cases} \Delta_\infty^N u + \xi(x) \cdot Du + \eta(x)u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial\Omega, \end{cases}$$

and showed the Lipschitz regularity in the whole $\bar{\Omega}$ of the viscosity solutions and obtained the existence of the principal eigenvalue.

Aronsson [25] obtained the specific form of a viscosity solution to the infinity Laplace equation ($\Delta_\infty u = 0$) in two-dimensional space: $u(x_1, x_2) = x_1^{\frac{4}{3}} - x_2^{\frac{4}{3}}$. Thus, the regularity of infinity harmonic functions (viscosity solutions to $\Delta_\infty u = 0$) is at most $C^{1,\frac{1}{3}}$. In [26], the C^1 regularity of infinity harmonic functions was proved by Savin in dimension two. Later, Evans and Savin [27] established the $C^{1,\alpha}$ regularity of infinity harmonic functions for some $\alpha > 0$ in dimension two. For $\mathbb{N} \geq 3$, Evans and Smart [28] [29] proved that infinity harmonic functions in $\mathbb{R}^{\mathbb{N}}$ are differentiable everywhere.

In this paper, we study the Lipschitz regularity of viscosity solutions of the Neumann problem (1). The main result can be summarized as the following theorem.

Theorem 1 Assume that Ω is a bounded domain of class C^2, $\beta \in \mathbb{R}$, $\xi(x):\Omega \to \mathbb{R}^{\mathbb{N}}$, $\eta(x):\Omega \to \mathbb{R}$ are continuous in $\bar{\Omega}$, g is a bounded function in $\bar{\Omega}$. If $u \in C(\bar{\Omega})$ is a viscosity solution of

$$\begin{cases} \Delta_\infty^N u + \beta|Du| + \xi(x)\cdot Du + \eta(x)u = g(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

then there exists a constant C_0 depending on $\Omega, \beta, \|g\|_\infty$ and $\|u\|_\infty$ such that

$$|u(x) - u(y)| \le C_0|x - y|, \quad \forall x, y \in \bar{\Omega}. \tag{5}$$

2. Definitions of the Viscosity Solutions

Since the normalized infinity Laplacian Δ_∞^N is singular at the points where the gradient vanishes, we give a proper explanation to the operator by the viscosity solutions theory according to Crandall, Ishii and Lions [30].

We denote $S(\mathbb{N})$ as the set of symmetric matrices on $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ and define $\|X\|$ in $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ by letting $\|X\| = \sup\{|X\theta| \mid \theta \in \mathbb{R}^{\mathbb{N}}, |\theta| \le 1\}$.

Denote $\sigma:\mathbb{R}^{\mathbb{N}} \to S(\mathbb{N})$:

$$\sigma(p) := \frac{p \otimes p}{|p|^2},$$

where $\otimes$ denotes the tensor product.

Then we get

$$\Delta_\infty^N u = \mathrm{tr}\left(\sigma(Du)D^2 u\right),$$

for any $u \in C^2(\Omega)$.

It is easy to check that the following properties are valid.

(1) $\sigma(p)$ is homogeneous of order 0, *i.e.*, for any $a \in \mathbb{R}$ and $p \in \mathbb{R}^{\mathbb{N}}$, one has

$$\sigma(ap) = \sigma(p).$$

(2) For all $p \in \mathbb{R}^{\mathbb{N}}$, one has

$$0 \le \sigma(p) \le I_{\mathbb{N}},$$

where $I_{\mathbb{N}}$ denotes the identity matrix in $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$.

(3) $\sigma(p)$ is idempotent, *i.e.*,

$$\left(\sigma(p)\right)^2 = \sigma(p).$$

Suppose that Ω is a bounded C^2 domain. Obviously, one has the interior sphere condition and the uniform exterior sphere condition, *i.e.*,

(Ω1) For $\forall x \in \partial\Omega$, there exist $R > 0$ and $y \in \Omega$ for which $|x - y| = R$ and $B_R(y) \subset \Omega$.

(Ω2) For $\forall x \in \partial\Omega$, there exists $r > 0$ such that $B_r\left(x + r\vec{n}(x)\right) \cap \Omega = \varnothing$.

