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Long Time Behavior of a Class of Generalized Beam-Kirchhoff Equations

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Abstract

In this paper, we study the long time behavior of a class of generalized Beam-Kirchhoff equation $u_{tt}+\beta\Delta^{2m}u_t+\delta\Psi\left(\|u_t\|^2\right)\Delta^{2m}u_t+\alpha\Delta^{2m}u+\gamma\Phi\left(\|D^m u\|_p^p\right)\Delta^{2m}u=g(x)$, and prove the existence and uniqueness of the global solution of this class of equation by Galerkin method by making some assumptions about the nonlinear function term $\Psi\left(\|u_t\|^2\right)$, $\Phi\left(\|D^m u\|_p^p\right)$, $g(x)$. The existence of the family of global attractor and its Hausdorff dimension and Fractal dimension estimation are proved.

Keywords

Beam-Kirchhoff Equation, Galerkin's Method, The Family of Global Attractor, Dimension Estimation

1. Introduction

We study the initial boundary value problem of the following higher order Beam-Kirchhoff equation:

$$u_{tt}+\beta\Delta^{2m}u_t+\delta\Psi\left(\|u_t\|^2\right)\Delta^{2m}u_t+\alpha\Delta^{2m}u+\gamma\Phi\left(\|D^m u\|_p^p\right)\Delta^{2m}u=g(x), \tag{1.1}$$

$$u(x,t)=0, \frac{\partial^i u}{\partial v^i}=0, i=1,2,\cdots,2m-1, x\in\partial\Omega, t>0, \tag{1.2}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega\subset R^n. \tag{1.3}$$

where $m>1$ is a positive integer, Ω is a bounded region in R^n with a smooth homogeneous Dirichlet boundary, $\partial\Omega$ represents the boundary of Ω, and $g(x)$ is the external force term. $\beta(-\Delta)^{2m}u_t$ is a strongly damped term,

$\alpha,\beta,\delta,\gamma$ is a constant greater than 0, $\Psi\left(\|u_t\|^2\right),\Phi\left(\|D^m u\|_p^p\right)$ is a given non-negative function, and $\|D^m u\|_p^p=\int_\Omega |D^m u|^p \,\mathrm{d}x$.

Kirchhoff type equation is a hot topic in the research of mathematical physics equations in recent years, and many scholars have done research on this kind of equation. Igor Chueshov [1] studied the long-term behavior of the solution of the following nonlinear strongly damped Kirchhoff wave equation:

$$u_{tt}-\sigma\left(\|\nabla u\|^2\right)\Delta u_t-\phi\left(\|\nabla u\|^2\right)\Delta u+g_1(u)=h(x)$$

In the energy space $H(\Omega)=H_0^1\cap L^{p+1}(\Omega)\times L^2(\Omega)$, the author can find that the growth of exponential p of strong nonlinear $g(u)$ is supercritical, that is, when $p^*\equiv\frac{N+2}{(N-2)^+}$, $p^{**}=\frac{N+4}{(N-4)^+}$, there is $p^*\le p\le p^{**}$, where when $p<p^*$, the growth of exponential p for $H^1(\Omega)\to L^{p+1}$ is supercritical. In $H(\Omega)$, under the locally strong topology, a global attractor is established that is finite-dimensional. Especially in the non-supercritical case: 1) The partial strong topology will become a strong topology; 2) In $H(\Omega)$, the exponential attractor can be obtained due to the feature of strong stability estimation. In addition, Chueshov [1] also considers the well-fitting solution of the Kirchhoff equation with structural damping term $\sigma\left(\|\nabla u\|^2\right)^\theta \Delta u_t$ at the abstract level and the long-term dynamic system, where $\frac{1}{2}\le\theta\le 1$.

Guoguang Lin, Yunlong Gao [2] studied the long time behavior of the solutions of a class of higher-order Kirchhoff type equation

$$u_{tt}+\left(\alpha+\beta\|D^m u\|^2\right)^q(-\Delta)^m+(-\Delta)^m u_t+g(u)=f(x).$$

For Kirchhoff dissipation term, Newton binomial theorem is used to process analysis. In dimension estimation, a novel method is used to deal with the variational problem. The lemma and theorem are obtained by adding $\alpha^q(-\Delta)^m u$ to both sides of the variational equation at the same time, and the existence of global attractor and exponential attractor and Hausdorff dimension estimation are proved.

Guoguang Lin and Zhuoqian Li [3] studied the family of global attractor and its dimension estimation of a class of high-order nonlinear Kirchhoff equation

$$u_{tt}+M\left(\|D^m u\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g(x,t)=f(x).$$

They use Sobolev embedding theorem $H_0^m(\Omega)\subset L^{2p}(\Omega)$ to deal with source term $g(x,t)$, analyzing Kirchhoff stress term by case treatment, and prove that compacting family of global attractor exists in solution semigroup by means of the family of global attractor correlation theory. Of course, there are many studies on the family of global attractor higher order Kirchhoff equations.

Zhijian Yang and Zhiming Liu [4] explored the long-term behavior of the solution of the Kirchhoff equation

$$u_{tt}-\Delta u_t-M\left(\|\nabla u\|^2\right)\Delta u+u_t+g(x,u)=f(x),(x,u)\in R^N\times R^+,$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in R^N$$

with nonlinear strong damping and critical term. Under the premise of critical nonlinearity, the adaptability of the solution, the existence of the global attractor and exponential attractor of the solution exist in the energy space $H=H^1\left(R^N\right)\times L^2\left(R^N\right)$. Their innovation is that the obtained results improve Yang's research results.

Guigui Xu, Libo Wang and Guoguang Lin [5] explored the inertial manifold of the strongly damped wave equation:

$$u_{tt}-\alpha\Delta u+\beta\Delta^2u-\gamma\Delta u_t+g(u)=f(x,t),(x,t)\in\Omega\times R^+,$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega,$$

$$u|_{\partial\Omega}=0,\Delta u|_{\partial\Omega}=0,(x,t)\in\partial\Omega\times R^+,$$

The Fadeo-Galerkin method and the uniformly compact method are used to prove the existence and uniqueness of the strong solution to this problem. The existence of the inertial manifold is proved under the assumption of a certain spectral interval and a sufficiently small delay.

Naimen [6] studied the Kirchhoff type elliptic boundary value problem with Sobolev critical growth:

$$-\left(a+b\int_\Omega|\nabla u|^2\,\mathrm{d}x\right)\Delta u=\mu g(x,u)+u^5.$$

When $u\to0^+,g(x,u)=o(u)$ and $u\to+\infty,g(x,u)=o\left(u^5\right),x\in\Omega$, The Sobolev embedding theorem $H_0^1(\Omega)\subseteq L^p(\Omega)$ is used to deal with the nonlinear source term $\mu g(x,u)$, and the corresponding energy functional is reduced to the critical value level, so the equation satisfies the compactness condition.

Masamro [7] studied the initial boundary value problem of a class of Kirchhoff type equation

$$u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u+\delta|u|^pu+\gamma u_t=f(x),x\in\Omega,t>0,$$

$$u(x,t)=0,x\in\partial\Omega,t\geq0,$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(t),x\in\Omega.$$

containing dissipative terms and damping terms, and proved the existence of the global solution of the equation under the initial boundary value condition by using Galerkin's method, where $\Omega\in R^n$ is a bounded region with a smooth boundary $\partial\Omega$, and

$$M(\gamma)\in C^1[0,\infty),M(\gamma)\geq m_0>0.$$

when $\delta>0,\alpha>0$ and $\forall\gamma\geq0$ are present.

Matsugama and Ikehata [8] used the potential well method to discuss the existence of the global solution of the Kirchhoff equation

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \delta|u_t|^{p-1}u_t = \mu|u|^{q-1}u,$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega,$$

$$u(x,t) = 0, x \in \partial\Omega, t > 0.$$

with damping term and the attenuation estimation of the global solution, where

$$\Omega \subset R^n, M(s) = a + bs^r \in C^1[0,\infty), a,b \geq 0, a+b > 0, r \geq 1, M(s) \geq m_0 > 0.$$

Nakao and Zhijian Yang [9] studied the long time behavior of the solution of the Kirchhoff type equation

$$u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u - \Delta u_t + g(x,u) = f(x),$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega,$$

$$u(x,t) = 0, x \in \partial\Omega, t > 0.$$

with strong damping term. When $g(x,u)$ satisfies the local Lipschitz condition, they prove the existence of the global attractor by proving the existence of the bounded absorption set and the asymptotic compactness of the corresponding continuous semigroup. Where $g(x,u) \in C^1(R^n, R)$ and $g(x,u)$ have order $p: 1 \leq p < \frac{4}{(N-4)^+}$ with respect to *u*.

2. Background Knowledge and Assumptions

In this paper, we need the following mathematical notation:

$f = f(x)$, $D = \nabla$, $H = L^2(\Omega)$, $H_0^{2m}(\Omega) = H^{2m}(\Omega) \cap H_0^1(\Omega)$.