From (Ω2), one has

$$\left\langle y - x, \vec{n}(x)\right\rangle \le \frac{1}{2r}|y - x|^2, x \in \partial\Omega, y \in \bar{\Omega}. \tag{6}$$

Due to the C^2-regularity of Ω, we obtain the existence of a neighborhood of $\partial\Omega$ in $\bar{\Omega}$ on which the distance to the boundary

$$d(x) := \inf\{|x-y|, y \in \partial\Omega\}, \ x \in \bar{\Omega}$$

is of class C^2. Without loss of generality, we assume that $|Dd(x)| \le 1$ on $\bar{\Omega}$.

The $\mathrm{USC}(\bar{\Omega})$ denotes the set of upper semicontinuous functions on $\bar{\Omega}$ and the $\mathrm{LSC}(\bar{\Omega})$ denotes the set of lower semicontinuous functions on $\bar{\Omega}$. We define $B : \partial\Omega \times \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$. Now we give the definitions of the viscosity solutions of the Neumann problem according to [30] [31].

Definition 1 Any function $u \in \mathrm{USC}(\bar{\Omega})$ (resp., $u \in \mathrm{LSC}(\bar{\Omega})$) is called a viscosity subsolution (resp., viscosity supersolution) of

$$\begin{cases} \Delta_\infty^N u + \beta|Du| + \xi(x)\cdot Du + \eta(x)u = g(x) & \text{in } \Omega, \\ B(x,u,Du) = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

if the following conditions hold:

(1) For every $x_0 \in \Omega$, for all $\varphi \in C^2(\bar{\Omega})$, such that $u-\varphi$ has a local strict maximum (resp., strict minimum) at x_0 with $u(x_0) = \varphi(x_0)$ and $D\varphi(x_0) \ne 0$, one has

$$\Delta_\infty^N \varphi(x_0) + \beta|D\varphi(x_0)| + \xi(x_0)\cdot D\varphi(x_0) + \eta(x_0)u(x_0) \ge (\text{resp.}, \le)\, g(x_0).$$

If $u \equiv k$ (k is a constant) in a neighborhood of x_0, then

$$\eta(x_0)k \ge (\text{resp.}, \le)\, g(x_0).$$

(2) For every $x_0 \in \partial\Omega$, for all $\varphi \in C^2(\bar{\Omega})$, such that $u-\varphi$ has a local maximum (resp., minimum) at x_0 with $u(x_0) = \varphi(x_0)$ and $D\varphi(x_0) \ne 0$, one has

$$\begin{aligned} \min\{&-\Delta_\infty^N \varphi(x_0) - \beta|D\varphi(x_0)| - \xi(x_0)\cdot D\varphi(x_0) - \eta(x_0)u(x_0) \\ &+ g(x_0), B(x_0, u(x_0), D\varphi(x_0))\} \le 0. \end{aligned}$$

(resp.,

$$\begin{aligned} \max\{&-\Delta_\infty^N \varphi(x_0) - \beta|D\varphi(x_0)| - \xi(x_0)\cdot D\varphi(x_0) - \eta(x_0)u(x_0) \\ &+ g(x_0), B(x_0, u(x_0), D\varphi(x_0))\} \ge 0. \end{aligned})$$

If $u \equiv k$ (k is a constant) in a neighborhood of x_0 in $\bar{\Omega}$, then

$$\min\{-\eta(x_0)k + g(x_0), B(x_0, k, 0)\} \le 0.$$

(resp.,

$$\max\{-\eta(x_0)k + g(x_0), B(x_0, k, 0)\} \ge 0.)$$

We call that u is a viscosity solution if u is both a viscosity supersolution and a viscosity subsolution.

The definition of the viscosity solutions can be also given by semijets $\bar{J}^{2,+}u(x_0)$ and $\bar{J}^{2,-}u(x_0)$ according to [32].

Definition 2 The second-order superjet of u at x_0 is defined to be the set

$$J^{2,+}u(x_0) = \{(D\varphi(x), D^2\varphi(x)) : \varphi \text{ is } C^2 \text{ and } u-\varphi \text{ has a local maximum at } x_0\},$$

whose closure is defined as

$$\bar{J}^{2,+}u(x_0)=\{(p,X)\in\mathbb{R}^{\mathbb{N}}\times S(\mathbb{N}):\exists(x_n,p_n,X_n)\in\Omega\times\mathbb{R}^{\mathbb{N}}\times S(\mathbb{N}) \text{ such that } (p_n,X_n)\in J^{2,+}u(x_n) \text{ and } (x_n,u(x_n),p_n,X_n)\to(x_0,u(x_0),p,X)\},$$

and the second-order subjet of u at x_0 is defined to be the set

$$J^{2,-}u(x_0)=\{(D\varphi(x),D^2\varphi(x)):\varphi \text{ is } C^2 \text{ and } u-\varphi \text{ has a local minimum at } x_0\},$$

whose closure is defined as

$$\bar{J}^{2,-}u(x_0)=\{(p,X)\in\mathbb{R}^{\mathbb{N}}\times S(\mathbb{N}):\exists(x_n,p_n,X_n)\in\Omega\times\mathbb{R}^{\mathbb{N}}\times S(\mathbb{N}) \text{ such that } (p_n,X_n)\in J^{2,-}u(x_n) \text{ and } (x_n,u(x_n),p_n,X_n)\to(x_0,u(x_0),p,X)\}.$$

Next we give the definitions of viscosity solutions by semijets.

Definition 3 Any function $u\in \mathrm{USC}(\bar{\Omega})$ (resp., $u\in \mathrm{LSC}(\bar{\Omega})$) is called a viscosity subsolution (resp., viscosity supersolution) of

$$\begin{cases}\Delta_\infty^N u+\beta|Du|+\xi(x)\cdot Du+\eta(x)u=g(x) & \text{in } \Omega,\\ B(x,u,Du)=0 & \text{on } \partial\Omega,\end{cases}$$

if the following conditions hold:

(1) For every $x_0\in\Omega$, $\forall(p,X)\in\bar{J}^{2,+}u(x_0)$ (resp., $(p,X)\in\bar{J}^{2,-}u(x_0)$) and $p\neq 0$, one has

$$\frac{1}{|p|^2}\langle Xp,p\rangle+\beta|p|+\xi(x_0)\cdot p+\eta(x_0)u(x_0)\geq(\text{resp.},\leq)g(x_0).$$

If $u\equiv k$ (k is a constant) in a neighborhood of x_0, then

$$\eta(x_0)k\geq(\text{resp.},\leq)g(x_0).$$

(2) For every $x_0\in\partial\Omega$, $\forall(p,X)\in\bar{J}^{2,+}u(x_0)$ (resp., $(p,X)\in\bar{J}^{2,-}u(x_0)$) and $p\neq 0$, one has

$$\min\left\{-\frac{1}{|p|^2}\langle Xp,p\rangle-\beta|p|-\xi(x_0)\cdot p-\eta(x_0)u(x_0)+g(x_0),B(x_0,u(x_0),p)\right\}\leq 0.$$

(resp.,

$$\max\left\{-\frac{1}{|p|^2}\langle Xp,p\rangle-\beta|p|-\xi(x_0)\cdot p-\eta(x_0)u(x_0)+g(x_0),B(x_0,u(x_0),p)\right\}\geq 0.)$$

If $u\equiv k$ (k is a constant) in a neighborhood of x_0 in $\bar{\Omega}$, then

$$\min\{-\eta(x_0)k+g(x_0),B(x_0,k,0)\}\leq 0.$$

(resp.,

$$\max\{-\eta(x_0)k+g(x_0),B(x_0,k,0)\}\geq 0.)$$

We call that u is a viscosity solution if u is both a viscosity supersolution and a viscosity subsolution.

3. Lipschitz Regularity of Viscosity Solutions

In this section, we show the Lipschitz regularity of the viscosity solutions of the Neumann problem (1).

Theorem 2 Assume that Ω is a bounded domain of class C^2, $\beta\in\mathbb{R}$, $\xi(x):\Omega\to\mathbb{R}^{\mathbb{N}}$, $\eta(x):\Omega\to\mathbb{R}$ are continuous in $\bar{\Omega}$, g and h are bounded functions in $\bar{\Omega}$. Let $u\in\mathrm{USC}(\bar{\Omega})$ be a viscosity subsolution of

$$\begin{cases}\Delta_\infty^N u+\beta|Du|+\xi(x)\cdot Du+\eta(x)u=g(x) & \text{in }\Omega,\\ \dfrac{\partial u}{\partial\vec{n}}=0 & \text{on }\partial\Omega,\end{cases}$$