$H_0^{m+k}(\Omega) = H^{m+k}(\Omega) \cap H_0^1(\Omega)$, $(k = 0,1,2,\cdots,2m)$

$H_0^{m+k}(\Omega) = D(\Omega)$, Closure of infinitely differentiable Spaces in $H^{m+k}(\Omega)$.

$E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega)$ and $E_0 = H_0^{2m}(\Omega) \times H$, where $C_i \geq 0 (i = 1,2,\cdots)$ is constant.

λ_1 is the first eigenvalue of $-\Delta$ on Ω,

The inner product and norm expressed by $(\bullet,\bullet)$ and $\|\bullet\|$ respectively, that is, $(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x$, $(u,u) = \|u\|^2$.

Note the global attractor from E_0 to E_k as A_k, while B_{0k} is the bounded absorption set in E_k, where $k = 1,2,\cdots,2m$.

Lemma 2.1. [10] (Holder inequality) set

$$\frac{1}{p} + \frac{1}{q} = 1, (p \geq 1, q \geq 1), \forall f(x) \in L^p(\Omega), \forall g(x) \in L^q(\Omega)$$

We have $\int_\Omega |f(x)g(x)\mathrm{d}x| \leq \left(\int_\Omega |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q \mathrm{d}x\right)^{\frac{1}{q}}$.

Lemma 2.2. [10] (Poincare inequality) If $\Omega \in R^n$ is a bounded open subset, then

$$\|u\|_{L^2(\Omega)} \le \lambda_1^{-\frac{1}{2}} \|\nabla u\|_{L^2(\Omega)}, \forall u \in H_0^1(\Omega)$$

where λ_1 is the first eigenvalue of $-\Delta$ on Ω.

Lemma 2.3. [10] (Young Inequality) For any real number $a,b \ge 0$, then

$$ab \le \frac{\varepsilon^p}{p} a^p + \frac{1}{q\varepsilon^q} b^q, \left(\frac{1}{p} + \frac{1}{q} = 1,\ p > 1,\ q > 1\right),$$

Let function $\Psi\left(\|u_t\|^2\right), \Phi\left(\|D^m u\|_p^p\right)$ satisfy the condition:

(A$_1$) $\Phi\left(\|D^m u\|_p^p\right) \in C^2\left([0,+\infty), R\right)$, $0 \le \sigma_0 \le \Phi\left(\|D^m u\|_p^p\right) \le \sigma_1$,

$$\delta_2 = \begin{cases} \sigma_0, \frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m} u\|^2 \ge 0 \\ \sigma_1, \frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m} u\|^2 \le 0 \end{cases}$$

Where σ_0, σ_1 is the positive constant;

(A$_2$) $\Psi\left(\|u_t\|^2\right) \in C^2\left([0,+\infty), R\right)$, $0 \le B_0 \le \Psi\left(\|u_t\|^2\right) \le B_1$,

Where B_0, B_1 is the positive constant;

(A$_3$) $\frac{2n}{n+2m} \le p \begin{cases} < \frac{2n}{n-2m}, n > 2m \\ < \infty, n \le 2m \end{cases}$

3. The Existence and Uniqueness of the Global Solution

Lemma 3.1. Assuming that condition (A$_1$), (A$_2$), (A$_3$) is true, $g(x) \in H$, $(u_0, u_1) \in E_0$, then the initial boundary value problem (1.1)-(1.3) has a global smooth solution $(u,v) \in E_0$ and is satisfied

$$\|(u,v)\|_{E_0}^2 = \|D^{2m} u\|^2 + \|v\|^2 \le \left(\|D^{2m} u_0\|^2 + \|v_0\|^2\right) \mathrm{e}^{-a_1 t} + \frac{C_1}{a_1}\left(1 - \mathrm{e}^{-a_1 t}\right),$$

where $v = u_t + \varepsilon u$,

Ream $a_1 = \min\left\{\beta\lambda_1^{2m} + \delta B_0 \lambda_1^{2m} - 2\varepsilon - 2\varepsilon^2, \frac{2\varepsilon(\alpha + r\delta_0) - \varepsilon^2\left(\lambda_1^{-2m} + \beta + \delta B_1\right)}{\alpha + r\delta}\right\}$

So there is a non-negative real number R_0 and t_0 that makes

$$\|(u,v)\|_{E_0}^2 = \|D^{2m} u\|^2 + \|v\|^2 \le R_0^2, (t > t_0)$$

Prove we set $v = u_t + \varepsilon u$ and take the inner product of both sides of Equation (1.1) and v, we get

$$\left(u_{tt} + \beta\Delta^{2m} u_t + \delta\Psi\left(\|u_t\|^2\right)\Delta^{2m} u_t + \alpha\Delta^{2m} u + \gamma\Phi\left(\|D^m u\|_p^p\right)\Delta^{2m} u, v\right) = \left(g(x), v\right) \quad (3.1)$$

This is obtained by using the Holder inequality, the Young inequality, the Poincare inequality and the terms of condition (A$_1$), (A$_2$), (A$_3$) in successive processing (3.1)

$$\begin{aligned}(u_{tt},v)&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2-\varepsilon\|v\|^2+\varepsilon^2(u,v)\geq\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2-\varepsilon\|v\|^2-\frac{\varepsilon^2}{2}\|v\|^2-\frac{\varepsilon^2}{2}\|u\|^2\\&\geq\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2-\frac{2\varepsilon+\varepsilon^2}{2}\|v\|^2-\frac{\varepsilon^2\lambda_1^{-2m}}{2}\left\|D^{2m}u\right\|^2\end{aligned}\tag{3.2}$$

$$\left(\alpha(-\Delta)^{2m}u,v\right)=\left(\alpha(-\Delta)^{2m}u,u_t+\varepsilon u\right)=\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2+\alpha\varepsilon\left\|D^{2m}u\right\|^2\tag{3.3}$$

$$\begin{aligned}\left(\beta(-\Delta)^{2m}u_t,v\right)&=\left(\beta(-\Delta)^{2m}(v-\varepsilon u),v\right)=\beta\left\|D^{2m}v\right\|^2-\beta\varepsilon\left(D^{2m}u,D^{2m}v\right)\\&\geq\beta\left\|D^{2m}v\right\|^2-\frac{\beta\varepsilon^2}{2}\left\|D^{2m}u\right\|^2-\frac{\beta}{2}\left\|D^{2m}v\right\|^2\\&\geq\frac{\beta\lambda_1^{2m}}{2}\|v\|^2-\frac{\beta\varepsilon^2}{2}\left\|D^{2m}u\right\|^2\end{aligned}\tag{3.4}$$

$$\begin{aligned}\left(\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}u_t,v\right)&=\delta\psi\left(\|u_t\|^2\right)\left\|D^{2m}v\right\|^2-\varepsilon\delta\psi\left(\|u_t\|^2\right)\left(D^{2m}u,D^{2m}v\right)\\&\geq\delta B_0\left\|D^{2m}v\right\|^2-\varepsilon\delta B_1\left(\frac{\varepsilon\left\|D^{2m}u\right\|^2+\frac{1}{\varepsilon}\left\|D^{2m}v\right\|^2}{2}\right)\\&\geq\frac{\delta B_0\lambda_1^{2m}}{2}\|v\|^2-\frac{\varepsilon^2\delta B_1}{2}\left\|D^{2m}u\right\|^2\end{aligned}\tag{3.5}$$

$$\begin{aligned}&\left(r\phi\left(\left\|D^m u\right\|_p^D\right)(-\Delta)^{2m}u,u_t+\varepsilon u\right)\\&=\frac{r\phi\left(\left\|D^m u\right\|_p^D\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2+\varepsilon r\phi\left(\left\|D^m u\right\|_p^D\right)\left\|D^{2m}u\right\|^2\\&\geq\frac{r\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2+\varepsilon r\delta_0\left\|D^{2m}u\right\|^2\end{aligned}\tag{3.6}$$

$$\left(g(x),v\right)\leq\frac{\varepsilon^2}{2}\|v\|^2+\frac{1}{2\varepsilon^2}\|g(x)\|^2\tag{3.7}$$

Synthetically available

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2+\frac{1}{2}(\alpha+r\delta)\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2+\left(\frac{\beta\lambda_1^{2m}-2\varepsilon-2\varepsilon^2+\delta B_0\lambda_1^{2m}}{2}\right)\|v\|^2\\&+\left(\alpha\varepsilon-\frac{\varepsilon^2\lambda_1^{-2m}}{2}-\frac{\beta\varepsilon^2}{2}-\frac{\varepsilon^2\delta B_1}{2}+\varepsilon r\delta_0\right)\left\|D^{2m}u\right\|^2\leq\frac{1}{2\varepsilon^2}\|g(x)\|^2\end{aligned}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+(\alpha+r\delta)\left\|D^{2m}u\right\|^2\right)+\left(\beta\lambda_1^{2m}-2\varepsilon-2\varepsilon^2+\delta B_0\lambda_1^{2m}\right)\|v\|^2\\&+\left(\frac{-\varepsilon^2\left(\lambda_1^{-2m}+\beta+\delta B_1\right)+2\varepsilon\left(\alpha+rB_0\right)}{\alpha+r\delta}\right)(\alpha+r\delta)\left\|D^{2m}u\right\|^2\leq\frac{1}{\varepsilon^2}\|g(x)\|^2\end{aligned}\tag{3.8}$$