and $v\in\mathrm{LSC}(\bar{\Omega})$ be a viscosity supersolution of

$$\begin{cases}\Delta_\infty^N v+\beta|Dv|+\xi(x)\cdot Dv+\eta(x)v=h(x) & \text{in }\Omega,\\ \dfrac{\partial v}{\partial\vec{n}}=0 & \text{on }\partial\Omega,\end{cases}$$

with u and v bounded, or $v\ge 0$ and v bounded. If $m=\max_{\bar{\Omega}}(u-v)\ge 0$, then there exists $C_0>0$ such that

$$u(x)-v(y)\le m+C_0|x-y|,\quad \forall x,y\in\bar{\Omega}, \tag{8}$$

where C_0 depends on $\Omega,\beta,\mathbb{N},\|\xi\|_\infty,\|\eta\|_\infty,\|g\|_\infty,\|h\|_\infty,\|v\|_\infty,m$ and $\|u\|_\infty$ or $\sup_{\bar{\Omega}}u$.

Proof. We set

$$\Psi(x)=PQ|x|-P(Q|x|)^2,$$

and

$$\psi(x,y)=m+\mathrm{e}^{-(|\beta|+1)M(d(x)+d(y))}\Psi(x-y),$$

where M is a fixed constant, P and Q are two positive constants to be chosen later.

If $Q|x|\le\frac{1}{4}$, then

$$\Psi(x)\ge\frac{3}{4}PQ|x|. \tag{9}$$

Define

$$\Delta_Q:=\left\{(x,y)\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}:|x-y|\le\frac{1}{4Q}\right\}.$$

Fix P such that

$$\max_{\bar{\Omega}^2}(u(x)-v(y))\le m+\frac{P}{8}\mathrm{e}^{-2(|\beta|+1)Md_0}, \tag{10}$$

where $d_0=\max_{x\in\bar{\Omega}}d(x)$. If we take Q large enough, there holds

$$u(x)-v(y)-\psi(x,y)\le 0,\quad (x,y)\in\Delta_Q\cap\bar{\Omega}^2.$$

Step 1. Suppose by contradiction that for each Q there exists a point $(\bar{x},\bar{y})\in\Delta_Q\cap\bar{\Omega}^2$ such that

$$u(\bar{x})-v(\bar{y})-\psi(\bar{x},\bar{y})=\max_{\Delta_Q\cap\bar{\Omega}^2}\left(u(x)-v(y)-\psi(x,y)\right)>0.$$

Here we have dropped the dependence of $\bar{x},\bar{y}$ on Q for simplicity of notations.

If $v\geq 0$, we obtain that $\Psi(x-y)$ is non-negative in Δ_Q and $m\geq 0$ by the inequality (9). Then $u(\bar{x})>0$.

Clearly $\bar{x}\neq\bar{y}$. For any $x,y\in\bar{\Omega}$ with $|x-y|=\frac{1}{4Q}$, we get

$$\begin{aligned}u(x)-v(y)&\leq m+\frac{P}{8}\mathrm{e}^{-2(|\beta|+1)Md_0}\\&\leq m+\frac{P}{2}\mathrm{e}^{-(|\beta|+1)M(d(x)+d(y))}Q|x-y|\\&\leq\psi(x,y).\end{aligned}$$

Thus, $(\bar{x},\bar{y})\in\operatorname{int}(\Delta_Q)\cap\bar{\Omega}^2$.

Next we compute the derivatives of ψ at $(\bar{x},\bar{y})$,

$$\begin{aligned}D_x\psi(\bar{x},\bar{y})=\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}PQ\Big\{&-(|\beta|+1)M|\bar{x}-\bar{y}|(1-Q|\bar{x}-\bar{y}|)Dd(\bar{x})\\&+(1-2Q|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\Big\},\end{aligned}$$

and

$$\begin{aligned}D_y\psi(\bar{x},\bar{y})=\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}PQ\Big\{&-(|\beta|+1)M|\bar{x}-\bar{y}|(1-Q|\bar{x}-\bar{y}|)Dd(\bar{y})\\&-(1-2Q|\bar{x}-\bar{y}|)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}\Big\}.\end{aligned}$$

For large Q, one has

$$0<\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}PQ\left(\frac{1}{2}-(|\beta|+1)M|\bar{x}-\bar{y}|\right)\leq|D_x\psi(\bar{x},\bar{y})|\leq 2PQ,\tag{11}$$

and

$$0<\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}PQ\left(\frac{1}{2}-(|\beta|+1)M|\bar{x}-\bar{y}|\right)\leq|D_y\psi(\bar{x},\bar{y})|\leq 2PQ.\tag{12}$$