Ream $a_1=\min\left\{\beta\lambda_1^{2m}+\delta B_0\lambda_1^{2m}-2\varepsilon-2\varepsilon^2,\frac{2\varepsilon\left(\alpha+r\delta_0\right)-\varepsilon^2\left(\lambda_1^{-2m}+\beta+\delta B_1\right)}{\alpha+r\delta}\right\}$

In a result $\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+(\alpha+r\delta)\left\|D^{2m}u\right\|^2\right)+a_1\left(\|v\|^2+(\alpha+r\delta)\left\|D^{2m}u\right\|^2\right)\leq C_1$ (3.9)

It's derived from the Gronwall inequality

$$\|v\|^2+(\alpha+r\delta)\|D^{2m}u\|^2\le\left(\|v_0\|^2+(\alpha+r\delta)\|D^{2m}u_0\|^2\right)\mathrm{e}^{-a_1t}+\frac{C_1}{a}\left(1-\mathrm{e}^{-a_1t}\right) \quad (3.10)$$

So there exist R_0 and t_0 to make $\|(u,v)\|_{E_0}^2=\|D^{2m}u\|+\|v\|^2\le R_0^2\,(t>t_0)$.

Lemma 3.2. Assuming that condition (A_1), (A_2), (A_3) is true and $g(x)\in H^k$, $(u_0,u_1)\in E_k$ is true, then the global solution $(u,v)\in E_k$ of the initial boundary value problem (1.1)-(1.3) is satisfied

$$\|(u,v)\|_{E_k}^2=\|D^{2m+k}u\|^2+\|D^kv\|^2\le\left(\|D^{2m+k}u_0\|^2+\|D^kv_0\|^2\right)\mathrm{e}^{-a_1t}+\frac{C_2}{a_1}\left(1-\mathrm{e}^{-a_1t}\right),$$

where $C_2=\frac{1}{\varepsilon^2}\|D^kg(x)\|^2$,

$$a_1=\min\left\{\beta\lambda_1^{2m}+\delta B_0\lambda_1^{2m}-2\varepsilon-2\varepsilon^2,\frac{2\varepsilon(\alpha+r\delta_0)-\varepsilon^2\left(\lambda_1^{-2m}+\beta+\delta B_1\right)}{\alpha+r\delta}\right\}$$

Then there are positive constants R_k and t_k that make

$$\|(u,v)\|_{E_k}^2=\|D^{2m+k}u\|^2+\|D^kv\|^2\le R_k^2,(t>t_k)$$

Prove set $(-\Delta)^kv=(-\Delta)^ku_t+\varepsilon(-\Delta)^ku$, and take the inner product of both sides of Equation (1.1) with $(-\Delta)^kv$, *i.e.*

$$\begin{aligned}&\left(u_{tt}+\beta\Delta^{2m}u_t+\delta\Psi\left(\|u_t\|^2\right)\Delta^{2m}u_t+\alpha\Delta^{2m}u+\gamma\Phi\left(\|D^mu\|_p^p\right)\Delta^{2m}u,(-\Delta)^kv\right)\\&=\left(g(x),(-\Delta)^kv\right).\end{aligned} \quad (3.11)$$

By using Holder inequality, Young inequality, Poincare inequality and conditions (A_1), (A_2), (A_3), the terms in Equation (3.11) are processed successively

$$\begin{aligned}\left(u_{tt},(-\Delta)^kv\right)&=\left(v_t-\varepsilon u_t,(-\Delta)^kv\right)=\left(D^kv_t,D^kv\right)-\varepsilon\left(D^ku_t,D^kv\right)\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^kv\|^2-\varepsilon\|D^kv\|^2+\varepsilon^2\left(D^ku,D^kv\right)\\&\ge\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^kv\|^2-\frac{\varepsilon+\varepsilon^2}{2}\|D^kv\|^2-\frac{\varepsilon^2\lambda_1^{-2m}}{2}\|D^{2m+k}u\|^2\end{aligned} \quad (3.12)$$

$$\begin{aligned}\left(\alpha(-\Delta)^{2m}u,(-\Delta)^kv\right)&=\alpha\left(D^{2m+k}u,D^{2m+k}u_t\right)+\alpha\varepsilon\|D^{2m+k}u\|^2\\&=\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m+k}u\|^2+\alpha\varepsilon\|D^{2m+k}u\|^2\end{aligned} \quad (3.13)$$

$$\begin{aligned}\left(\beta(-\Delta)^{2m}u_t,(-\Delta)^kv\right)&=\beta\left((-\Delta)^{2m}(v-\varepsilon u),(-\Delta)^kv\right)\\&=\beta\|D^{2m+k}v\|^2-\varepsilon\beta\left(D^{2m+k}u,D^{2m+k}v\right)\\&\ge\beta\|D^{2m+k}v\|^2-\frac{\varepsilon\beta}{2}\left(\varepsilon\|D^{2m+k}u\|^2+\frac{\|D^{2m+k}v\|^2}{\varepsilon}\right)\\&\ge\frac{\beta\lambda_1^{2m}}{2}\|D^kv\|^2-\frac{\varepsilon^2\beta}{2}\|D^{2m+k}u\|^2\end{aligned} \quad (3.14)$$

$$\begin{aligned}&\left(\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}u_t,(-\Delta)^k v\right)\\&=\delta\psi\left(\|u_t\|^2\right)\left\|D^{2m+k}v\right\|^2-\varepsilon\delta\psi\left(\|u_t\|^2\right)\left(D^{2m+k}u,D^{2m+k}v\right)\\&\ge\delta B_0\left\|D^{2m+k}v\right\|^2-\frac{\varepsilon\delta B_1}{2}\left(\varepsilon\left\|D^{2m+k}u\right\|^2+\frac{\left\|D^{2m+k}v\right\|^2}{\varepsilon}\right)\\&\ge\frac{\delta B_0\lambda_1^{2m}}{2}\left\|D^k v\right\|^2-\frac{\varepsilon^2\delta B_1}{2}\left\|D^{2m+k}u\right\|^2\end{aligned}\tag{3.15}$$

$$\begin{aligned}&\left(r\phi\left(\left\|D^m u\right\|_p^D\right)(-\Delta)^{2m}u,(-\Delta)^k v\right)\\&=\frac{r\phi\left(\left\|D^m u\right\|_p^D\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\varepsilon r\phi\left(\left\|D^m u\right\|_p^D\right)\left\|D^{2m+k}u\right\|^2\\&\ge\frac{r\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\varepsilon r\delta_0\left\|D^{2m+k}u\right\|^2\end{aligned}\tag{3.16}$$

$$\left(g(x),(-\Delta)^k v\right)=\left(D^k g(x),D^k v\right)\le\frac{\varepsilon^2}{2}\left\|D^k v\right\|^2+\frac{1}{2\varepsilon^2}\left\|D^k g(x)\right\|^2\tag{3.17}$$

In summary can be obtained

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k v\right\|^2+\frac{\alpha+r\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\frac{\beta\lambda_1^{2m}+\delta B_0\lambda_1^{2m}-2\varepsilon^2-2\varepsilon}{2}\left\|D^k v\right\|^2\\&+\frac{2\varepsilon(\alpha+r\delta_0)-\varepsilon^2\left(\lambda_1^{2m}+\beta+\delta B_1\right)}{2}\left\|D^{2m+k}u\right\|^2\le C_2\end{aligned}$$

From the Gronwall inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k v\right\|^2+(\alpha+r\delta)\left\|D^{2m+k}u\right\|^2\right)+a_1\left(\left\|D^k v\right\|^2+(\alpha+r\delta)\left\|D^{2m+k}u\right\|^2\right)\le C_2$$

$$\left\|D^k v\right\|^2+(\alpha+r\delta)\left\|D^{2m+k}u\right\|^2\le\left\|v_0\right\|^2+(\alpha+r\delta)\left\|D^{2m+k}u_0\right\|^2\mathrm{e}^{-a_1 t}+\frac{C_2}{a_1}\left(1-\mathrm{e}^{-a_1 t}\right)\tag{3.18}$$

So there exist R_k and t_k to make $\|(u,v)\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^k v\right\|^2\le R_k^2\ (t>t_k)$

Theorem 3.3. (existence and uniqueness of solutions) Under the conditions of lemma 3.1 and Lemma 3.2, and assuming $\psi'<0$, $g\in H^k$, $(u_0,u_1)\in E_k$, then the initial boundary value problem (1.1)-(1.3) has a unique global solution $(u,v)\in L^\infty([0,+\infty);E_k)$

Proof: Galerkin's finite element method is used to prove the existence of the global solution.

The first step is to construct an approximate solution.

Let $(-\Delta)^{2m+k}w_j=\lambda_j^{2m+k}w_j, k=1,2,\cdots,2m$, where λ_j is the eigenvalue of $-\Delta$ with a homogeneous Dirichlet boundary on Ω, and w_j is the eigenfunction determined by the corresponding eigenvalue λ_j, and the orthonormal basis of H is formed by the eigenvalue theory known $w_1,w_2,\cdots,w_l$.

Construct the approximate solution $u_l(t)=\sum_{j=1}^{l}g_{jl}(t)w_j$ of problem where $g_{jl}(t)$ is determined by the following nonlinear system of ordinary differential equations

$$\begin{aligned}&\left(u_{tt}+\beta\Delta^{2m}u_t+\delta\Psi\left(\|u_t\|^2\right)\Delta^{2m}u_t+\alpha\Delta^{2m}u+\gamma\Phi\left(\|D^m u\|_p^p\right)\Delta^{2m}u,w_j\right)\\&=\left(g(x),w_j\right),j=1,2,\cdots,l.\end{aligned}\tag{3.19}$$

When the initial condition $u_{l0}(0)=u_{l0},u_{lt}(0)=u_{l1}$ is satisfied and $l\to+\infty$ is satisfied, $(u_{l0},u_{l1})\to(u_0,u_1)$, in E_k is known from the basic theory of ordinary differentiation that the approximate solution u_{lt} exists on $(0,t_l)$.

The second step is prior estimation.

Recording $v_l(t)=u_{lt}(t)+\varepsilon u_l(t)$, multiply both sides of the equation $v_l(t)=u_{lt}(t)+\varepsilon u_l(t)$, by $\lambda_j^k\left(g'_{jl}(t)+\varepsilon g_{jl}(t)\right)$, and sum over *j* to get

1) $k=0$, have

$$\|(u_l,v_l)\|_{E_0}^2=\|D^{2m}u_l\|^2+\|v_l\|^2\le R_0^2.$$

2) $k=1,2,\cdots,2m$, have

$$\|(u_l,v_l)\|_{E_k}^2=\|D^{2m+k}u_l\|^2+\|v_l\|^2\le R_k^2.$$

then (u_l,v_l) in $L^\infty([0,+\infty];E_k)$ Middle bounded.

The third step is the limiting process.

In space $E_k(k=1,2,\cdots,2m)$, choosing a subcolumn u_μ from sequence u_l causes $(u_\mu,v_\mu)\to(u,v)$ to converge weakly * in L^∞.

From the space compact embedding theorem we know that E_k is compactly embedded in E_0, and $(u_\mu,v_\mu)\to(u,v)$ is strongly convergent almost everywhere in E_0.

Let $l=\mu$ and take the limit, which can be obtained from the above equation

$$\left(u_{\mu t},(-\Delta)^k w_j\right)=\left(v_\mu,\lambda_j^k w_j\right)-\left(\varepsilon u_\mu,\lambda_j^k w_j\right)\to\left(v,\lambda_j^k w_j\right)-\left(\varepsilon u,\lambda_j^k w_j\right)\tag{3.20}$$

Weakly convergent in $L^\infty[0,+\infty)$. Due to

$$\left(u_{\mu tt},(-\Delta)^k w_j\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(u_{\mu t},(-\Delta)^k w_j\right),$$

Thus $\left(u_{\mu tt},(-\Delta)^k w_j\right)\to\left(u_{tt},(-\Delta)^k w_j\right)$ converges in $D'[0,+\infty)$, and $D'[0,+\infty)$ is the conjugate space of the infinitely differentiable space of $D[0,+\infty)$.

$$\left(\beta\Delta^{2m}u_t,(-\Delta)^k w_j\right)\to\beta\left(\Delta^{\frac{2m+k}{2}}v-\varepsilon\Delta^{\frac{2m+k}{2}}u,\lambda_j^{\frac{2m+k}{2}}w_j\right)$$

is weakly convergent in $L^\infty[0,+\infty)$.

$$\left(\delta\Psi\left(\|u_t\|^2\right)\Delta^{2m}u_{\mu t},(-\Delta)^k w_j\right)\to\delta\Psi\left(\|u_t\|^2\right)\left(\Delta^{\frac{2m+k}{2}}v-\varepsilon\Delta^{\frac{2m+k}{2}}u,\lambda_j^{\frac{2m+k}{2}}w_j\right)$$

is weakly convergent in $L^\infty[0,+\infty)$.

$$\left(\gamma\Phi\left(\|D^m u\|_p^p\right)\Delta^{2m}u_\mu,(-\Delta)^k w_j\right)\to\left(\gamma\Phi\left(\|D^m u\|_p^p\right)(-\Delta)^{\frac{2m+k}{2}}u,\lambda_j^{\frac{2m+k}{2}}w_j\right)$$

is weakly convergent in $L^\infty[0,+\infty)$.

$$\left(\alpha\Delta^{2m}u_{\mu},(-\Delta)^{k}w_j\right)\to\left(\alpha\Delta^{\frac{2m+k}{2}}u,\lambda_j^{\frac{2m+k}{2}}w_j\right)$$

is weakly convergent in $L^{\infty}[0,+\infty)$.

In particular, $u_{\mu0}\to u_0$ is weakly convergent in E_k, and $u_{\mu t}\to u_t=u_1$ is weakly convergent in E_k. For all *j* and $\mu\to+\infty$,

$$\begin{aligned}&\left(u_{tt}+\beta\Delta^{2m}u_t+\delta\Psi\left(\|u_t\|^2\right)\Delta^{2m}u_t+\alpha\Delta^{2m}u+\gamma\Phi\left(\|D^m u\|_p^p\right)\Delta^{2m}u,w_j\right)\\&=\left(g(x),w_j\right),j=1,2,\cdots,l\end{aligned}\tag{3.21}$$

can be derived.

Therefore, the existence of the weak solution of the problem (1.1)-(1.3) is obtained, the existence is proved, and the uniqueness of the solution below is obtained.

Let u,v be two solutions to the problem (1.1)-(1.3), and let $w=u-v$ have

$$\left(w_{tt},w_t+\varepsilon w\right)=\left(w_{tt},w_t\right)+\varepsilon\left(w_{tt},w\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t\|^2+\varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left(w_t,w\right)-\varepsilon^2\|w_t\|^2\tag{3.22}$$

$$\left(\beta(-\Delta)^{2m}w_t,w_t+\varepsilon w\right)=\frac{\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}w\|^2+\beta\|D^{2m}w_t\|^2\tag{3.23}$$

$$\left(\alpha(-\Delta)^{2m}w,w_t+\varepsilon w\right)=\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}w\|^2+\alpha\varepsilon\|D^{2m}w\|^2\tag{3.24}$$

$$\begin{aligned}&\left(\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}u_t-\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}v_t,w_t+\varepsilon w\right)\\&=\left(\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}w_t,w_t+\varepsilon w\right)+\left(\left[\delta\psi\left(\|u_t\|^2\right)-\delta\psi\left(\|v_t\|^2\right)\right](-\Delta)^{2m}v_t,w_t+\varepsilon w\right)\\&=\left(\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}w_t,w_t+\varepsilon w\right)\\&\quad+\left(\delta\psi'(\xi)\left(\|u_t\|+\|v_t\|\right)\left(\|u_t\|-\|v_t\|\right)(-\Delta)^{2m}u_t,w_t+\varepsilon w\right)\end{aligned}$$

By $\psi'<0$, then the original formula

$$\begin{aligned}&\ge\delta B_0\left(D^{2m}w_t,D^{2m}w_t+\varepsilon D^{2m}w\right)+C_3\left(\|w_t\|,w_t+\varepsilon w\right)\\&\ge\delta B_0\|D^{2m}w_t\|^2+\frac{\varepsilon\delta B}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}w\|^2+C_3\left(\frac{2+\varepsilon}{2}\right)\|w_t\|^2+\frac{C_3\varepsilon}{2}\|w\|^2\end{aligned}\tag{3.25}$$

$$\begin{aligned}&\left(r\phi\left(\|D^m u\|_p^D\right)(-\Delta)^{2m}u-r\phi\left(\|D^m v\|_p^D\right)(-\Delta)^{2m}v,w_t+\varepsilon w\right)\\&=\left(r\phi\left(\|D^m u\|_p^D\right)(-\Delta)^{2m}u-r\phi\left(\|D^m u\|_p^D\right)(-\Delta)^{2m}v+r\phi\left(\|D^m v\|_p^D\right)(-\Delta)^{2m}v,w_t+\varepsilon w\right)\\&=r\phi\left(\|D^m u\|_p^D\right)\left((-\Delta)^{2m}w,w_t+\varepsilon w\right)\\&\quad+r\phi'\left(\|D^m\xi\|_p^p\right)\left(\|D^m\xi\|_p^p\right)'\left(D^m w(-\Delta)^{2m}w,w_t+\varepsilon w\right)\\&\ge r\delta\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}w\|^2+\varepsilon r\delta_0\|D^{2m}w\|^2+C_4\left(D^m w\cdot(-\Delta)^{2m}w,w_t+\varepsilon w\right)\\&\ge r\delta\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}w\|^2+\varepsilon r\delta_0\|D^{2m}w\|^2-\frac{C_4}{2}\left(\|D^m w\cdot(-\Delta)^{2m}w\|^2+\|w_t\|^2\right)\end{aligned}$$