By the inequality (6), if $\bar{x}\in\partial\Omega$, one has

$$\begin{aligned}&\langle D_x\psi(\bar{x},\bar{y}),\vec{n}(x)\rangle\\&=\mathrm{e}^{-(|\beta|+1)Md(\bar{y})}PQ\Big\{(|\beta|+1)M|\bar{x}-\bar{y}|(1-Q|\bar{x}-\bar{y}|)\\&\quad+(1-2Q|\bar{x}-\bar{y}|)\left\langle\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|},\vec{n}(\bar{x})\right\rangle\Big\}\\&\geq\mathrm{e}^{-(|\beta|+1)Md(\bar{y})}PQ\left\{\frac{3}{4}(|\beta|+1)M|\bar{x}-\bar{y}|-(1-2Q|\bar{x}-\bar{y}|)\frac{|\bar{x}-\bar{y}|}{2r}\right\}\\&\geq\frac{1}{2}\mathrm{e}^{-(|\beta|+1)Md(\bar{y})}PQ|\bar{x}-\bar{y}|\left(\frac{3}{2}M(|\beta|+1)-\frac{1}{r}\right)>0,\end{aligned}$$

where r is the radius in the uniform exterior sphere condition (Ω2) and we have chosen $M > \frac{2}{3(|\beta|+1)r}$.

Similarly, if $\bar{y} \in \partial\Omega$, one has

$$\left\langle -D_y\psi(\bar{x},\bar{y}), \vec{n}(\bar{y}) \right\rangle \le \frac{1}{2}\mathrm{e}^{-(|\beta|+1)Md(\bar{x})}PQ|\bar{x}-\bar{y}|\left(-\frac{3}{2}M(|\beta|+1)+\frac{1}{r}\right) < 0.$$

Since u is a viscosity subsolution and v is a viscosity supersolution, we obtain

$$\operatorname{tr}\left(\sigma\left(D_x\psi(\bar{x},\bar{y})\right)X\right) + \beta\left|D_x\psi(\bar{x},\bar{y})\right| + \xi(\bar{x})\cdot D_x\psi(\bar{x},\bar{y}) + \eta(\bar{x})u(\bar{x}) \ge g(\bar{x}),$$
$$\text{if } \left(D_x\psi(\bar{x},\bar{y}), X\right) \in \bar{J}^{2,+}u(\bar{x}),$$

and

$$\operatorname{tr}\left(\sigma\left(D_y\psi(\bar{x},\bar{y})\right)Y\right) + \beta\left|D_y\psi(\bar{x},\bar{y})\right| - \xi(\bar{y})\cdot D_y\psi(\bar{x},\bar{y}) + \eta(\bar{y})v(\bar{y}) \le h(\bar{y}),$$
$$\text{if } \left(-D_y\psi(\bar{x},\bar{y}), Y\right) \in \bar{J}^{2,-}v(\bar{y}).$$

Then the inequalities (11) and (12) hold for any maximum point $(\bar{x},\bar{y}) \in \Delta_Q \cap \bar{\Omega}^2$, provided Q is large enough.

Step 2. For every $\varepsilon > 0$, there exist $X, Y \in S(\mathbb{N})$ such that $\left(D_x\psi(\bar{x},\bar{y}), X\right) \in \bar{J}^{2,+}u(\bar{x})$, $\left(-D_y\psi(\bar{x},\bar{y}), Y\right) \in \bar{J}^{2,-}v(\bar{y})$ and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le D^2\psi(\bar{x},\bar{y}) + \varepsilon\left(D^2\psi(\bar{x},\bar{y})\right)^2. \tag{13}$$

Next we estimate the right-hand side of the inequality (13):

$$\begin{aligned} D^2\psi(\bar{x},\bar{y}) = {} & \Psi(\bar{x}-\bar{y})D^2\left(\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}\right) + D\left(\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}\right) \\ & \otimes D\left(\Psi(\bar{x}-\bar{y})\right) + D\left(\Psi(\bar{x}-\bar{y})\right) \otimes D\left(\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}\right) \\ & + \mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}D^2\left(\Psi(\bar{x}-\bar{y})\right). \end{aligned}$$