$$
\begin{aligned}
&-C_4\varepsilon\left(\left\|D^m w\cdot(-\Delta)^{2m} w\right\|^2+\left\|w_t\right\|^2\right)\\
&\geq r\delta\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2+\varepsilon r\delta_0\left\|D^{2m}w\right\|^2-C_4\left(\frac{1+2\varepsilon}{2}\right)\left\|(-\Delta)^{2m} w\right\|_\infty\left\|D^m w\right\|^2-\frac{C_4}{2}\left\|w_t\right\|^2 \quad (3.26)\\
&> r\delta\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2+\varepsilon r\delta_0\left\|D^{2m}w\right\|^2-C_5\left\|D^m w\right\|^2-\frac{C_4}{2}\left\|w_t\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|w_t\right\|^2+\varepsilon\left(w_t,w\right)+\frac{\alpha+\beta\varepsilon+\varepsilon\delta B+2r\delta}{2}\left\|D^{2m}w\right\|^2\right)\\
&+\left(-\frac{2\varepsilon-\varepsilon C_3-2C_3+C_4}{2}\right)\left\|w_t\right\|^2+\frac{\varepsilon C_3}{2}\left\|w\right\|^2+\left(\alpha\varepsilon+\varepsilon r\delta-\frac{C_5}{\lambda_1^m}\right)\left\|D^{2m}w\right\|^2\leq 0
\end{aligned}
$$

So there exist a_2 to make that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1+\varepsilon}{2}\left\|w_t\right\|^2+\frac{\varepsilon}{2}\left\|w\right\|^2+\frac{\alpha+\beta\varepsilon+\varepsilon\delta B+2r\delta}{2}\left\|D^{2m}w\right\|^2\right)\\
&+a_2\left(\frac{1+\varepsilon}{2}\left\|w_t\right\|^2+\frac{\varepsilon}{2}\left\|w\right\|^2+\frac{\alpha+\beta\varepsilon+\varepsilon\delta B+2r\delta}{2}\left\|D^{2m}w\right\|^2\right)\leq 0
\end{aligned} \tag{3.27}
$$

From Gronwall's inequality:

$$
\begin{aligned}
&\frac{1+\varepsilon}{2}\left\|w_t\right\|^2+\frac{\varepsilon}{2}\left\|w\right\|^2+\frac{\alpha+\beta\varepsilon+\varepsilon\delta B+2r\delta}{2}\left\|D^{2m}w_1\right\|^2\\
&\leq\frac{1+\varepsilon}{2}\left\|w_{0t}\right\|^2+\frac{\varepsilon}{2}\left\|w_0\right\|^2+\frac{\alpha+\beta\varepsilon+\varepsilon\delta B+2r\delta}{2}\left\|D^{2m}w_0\right\|^2=0
\end{aligned} \tag{3.28}
$$

In a result, we have $u=v$ and the Uniqueness obtained.

4. The Existence and Dimension Estimation of the Family of Global Attractor

Having studied the existence and uniqueness of global solutions for problems (1.1)-(1.3), the following proves the existence of the family of global attractor and estimates the hausdorff dimension and fractal dimension.

Theorem 4.1. [10] [11] Let E be a Banach space and semigroup $S(t):E\to E$ satisfy the following conditions:

1) The semigroup $S(t)$ is uniformly bounded in *E*, that is $\forall r>0$, when $\|u\|_E\leq r$, there is a constant $C(r)$, such that

$$
\left\|S(t)u\right\|_E\leq C(r),\forall t\in[0,+\infty)
$$

is true;

2) There exists a bounded absorption set B_0 in *E*;

3) $S(t)$ is a completely continuous operator, and the semigroup $S(t)$ has a compact complete attractor A_0.

By changing the Banach space E from the above theorem to the Hilbert space E_k, the existence theorem of the family of global attractor is obtained.

Theorem 4.2. If the global smooth solution of the problem (1.1)-(1.3) satisfies the conditions of Lemma 3.1, Lemma 3.2 and theorem 3.3 in the existence and uniqueness of the solution, then the problem (1.1)-(1.3) has a family of global attractor

$$A_k=\omega\left(K_{0k}\right)=\bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}S\left(t\right)K_{0k}},k=1,2,\cdots,2m,$$

where $K_{0k}=\left\{\left(u,v\right)\in E_k:\left\|\left(u,v\right)\right\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^kv\right\|^2\le R_k^2\right\}$ is the bounded absorption set in E_k and satisfies:

1) $S\left(t\right)A_k=A_k,t>0$.
2) $\lim_{t\to\infty}dist\left(S\left(t\right)B_k,A_k\right)=0$ ($\forall B_k\subset E_k$ is a bounded set), where

$$\lim_{t\to\infty}dist\left(S\left(t\right)B_k,A_k\right)=\sup_{x\in B_k}\inf_{y\in A_k}\left\|S\left(t\right)x-y\right\|_{E_k},$$

where $S\left(t\right)$ is the solution semigroup generated by problem (1.1)-(1.3).

Proof: It is necessary to verify the condition (1), (2), (3) of theorem 4.1, under the condition of theorem, the equation has a solution semigroup $S\left(t\right):E_k\to E_k$. The bounded set known as $\forall K_{0k}\subset E_k$ and contained in ball $\left\{\left\|\left(u,v\right)\right\|_{E_k}^2\le R_k^2\right\}$,

$$\left\|S\left(t\right)\left(u_0,v_0\right)\right\|_{E_k}^2=\left\|u\right\|_{H_0^{2m+k}(\Omega)}^2+\left\|v\right\|_{H_0^k(\Omega)}^2\le\left\|u_0\right\|_{H_0^{2m+k}(\Omega)}^2+\left\|v_0\right\|_{H_0^k(\Omega)}^2\le R_k^2 \tag{4.1}$$

Then $\left\{S\left(t\right)\right\}\left(t\ge 0\right)$ is uniformly bounded within E_k. For any $\left(u_0,v_0\right)\in E_k$,

$$\left\|S\left(t\right)\left(u_0,v_0\right)\right\|_{E_k}^2=\left\|u\right\|_{H^{2m+k}(\Omega)}^2+\left\|v\right\|_{H^k(\Omega)}^2\le R_k^2$$

It follows that $K_{0k}=\left\{\left(u,v\right)\in E_k:\left\|\left(u,v\right)\right\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^kv\right\|^2\le R_k^2\right\}$ is the bounded absorption set of a semigroup $S\left(t\right)$.

Since $E_k\subset E_0$ is compactly embedded, that is the bounded set in E_k is the compactly set in E_0, the solution semigroup $S\left(t\right)$ is a completely continuous operator, so there exists a family of global attractor $A_k=\omega\left(K_{0k}\right)=\bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}S\left(t\right)K_{0k}}$ of the solution semigroup $S\left(t\right)$.

Theorem 4.2 is proved.

To linearize Equation (1.1)-(1.3), consider the following initial boundary value problem:

$$\begin{aligned}&U_{tt}+\delta\psi\left(\left\|u_t\right\|^2\right)\Delta^{2m}U_t+\delta\psi'\left(\left\|u_t\right\|^2\right)\left(\left\|u_t\right\|^2\right)'U_t\cdot\Delta^{2m}u_t+\beta\Delta^{2m}U_t\\&+\gamma\Phi\left(\left\|D^mu\right\|_p^p\right)\Delta^{2m}U+\gamma\Phi'\left(\left\|D^mu\right\|_p^p\right)\left(\left\|D^mu\right\|_p^p\right)'D^mU\cdot\Delta^{2m}u+\alpha\Delta^{2m}U=0\end{aligned} \tag{4.2}$$

$$U\left(x,t\right)=0,\frac{\partial^iU}{\partial V^i}=0,i=1,2,\cdots,2m-1,x\in\partial\Omega,t>0, \tag{4.3}$$

$$U\left(x,0\right)=\xi,U_t\left(x,0\right)=\eta. \tag{4.4}$$

where $\left(u_0,u_1\right)\in A_k$, $S\left(t\right):E_k\to E_k$, $\left(\xi,\eta\right)\in E_k$, $\left(u,u_t\right)=S\left(t\right)\left(u_0,u_1\right)$ is the solution of problem (1.1)-(1.3) obtained by $\left(u_0,u_1\right)\in A_k$, it can be shown that for any $\left(\xi,\eta\right)\in E_k$, the linearized initial boundary value problem (4.2)-(4.3) has a unique solution $\left(U\left(t\right),U_t\left(t\right)\right)\in L^\infty\left(\left[0,+\infty\right);E_k\right)$.

Lemma 4.3. $\forall t>0,R>0$, The map $E_k\to E_k$ is Frechet differentiable on E_k, and the Frechet differential in $\phi_0=\left(u_0,u_1\right)^{\mathrm{T}}$ is a linear operator $F:\left(\xi,\eta\right)^{\mathrm{T}}\to\left(U\left(t\right),U_t\left(t\right)\right)^{\mathrm{T}}$.