We denote

$$A_1 := \Psi(\bar{x}-\bar{y})D^2\left(\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}\right),$$

$$\begin{aligned} A_2 := {} & D\left(\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}\right) \otimes D\left(\Psi(\bar{x}-\bar{y})\right) \\ & + D\left(\Psi(\bar{x}-\bar{y})\right) \otimes D\left(\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}\right), \end{aligned}$$

$$A_3 := \mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}D^2\left(\Psi(\bar{x}-\bar{y})\right).$$

One has

$$A_1 \le CQ|\bar{x}-\bar{y}|\begin{pmatrix} I_{\mathbb{N}} & 0 \\ 0 & I_{\mathbb{N}} \end{pmatrix}, \tag{14}$$

and

$$A_2 \le CQ\begin{pmatrix} I_{\mathbb{N}} & 0 \\ 0 & I_{\mathbb{N}} \end{pmatrix} + CQ\begin{pmatrix} I_{\mathbb{N}} & -I_{\mathbb{N}} \\ -I_{\mathbb{N}} & I_{\mathbb{N}} \end{pmatrix}. \tag{15}$$

Indeed, for $\rho,\tau\in\mathbb{R}^{\mathbb{N}}$, we have

$$\begin{aligned}&\left\langle A_2(\rho,\tau),(\rho,\tau)\right\rangle\\&=2(|\beta|+1)M\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}\left\{\left\langle Dd(\bar{x})\otimes D\Psi(\bar{x}-\bar{y})(\tau-\rho),\rho\right\rangle\right.\\&\quad\left.+\left\langle Dd(\bar{y})\otimes D\Psi(\bar{x}-\bar{y})(\tau-\rho),\tau\right\rangle\right\}\\&\le CQ(|\rho||+|\tau|)|\tau-\rho|\\&\le CQ\left(|\rho|^2+|\tau|^2\right)+CQ|\tau-\rho|^2,\end{aligned}$$

where C denotes various positive constants independent of Q.

Now we are ready to estimate A_3. For $D^2\left(\Psi(\bar{x}-\bar{y})\right)$, one has

$$D^2\left(\Psi(\bar{x}-\bar{y})\right)=\begin{pmatrix}D^2\Psi(\bar{x}-\bar{y}) & -D^2\Psi(\bar{x}-\bar{y})\\ -D^2\Psi(\bar{x}-\bar{y}) & D^2\Psi(\bar{x}-\bar{y})\end{pmatrix},$$

and the Hessian matrix of $\Psi(x)$ is

$$D^2\Psi(x)=\frac{PQ}{|x|}\left(I_{\mathbb{N}}-\frac{x\otimes x}{|x|^2}\right)-2PQ^2I_{\mathbb{N}}. \tag{16}$$

Denoting

$$\varepsilon=\frac{|\bar{x}-\bar{y}|}{2PQ\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}}, \tag{17}$$

we obtain

$$\begin{aligned}&\varepsilon A_1^2\le CQ|\bar{x}-\bar{y}|^3 I_{2\mathbb{N}},\quad \varepsilon A_2^2\le CQ|\bar{x}-\bar{y}|I_{2\mathbb{N}},\\&\varepsilon(A_1A_2+A_2A_1)\le CQ|\bar{x}-\bar{y}|^2 I_{2\mathbb{N}},\\&\varepsilon(A_1A_3+A_3A_1)\le CQ|\bar{x}-\bar{y}|I_{2\mathbb{N}},\quad \varepsilon(A_2A_3+A_3A_2)\le CQI_{2\mathbb{N}},\end{aligned} \tag{18}$$

where

$$I_{2\mathbb{N}}=\begin{pmatrix}I_{\mathbb{N}} & 0\\ 0 & I_{\mathbb{N}}\end{pmatrix}.$$