Proof: Let $\phi_0=(u_0,u_1)^{\mathrm{T}}\in E_k$, $\bar{\phi}_0=(u_0+\xi,u_1+\eta)^{\mathrm{T}}\in E_k$ and $\|\phi_0\|_{E_k},\|\bar{\phi}_0\|_{E_k}\le R$ give the Lipchitz continuity of $S(t)$ on E_k, *i.e.*

$$\left\|S(t)\phi_0-S(t)\bar{\phi}_0\right\|_{E_k}^2\le \mathrm{e}^{c_{16}t}\left\|(\xi,\eta)^{\mathrm{T}}\right\|_{E_k}^2 .$$

If $h=\bar{u}-u-U,\mu=h_t+\varepsilon h$, then

$$\begin{cases}\bar{u}_{tt}+\beta\Delta^{2m}\bar{u}_t+\delta\Psi\left(\|\bar{u}_t\|^2\right)\Delta^{2m}\bar{u}_t+\alpha\Delta^{2m}\bar{u}+\gamma\Phi\left(\left\|D^m\bar{u}\right\|_p^p\right)\Delta^{2m}\bar{u}=g(x)\\ u_{tt}+\beta\Delta^{2m}u_t+\delta\Psi\left(\|u_t\|^2\right)\Delta^{2m}u_t+\alpha\Delta^{2m}u+\gamma\Phi\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u=g(x)\\ U_{tt}+\delta\psi\left(\|u_t\|^2\right)\Delta^{2m}U_t+\delta\psi'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'U_t\cdot\Delta^{2m}u_t+\beta\Delta^{2m}U_t\\ +\gamma\Phi\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}U+\gamma\Phi'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)'D^mU\cdot\Delta^{2m}u+\alpha\Delta^{2m}U=0\end{cases}$$

Subtract from the three formulas to get

$$(\bar{u}-u-U)_{tt}+\beta(-\Delta)^{2m}(\bar{u}-u-U)_t+\alpha(-\Delta)^{2m}(\bar{u}-u-U)+g_1+g_2=0 \tag{4.5}$$

where

$$\begin{aligned}g_1=&\,\delta\psi\left(\|\bar{u}_t\|^2\right)(-\Delta)^{2m}\bar{u}_t-\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}u_t-\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}U_t\\&-\delta\psi'\left(\left\|D^m u_t\right\|^2\right)\left(\left\|D^m u_t\right\|^2\right)'D^mU_t(-\Delta)^{2m}u_t\end{aligned} \tag{4.6}$$

$$\begin{aligned}g_2=&\,r\phi\left(\left\|D^m\bar{u}\right\|_p^p\right)(-\Delta)^{2m}\bar{u}-r\phi\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^{2m}u-r\phi\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^{2m}U\\&-r\phi'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)'D^mU(-\Delta)^{2m}u\end{aligned} \tag{4.7}$$

$\bar{u}-u-U=h$, Use the mean value theorem to deal with

$$\begin{aligned}g_1=&\,\delta\psi'\left(\|\xi_t\|^2\right)\left(\|\xi_t\|^2\right)'(\bar{u}_t-u_t)(-\Delta)^{2m}\bar{u}_t+\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}ht\\&-\delta\psi'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'U_t(-\Delta)^{2m}u_t\\=&\,g_{11}+g_{12}\end{aligned} \tag{4.8}$$

where

$$g_{11}=\delta\psi'\left(\|\xi_t\|^2\right)\left(\|\xi_t\|^2\right)'(-\Delta)^{2m}\bar{u}_t(\bar{u}_t-u_t)-\delta\psi'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'(-\Delta)^{2m}u_t(\bar{u}_t-u_t)$$

$$g_{12}=\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}ht+\delta\psi'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'ht(-\Delta)^{2m}u_t$$

$$\begin{aligned}g_{11}=&\,\delta\psi'\left(\|\xi_t\|^2\right)\left(\|\xi_t\|^2\right)'(-\Delta)^{2m}\bar{u}_t-\delta\psi'\left(\|u_t\|^2\right)(-\Delta)^{2m}u_t(\bar{u}_t-u_t)\\=&\left[\left(\delta\psi'\left(\|\xi_t\|^2\right)\left(\|\xi_t\|^2\right)'-\delta\psi'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'\right)(-\Delta)^{2m}\bar{u}_t\right.\\&\left.+\delta\psi'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'(-\Delta)^{2m}(\bar{u}_t-u_t)\right](\bar{u}_t-u_t)\end{aligned}$$

Make $f(x)=\delta\psi'\left(\|x\|^2\right)\left(\|x\|^2\right)'$ and $\xi_t=\lambda_1u_t+(1-\lambda_1)\bar{u}_t,\lambda_1\in(0,1)$, Then the above formula is

$$\begin{aligned} g_1 &= f'(s)\lambda_1(-\Delta)^{2m}\bar{u}_t(\bar{u}_t-u_t)^2 + f(u_t)(-\Delta)^{2m}(\bar{u}_t-u_t)\cdot(\bar{u}_t-u_t) \\ &\quad + \delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}ht + f(u_t)(-\Delta)^{2m}u_t\cdot ht \end{aligned} \tag{4.9}$$

Similarly deal with

$$\begin{aligned} g_2 &= r\phi'\left(\|D^m\xi_t\|_p^p\right)\left(\|D^m\xi_t\|_p^p\right)'\left(D^m\bar{u}-D^m u\right)(-\Delta)^{2m}\bar{u} \\ &\quad - r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)' D^m\bar{u}(-\Delta)^{2m}u \\ &\quad + r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)' D^m u\cdot(-\Delta)^{2m}u + r\phi\left(\|D^m u\|_p^p\right)(-\Delta)^{2m}h \\ &\quad + r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)' D^m h\cdot(-\Delta)^{2m}u \\ &= g_{21}+g_{22} \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} g_{21} &= r\phi'\left(\|D^m\xi_t\|_p^p\right)\left(\|D^m\xi_t\|_p^p\right)'\left(D^m\bar{u}-D^m u\right)(-\Delta)^{2m}\bar{u} \\ &\quad - r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)'\left(D^m\bar{u}-D^m u\right)(-\Delta)^{2m}u \end{aligned}$$

$$g_{22} = r\phi\left(\|D^m u\|_p^p\right)(-\Delta)^{2m}h + r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)' D^m h\cdot(-\Delta)^{2m}u$$

Similarly deal with

$$\begin{aligned} g_{21} &= \left[r\phi'\left(\|D^m\|_p^p\right)\left(\|D^m\|_p^p\right)'(-\Delta)^{2m}\bar{u} - r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)'(-\Delta)^{2m}u\right]\left(D^m\bar{u}-D^m u\right) \\ &= \left[r\phi'\left(\|D^m\|_p^p\right)\left(\|D^m\|_p^p\right)' - r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)'(-\Delta)^{2m}\bar{u}\right. \\ &\quad \left. + r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)'\cdot(-\Delta)^{2m}(\bar{u}-u)\right]\left(D^m\bar{u}-D^m u\right) \end{aligned}$$

Recording $m(y) = r\phi'\left(\|b\|_p^p\right)\left(\|b\|_p^p\right)'$, $D^m\xi = D^m u + \lambda_2\left(D^m\bar{u}-D^m u\right), \lambda_2\in(0,1)$.
So we can obtain

$$\begin{aligned} g_{21} &= m'(s)\lambda_2(-\Delta)^{2m}\bar{u}\left(D^m\bar{u}-D^m u\right)^2 \\ &\quad + r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)'(-\Delta)^{2m}(\bar{u}-u)\left(D^m\bar{u}-D^m u\right) \end{aligned}$$

$$\begin{aligned} g_2 &= m'(s)\delta(-\Delta)^{2m}\bar{u}\left(D^m\bar{u}-D^m u\right)^2 \\ &\quad + r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)'(-\Delta)^{2m}(\bar{u}-u)\left(D^m\bar{u}-D^m u\right) \\ &\quad + r\phi\left(\|D^m u\|_p^p\right)(-\Delta)^{2m}h + r\phi'\left(\|D^m u\|_p^p\right)\left(\|D^m u\|_p^p\right)'(-\Delta)^{2m}u\cdot D^m h \end{aligned} \tag{4.11}$$

Let $(-\Delta)^k\mu$ and each of them take the inner product: where $\mu = h_t + \varepsilon h$.