By the inequalities (14), (15), (18) and

$$\left(D^2\left(\Psi(\bar{x}-\bar{y})\right)\right)^2=\begin{pmatrix}2\left(D^2\Psi(\bar{x}-\bar{y})\right)^2 & -2\left(D^2\Psi(\bar{x}-\bar{y})\right)^2\\ -2\left(D^2\Psi(\bar{x}-\bar{y})\right)^2 & 2\left(D^2\Psi(\bar{x}-\bar{y})\right)^2\end{pmatrix},$$

one obtains

$$\begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix}\le O(Q)\begin{pmatrix}I_{\mathbb{N}} & 0\\ 0 & I_{\mathbb{N}}\end{pmatrix}+\begin{pmatrix}B & -B\\ -B & B\end{pmatrix}, \tag{19}$$

where

$$B=CQI_{\mathbb{N}}+\mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}\left[D^2\Psi(\bar{x}-\bar{y})+\frac{|\bar{x}-\bar{y}|}{PQ}\left(D^2\Psi(\bar{x}-\bar{y})\right)^2\right].$$

Thus, we can rewrite the inequality (19) as

$$\begin{pmatrix}\tilde{X} & 0\\ 0 & -\tilde{Y}\end{pmatrix}\le\begin{pmatrix}B & -B\\ -B & B\end{pmatrix}, \tag{20}$$

where $\tilde{X} = X - O(Q)I_{\mathbb{N}}$, $\tilde{Y} = Y + O(Q)I_{\mathbb{N}}$.

Multiplying on the left of the inequality (20) by the non-negative symmetric matrix

$$\begin{pmatrix} \sigma\left(D_x\psi(\bar{x},\bar{y})\right) & 0 \\ 0 & \sigma\left(D_y\psi(\bar{x},\bar{y})\right) \end{pmatrix},$$

one has

$$\begin{aligned} &\operatorname{tr}\left(\sigma\left(D_x\psi(\bar{x},\bar{y})\right)\tilde{X}\right) - \operatorname{tr}\left(\sigma\left(D_y\psi(\bar{x},\bar{y})\right)\tilde{Y}\right) \\ &\le \operatorname{tr}\left(\sigma\left(D_x\psi(\bar{x},\bar{y})\right)B\right) + \operatorname{tr}\left(\sigma\left(D_y\psi(\bar{x},\bar{y})\right)B\right). \end{aligned} \tag{21}$$

We aim to get the estimate on the right side of the inequality (21). Next we define

$$0 \le H := \frac{(\bar{x}-\bar{y}) \otimes (\bar{x}-\bar{y})}{|\bar{x}-\bar{y}|^2} \le I_{\mathbb{N}},$$

and compute $\operatorname{tr}(HB)$. Since $\dfrac{x \otimes x}{|x|^2}$ is idempotent, one has

$$\left(D^2\Psi(x)\right)^2 = \frac{P^2Q^2}{|x|^2}\left(1 - 4Q|x|\right)\left(I_{\mathbb{N}} - \frac{x \otimes x}{|x|^2}\right) + 4P^2Q^4 I_{\mathbb{N}}.$$

For large Q, since $\operatorname{tr}H = 1$ and $4Q|\bar{x}-\bar{y}| \le 1$, we get

$$\begin{aligned} \operatorname{tr}(HB) &= CQ + \mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}\left(-2PQ^2 + 4PQ^3|\bar{x}-\bar{y}|\right) \\ &\le CQ - \mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}PQ^2 \le -CQ^2. \end{aligned}$$

Thus, we can write $D_x\psi(\bar{x},\bar{y})$ as

$$D_x\psi(\bar{x},\bar{y}) = \mathrm{e}^{-(|\beta|+1)M(d(\bar{x})+d(\bar{y}))}PQ(v_1 + v_2),$$

where

$$v_1 = -(|\beta|+1)M|\bar{x}-\bar{y}|\left(1 - Q|\bar{x}-\bar{y}|\right)Dd(\bar{x}),$$

and

$$v_2 = \left(1 - 2Q|\bar{x}-\bar{y}|\right)\frac{\bar{x}-\bar{y}}{|\bar{x}-\bar{y}|}.$$

Therefore,

$$\sigma\left(D_x\psi(\bar{x},\bar{y})\right) = \frac{v_1 \otimes v_1}{|v_1+v_2|^2} + \frac{v_1 \otimes v_2 + v_2 \otimes v_1}{|v_1+v_2|^2} + \frac{v_2 \otimes v_2}{|v_1+v_2|^2}.$$