$$\begin{aligned}&\left(h_{tt},(-\Delta)^{k}\mu\right)=\left(k_t-\varepsilon k+\varepsilon^2 h,(-\Delta)^{k}\mu\right)\\&\qquad=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k\mu\right\|^2-\varepsilon\left\|D^k\mu\right\|^2+\frac{\varepsilon^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k h\right\|^2+\varepsilon^3\left\|D^k h\right\|^2\end{aligned}\tag{4.12}$$

$$\begin{aligned}&\left(\beta+\delta\psi\left(\|u_t\|^2\right)(-\Delta)^{2m}h_t,(-\Delta)^k\mu\right)\\&=\beta+\delta\psi\left(\|u_t\|^2\right)\left(\left\|D^{2m+k}h_t\right\|^2+\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2\right)\geq\frac{(\beta+\delta B)\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2\end{aligned}\tag{4.13}$$

$$\begin{aligned}&\left(\alpha+r\phi\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^{2m}h,(-\Delta)^k\mu\right)\\&=\left(\alpha+r\phi\left(\left\|D^m u\right\|_p^p\right)\right)\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}h\right\|^2+\varepsilon\left\|D^{2m+k}h\right\|^2\right)\\&\geq\frac{\alpha+r\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}h\right\|^2+\varepsilon\left(\alpha+r\delta_0\right)\left\|D^{2m+k}h\right\|^2\end{aligned}\tag{4.14}$$

$$\begin{aligned}&\left(f'(s)\lambda_1(-\Delta)^{2m}\bar{u}_t\left(\bar{u}_t-u_t\right)^2,(-\Delta)^k\mu\right)\\&=f'(s)\lambda_1\int_\Omega D^{2m+k}\bar{u}\left(\bar{u}_t-u_t\right)^2D^{2m+k}k\,\mathrm{d}x\leq C_6\left\|\bar{u}_t-u_t\right\|^2\cdot\left\|D^{2m+k}\mu\right\|\end{aligned}$$

Due to

$$\begin{aligned}\|u_t\|&\leq\|v\|+\varepsilon\|u\|\leq\|v\|+\|u\|\leq\frac{1}{\lambda_1^{\frac{k}{2}}}\left\|D^k v\right\|+\frac{1}{\lambda_1^{\frac{2m+k}{2}}}\left\|D^{2m+k}u\right\|\\&\leq\lambda_0\left(\left\|D^k v\right\|+\left\|D^{2m+k}u\right\|\right)=\lambda_0\|u\|_{E_k}\end{aligned}$$

So there are

$$\begin{aligned}&\left(f'(s)\lambda_1(-\Delta)^{2m}\bar{u}_t\left(\bar{u}_t-u_t\right)^2,(-\Delta)^k\mu\right)\\&\leq C_6\lambda_0\left\|\bar{u}_t-u_t\right\|_{E_k}^2\cdot\left\|D^{2m+k}\mu\right\|\leq\frac{C_6\lambda_0}{2}\left\|\bar{u}_t-u_t\right\|_{E_k}^4+\frac{C_6\lambda_0}{2}\left\|D^{2m+k}\mu\right\|^2\end{aligned}\tag{4.15}$$

$$\left(f(u_t)(-\Delta)^{2m}u_t h_t,(-\Delta)^k\mu\right)\leq C_7\left\|\bar{u}_t-u_t\right\|^2\cdot\left\|D^{2m+k}\mu\right\|$$

$$\left(f(u_t)(-\Delta)^{2m}u_t\cdot h_t,(-\Delta)^k\mu\right)\geq C_8\left(\left\|D^k h_t\right\|^2+\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k h\right\|^2\right)$$

$$\begin{aligned}&\left(m'(s)\lambda_2(-\Delta)^{2m}\bar{u}\left(D^m\bar{u}-D^m u\right)^2,(-\Delta)^k\mu\right)\\&\leq C_9\left\|D^m\bar{u}-D^m u\right\|^2\cdot\left\|D^{2m+k}\mu\right\|\leq\frac{C_9}{\lambda_1^{m+k}}\left\|D^{2m+k}\bar{u}-D^{2m+k}u\right\|^2-\left\|D^{2m+k}w\right\|\\&\leq\frac{C_9}{\lambda_1^{m+k}}\left\|\bar{u}_t-u_t\right\|_{E_k}^2\cdot\left\|D^{2m+k}\mu\right\|\leq\frac{C_9}{2\lambda_1^{m+k}}\left\|\bar{u}_t-u_t\right\|_{E_k}^4+\frac{C_9}{2\lambda_1^{m+k}}\left\|D^{2m+k}\mu\right\|^2\end{aligned}\tag{4.16}$$

$$\begin{aligned}&\left(r\phi'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)'(-\Delta)^{2m}(\bar{u}-u)\left(D^m\bar{u}-D^m u\right),(-\Delta)^k\mu\right)\\&\leq C_{10}\left\|D^{2m+k}\bar{u}-D^{2m+k}u\right\|\cdot\left\|D^m(\bar{u}-u)\right\|\cdot\left\|D^{2m+k}\mu\right\|\\&\leq\frac{C_{10}}{\lambda_1^{\frac{m+k}{2}}}\left\|D^{2m+k}(\bar{u}-u)\right\|^2\cdot\left\|D^{2m+k}\mu\right\|\leq\frac{C_{10}}{2\lambda_1^{\frac{m+k}{2}}}\left(\left\|\bar{u}-u\right\|_{E_k}^4+\left\|D^{2m+k}\mu\right\|^2\right)\end{aligned}\tag{4.17}$$

$$\begin{aligned}&\left(r\phi'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)'(-\Delta)^{2m} uD^m h,(-\Delta)^k \mu\right)\\&\le C_{11}\left\|D^m h\right\|\cdot\left\|D^{2m+k}h\right\|\le\frac{C_{11}}{2}\left\|D^m h\right\|^2+\frac{C_{11}}{2}\left\|D^{2m+k}h\right\|^2\\&\le\left(\frac{C_{11}}{2}+\frac{C_{11}}{2\lambda_1^{m+k}}\right)\left\|D^{2m+k}h\right\|^2\end{aligned}\tag{4.18}$$

So we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|D^k\mu\right\|^2+\frac{1}{2}\left\|D^{2m+k}\mu\right\|^2+\frac{\varepsilon(C_8+\varepsilon)}{2}\left\|D^k h\right\|^2+\frac{\beta+\delta B\varepsilon}{2}\|h\|^2\right.\\&\left.+\frac{2\varepsilon+1}{2}(\alpha+r\delta)\left\|D^{2m+k}h\right\|^2\right)\\&\le C_{12}\left\|\bar{u}-u\right\|_{E_k}^4+C_{13}\left\|D^{2m+k}\mu\right\|^2+C_{14}\left\|D^k\mu\right\|^2\\&\quad+\frac{C_{11}}{2}\left(1+\frac{1}{\lambda_1^{m+k}}\right)\left\|D^{2m+k}h\right\|^2+\left(\varepsilon-\varepsilon^2\right)\left\|D^k h\right\|^2\end{aligned}\tag{4.19}$$

$$\begin{aligned}&\frac{1}{2}\left\|D^k\mu\right\|^2+\frac{1}{2}\left\|D^{2m+k}\mu\right\|^2+\frac{\varepsilon(C_8+\varepsilon)}{2}\left\|D^k h\right\|^2+\frac{\beta+\delta B\varepsilon}{2}\|h\|^2\\&+\frac{2\varepsilon+1}{2}(\alpha+r\delta)\left\|D^{2m+k}h\right\|^2\le C_{15}\mathrm{e}^{C_{16}t}\left\|\bar{u}-u\right\|_{E_k}^4\end{aligned}\tag{4.20}$$

when $\left\|(\xi,\eta)\right\|_{E_k}^2\to 0$, $\frac{1}{2}\frac{\left\|\bar{u}-u-U\right\|_{E_k}^2}{\left\|(\xi,\eta)\right\|_{E_k}^2}\le C_{15}\mathrm{e}^{C_{16}t}\left\|(\xi,\eta)\right\|_{E_k}^2\to 0$.

Theorem 4.4. Under the condition of Theorem 4.2, the family of global attractor A_k of the problem (1.1)-(1.3) has the Hausdorff dimension and the Fractal dimension and $d_H(A_k)<\frac{2}{3}n$, $d_F(A_k)<\frac{4}{3}n$.

Proof: Let $P_\varepsilon:\{u,u_t\}\to\{u,u_t+\varepsilon u\}$ be an isomorphic mapping, then

$$\Psi=P_\varepsilon\bar{\varphi}=(U,V)^{\mathrm{T}},\ \text{where}\ \bar{\varphi}=(U,U_t)^{\mathrm{T}},\ V=U_t+\varepsilon U,$$

The Frechet differentiability of $S(t):E_k\to E_k$ is known from lemma 4.3 to estimate the Hausdorff dimension and Fractal dimension of problem (1.1)-(1.3). Consider the variational equation $Y_t+A_\varepsilon Y=0$ of Equation (4.2) under initial conditions, where

$$Y=\begin{bmatrix}U\\V\end{bmatrix}$$

$$A_\varepsilon=\begin{bmatrix}\varepsilon I & -I\\ \alpha-\varepsilon\beta-r\phi\left(\left\|D^m u\right\|_p^p\right)-\varepsilon\delta\psi\left(\left\|u_t\right\|^2\right)A^{2m}+\varepsilon^2-2\delta\varepsilon B+prC & \beta+\delta\psi\left(\left\|u_t\right\|^2\right)A^{2m}-\varepsilon+2\delta D\end{bmatrix}\tag{4.21}$$

where ream $-\Delta=A$, $DV=\psi'\left(\left\|u_t\right\|^2\right)\int_\Omega u_t V\mathrm{d}x(-\Delta)^{2m}u_t$,

$$BU=\psi'\left(\left\|u_t\right\|^2\right)\int_\Omega u_t U\mathrm{d}x(-\Delta)^{2m}u_t,$$

$$CU=\phi'\left(\left\|D^m u\right\|_p^p\right)\int_\Omega\left|D^m u\right|^{p-2}D^m uD^m U\mathrm{d}x(-\Delta)^{2m}u$$

For A fixed $(u_0,v_0)\in E_k$, let $\gamma_1,\gamma_2,\cdots,\gamma_n$ be n elements of E_k, and let $U_1(t),U_2(t),\cdots,U_n(t)$ be n solutions of the linear Equation (4.2) with corresponding initial values of $U_1(0)=\gamma_1,U_2(0)=\gamma_2,\cdots,U_n(0)=\gamma_n$. Get

$$\begin{aligned}&\left\|U_1(t)\Lambda U_2(t)\Lambda\cdots\Lambda U_n(t)\right\|_{\Lambda E_k}^2\\&=\left\|\gamma_1\Lambda\gamma_1\Lambda\cdots\Lambda\gamma_n\right\|_{\Lambda E_k}^2\exp\left(\int_0^t trF'(\Psi(\tau))\cdot Q_n(\tau)\mathrm{d}\tau\right),t\in[0,1]\end{aligned}$$

where Λ represents the outer product, tr represents the trace of the operator, and Q_N is the orthogonal projection from space E_k to

$$span\{U_1(t),U_2(t),\cdots,U_n(t)\}.$$

Given τ, let $\omega_j(\tau)=(\xi_j(\tau),\eta_j(\tau))^{\mathrm{T}}, j=1,2,\cdots,n$ be the orthonormal basis of $span\{U_1(t),U_2(t),\cdots,U_n(t)\}$.

Define the inner product on E_k as

$$((\xi,\eta),(\bar{\xi},\bar{\eta}))=\left(\left(D^{2m+k}\xi,D^{2m+k}\bar{\xi}\right)+\left(D^k\eta,D^k\bar{\eta}\right)\right), \tag{4.22}$$

$$\begin{aligned}trF'(\Psi(\tau))\cdot Q_n(\tau)&=-\sum_{j=1}^n\left(\Lambda_\varepsilon(\Psi(\tau))\cdot Q_n(\tau)\omega_j(\tau),\omega_j(\tau)\right)_{E_k}\\&=-\sum_{j=1}^n\left(\Lambda_\varepsilon(\Psi(\tau))\omega_j(\tau),\omega_j(\tau)\right)_{E_k}\end{aligned} \tag{4.23}$$

$$\begin{aligned}&-\left(A_\varepsilon w_j,w_j\right)_{E_k}\\&=-\Big[\Big(\varepsilon\xi_j-\eta_j,\Big(\alpha-\varepsilon\beta-r\phi\left(\left\|D^mu\right\|_p^p\right)-\varepsilon\delta\psi\left(\left\|u_t\right\|^2\right)A^{2m}+\varepsilon^2-2\delta\varepsilon B+prC\Big)\xi_j\\&\quad+\left(\beta+\delta\psi\left(\left\|u_t\right\|^2\right)A^{2m}-\varepsilon+2\delta D\right)\mu_j\Big)\cdot\left(\xi_j,\mu_j\right)\Big]_{E_k}\\&=-\varepsilon\left\|D^{2m+k}\xi_j\right\|^2+\left(1-\alpha+\varepsilon\beta+r\phi\left(\left\|D^mu\right\|_p^p\right)-\varepsilon\delta\psi\left(\left\|u_t\right\|^2\right)\right)\left(D^{2m+k}\mu_j,D^{2m+k}\xi_j\right)\\&\quad-\left(\varepsilon^2-2\delta\varepsilon B+prC\right)\left(D^{2m+k}\mu_j,D^{2m+k}\xi_j\right)-(\beta+\delta\psi)\left(\left\|u_t\right\|^2\right)\left\|D^{2m+k}\mu_j\right\|^2\\&\quad+\left(\varepsilon-2\delta|D|\left\|D^k\mu_j\right\|^2\right)\\&\le-\varepsilon\left\|D^{2m+k}\xi_j\right\|^2+\frac{1-\alpha+\varepsilon\beta+r\delta_1-\varepsilon\delta B_0}{2}\left(\left\|D^{2m+k}\mu_j\right\|^2+\left\|D^{2m+k}\xi_j\right\|^2\right)\\&\quad+\frac{\varepsilon^2-2\delta\varepsilon|B|+pr|C|}{2}\left(\left\|D^k\xi_j\right\|^2+\left\|D^k\mu_j\right\|^2\right)\\&\le\left(-\varepsilon+\frac{1-\alpha+\varepsilon\beta+r\delta_1-\varepsilon\delta B_0}{2}+\frac{\varepsilon^2-2\delta\varepsilon|B|+pr|C|}{2\lambda_1^{2m+k}}\right)\left\|D^{2m+k}\xi_j\right\|^2\\&\quad+\frac{\varepsilon^2-2\delta\varepsilon|B|+pr|C|}{2}\left\|D^k\mu_j\right\|^2+\varepsilon\left\|D^k\mu_j\right\|^2\end{aligned}$$

Let

$$C_{17}=\max\left\{\left(-\varepsilon+\frac{1-\alpha+\varepsilon\beta+r\delta_1-\varepsilon\delta B_0}{2}+\frac{\varepsilon^2-2\delta\varepsilon|B|+pr|C|}{2\lambda_1^{2m}}\right),\frac{\varepsilon^2-2\delta\varepsilon|B|+pr|C|}{2}\right\}$$

$$\varepsilon=r,\quad\therefore-\left(A_\varepsilon w_j,w_j\right)_{E_k}\le C_{17}\left(\left\|D^k\mu_j\right\|^2+\left\|D^{2m+k}\xi_j\right\|^2\right)+r\left\|D^k\mu_j\right\|^2. \tag{4.24}$$

Since $\omega_j(\tau)=(\xi_j(\tau),\eta_j(\tau))^{\mathrm{T}}, j=1,2,\cdots,n$, $span\{U_1(t),U_2(t),\cdots,U_n(t)\}$ is an orthonormal basis,

$$\sum_{j=1}^{n}\left(F'(\psi(\tau))\omega_j(\tau),\omega_j(\tau)\right)_{E_k}\le -nC_{17}+r\sum_{j=1}^{n}\left\|D^k\xi_j\right\|^2 \tag{4.25}$$

For any *t*, there's a

$$\sum_{j=1}^{n}\left\|D^k\xi_j\right\|^2\le\sum_{j=1}^{n}\lambda_j^{s-1}, \tag{4.26}$$

So there is

$$TrF'(\Psi(\tau))\cdot Q_n(\tau)\le -nC_{17}+r\sum_{j=1}^{n}\lambda_j^{s-1}. \tag{4.27}$$

Let

$$q_n(t)=\sup_{\Psi_0\in K_{0k}}\sup_{\substack{\eta_j\in E_k\\ \|\eta_j\|_{E_k}\le 1}}\left(\frac{1}{t}\int_0^t trF'(S(\tau)\Psi_0)\cdot Q_n(\tau)\mathrm{d}\tau\right), q_n=\limsup_{t\to\infty} q_n(t)$$

Then $q_n\le -nC_{17}+r\sum_{j=1}^{n}\lambda_j^{s-1}$, Therefore, the Lyapunov exponent $\tilde{\mu}_1,\tilde{\mu}_2,\cdots,\tilde{\mu}_j$ of K_{0k} is uniformly bounded, and

$$\tilde{\mu}_1+\tilde{\mu}_2+\cdots+\tilde{\mu}_j\le -nC_{17}+r\sum_{j=1}^{n}\lambda_j^{s-1}$$

$$\exists s\in[0,1] \text{ make } (q_j)_+\le\frac{nC_{17}}{7},\quad q_n=-\frac{nC_{17}}{7}\left(1-\frac{2r}{nC_{17}}\sum_{j=1}^{n}\lambda_j^{s-1}\right)\le -\frac{3}{14}nC_{17} \tag{4.28}$$

$$\therefore \max\frac{(q_j)_+}{|q_n|}\le\frac{2}{3},\quad \therefore d_H(A_k)\le\frac{2}{3}n,\quad d_F(A_k)\le\frac{4}{3}n \tag{4.29}$$

It can be concluded that N-dimensional volume elements decay exponentially in E_k and $d_H(A_k)<\frac{2}{3}n$, $d_F(A_k)<\frac{4}{3}n$, then the Hausdorff dimension and Fractal dimension of the family of global attractor are finite, and theorem 4.4 can be proved.

5. Conclusion

In this paper, we studied a class of generalized Bean-Kirchhoff equations, proved the existence and uniqueness of the global solution of this class of equations by Galerkin method, and further proved the existence of the family of global attractor and its Hausdorff dimension and Fractal dimension estimation, which has certain scientific significance. And there's more to study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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