Since $Q|\bar{x}-\bar{y}| \le \dfrac{1}{4}$, for large Q, one has

$$\frac{1}{4} = \frac{1}{2} - \frac{1}{4} \le |v_2| - |v_1| \le |v_1 + v_2| \le |v_1| + |v_2| \le 2,$$

and

$$\|B\| \le \frac{CQ}{|\bar{x}-\bar{y}|}.$$

Then

$$\left|\operatorname{tr}\left(\frac{v_1 \otimes v_1}{|v_1+v_2|^2} B\right)\right| \le C|\bar{x}-\bar{y}|^2 \|B\| \le CQ|\bar{x}-\bar{y}|,$$

$$\left|\operatorname{tr}\left(\frac{v_1 \otimes v_2 + v_2 \otimes v_1}{|v_1+v_2|^2} B\right)\right| \le C|\bar{x}-\bar{y}|\|B\| \le CQ,$$

and

$$\operatorname{tr}\left(\frac{v_2 \otimes v_2}{|v_1+v_2|^2} B\right) = \frac{\operatorname{tr}(HB)}{|v_1+v_2|^2} \le -CQ^2.$$

We conclude that

$$\operatorname{tr}\left(\sigma\left(D_x\psi(\bar{x},\bar{y})\right)B\right) \le O(Q) - CQ^2.$$

Similarly, we can get the following estimate

$$\operatorname{tr}\left(\sigma\left(D_y\psi(\bar{x},\bar{y})\right)B\right) \le O(Q) - CQ^2.$$

Therefore, by the inequality (21) one has

$$\operatorname{tr}\left(\sigma\left(D_x\psi(\bar{x},\bar{y})\right)\tilde{X}\right) - \operatorname{tr}\left(\sigma\left(D_y\psi(\bar{x},\bar{y})\right)\tilde{Y}\right) \le O(Q) - CQ^2.$$

Step 3. By the definition of $\tilde{X}$ and $\tilde{Y}$ and the fact that u, v are respectively viscosity subsolution and viscosity supersolution, one has

$$\begin{aligned} g(\bar{x}) - \eta(\bar{x})u(\bar{x}) &\le \operatorname{tr}\left(\sigma(D_x\psi)X\right) + \beta|D_x\psi| + \xi(\bar{x})\cdot D_x\psi \\ &\le \operatorname{tr}\left(\sigma(D_x\psi)\tilde{X}\right) + O(Q) + \beta|D_x\psi| + \xi(\bar{x})\cdot D_x\psi \\ &\le \operatorname{tr}\left(\sigma(D_y\psi)Y\right) + O(Q) - CQ^2 + \beta|D_x\psi| + \xi(\bar{x})\cdot D_x\psi \\ &\le -\beta|D_y\psi| + \xi(\bar{y})\cdot D_y\psi - \eta(\bar{y})v(\bar{y}) + h(\bar{y}) + O(Q) \\ &\quad - CQ^2 + \beta|D_x\psi| + \xi(\bar{x})\cdot D_x\psi. \end{aligned}$$

According to the inequalities (11) and (12), one gets

$$g(\bar{x}) - h(\bar{y}) - \eta(\bar{x})u(\bar{x}) + \eta(\bar{y})v(\bar{y}) \le O(Q) - CQ^2. \tag{22}$$

If u and v are both bounded, the left-hand side of the inequality (22) is bounded from below by $-\|g\|_\infty - \|h\|_\infty - \|\eta\|_\infty\left(\|u\|_\infty + \|v\|_\infty\right)$. Otherwise, if v is non-negative and bounded, then $u(\bar{x}) \ge 0$ and that quantity is greater than $-\|g\|_\infty - \|h\|_\infty - \|\eta\|_\infty\left(\sup u + \|v\|_\infty\right)$. On the other hand, the right-hand side of the inequality (22) goes to $-\infty$ as $Q \to +\infty$. Hence, taking Q large enough, we can obtain a contradiction and this concludes the proof.

Theorem 1 is an immediate consequence of Theorem 2.

Proof of Theorem 1. Since $u(x) \in C(\bar{\Omega})$ is a viscosity solution of the problem (4), $u(x)$ is both a viscosity subsolution and a viscosity supersolution of the problem (4). Thus, we have $m = 0$. Since u is bounded, by Theorem 2, we can get immediately the Lipschizt estimate (5).

4. Conclusion

In this paper, we establish the Lipschitz regularity of the problem (1) arsing from the generalized random tug-of-war game. The Lipschitz regularity is an indispensable part and an important issue in the study of PDEs. The Lipschitz regularity also plays a vital role in applications, such as image processing, financial problems and physical engineering.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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