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Path Integral Quantization of Non-Natural Lagrangian

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Abstract

Path integral technique is discussed using Hamilton Jacobi method. The Hamilton Jacobi function of **non-natural** Lagrangian is obtained using separation of variables method. This function makes an important role in path integral quantization. The path integral is obtained as integration over the canonical phase space coordinates, which contains the generalized coordinate q and the generalized momentum p. One illustrative example is considered to explain the application of our formalism.

Keywords

Path Integral, Quantization, Hamilton Jacobi Equation, Non-Natural Lagrangian, Hamilton Jacobi Function

1. Introduction

Path integral technique was first investigated by [1] who presented the Wiener integral for solving problems in diffusion and Brownian motion. Then this work is followed by [2] [3] where the use of the Lagrangian in quantum mechanics appeared.

Also path integral quantization of constrained systems has been developed by [4] [5] [6] [7]. In this formalism, the equations of motion are obtained as total differential equations. These researchers find the Hamilton Jacobi equation, which helps them to find the Hamilton Jacobi function that is used in path integral quantization.

Path integral quantization of dissipative systems was studied by [8], using Hamilton Jacobi treatment where the dissipation is presence as time dependent factor. Then path integral quantization is used to quantize nonconservative systems which are characterized by irregular Lagrangian, using canonical approach which depends on Euler Lagrange equation [9]. Recently, path integral quantization of regular Lagrangian using Hamiltonian formalism is investigated [10], where we can find the Hamilton Jacobi function S as a function of position coordinate and time. As a continuation of the previous works we will find path integral quantization using **non-natural** Lagrangian which appears in $e^{\lambda q}$ factor.

Our present work is organized as follows. In Section 2, the canonical path integral formulation of **non-natural** Lagrangianis discussed. In Section 3, illustrative example is presented in detail. The work closes with some concluding remarks in Section 4.

2. Path Integral Formulation of Non-Natural Lagrangian

We start with Lagrangianof **non-natural** Lagrangianas follows:

$$L = L(q, \dot{q}, t)e^{\lambda q} \tag{1}$$

The linear momentum is [11]:

$$p_q = \frac{\partial L}{\partial \dot{q}} \tag{2}$$

The canonical Hamiltonian has the standard form:

$$H_0 = p\dot{q} - L \tag{3}$$

Then, the Hamilton Jacobi equation takes the following form:

$$H = p_0 + H_0 = 0 \tag{4}$$

where;

$$p_0 = \frac{\partial S}{\partial t} \tag{5}$$

Then, using Equation (5) the Hamilton Jacobi equation takes the standard form:

$$H = \frac{\partial S}{\partial t} + H_0 = 0 \tag{6}$$

Now, making use separation of variables method to formulate the Hamilton Jacobi function S

$$S(q, E, t) = W(q, E) - f(t) \tag{7}$$

where $W(q, E)$ is a function of coordinate q, and $f(t)$ is a function of time t.

Also we can write;

$$\frac{\partial f}{\partial t} = \frac{\partial S}{\partial t} \tag{8}$$

And

$$\frac{\partial f}{\partial t} = E \tag{9}$$

From the Hamilton Jacobi function, the equation of motion equals to:

$$\frac{\partial S}{\partial E}=\beta \tag{10}$$

The linear momentum reads:

$$\frac{\partial S}{\partial q}=p \tag{11}$$

The Hamilton Jacobi function S can be calculated using Lagrangian mechanics as:

$$S(q,\dot{q},t)=\int_{t_1}^{t_2} L(q,\dot{q},t)\mathrm{d}t \tag{12}$$

or, by using Hamiltonian mechanics using this formula $L=p\dot{q}-H$.

Then;

Equation (12) becomes:

$$S(q,\dot{q},t)=\int_{t_1}^{t_2} L(q,\dot{q},t)\mathrm{d}t=\int_{t_1}^{t_2}(p\dot{q}-H)\mathrm{d}t \tag{13}$$

The path integral representation may be written as:

$$D(q,p)=\int\left[\exp(iS)\right]\mathrm{d}p\mathrm{d}q \tag{14}$$

which is obtained as integration over the canonical phase space coordinates which contains the generalized coordinate q and the generalized momentum p.

3. Example

Let us discuss the motion of a pendulum of mass m and length L with angular displacement θ from the vertical [12].

The Lagrangian of our system is:

$$L=\frac{1}{2}ml^2\dot{\theta}^2-mgl(1-\cos\theta) \tag{15}$$

In **non-natural** Lagrangian form Equation (15) becomes

$$L=\left[\frac{1}{2}ml^2\dot{\theta}^2-mgl(1-\cos\theta)\right]\mathrm{e}^{\lambda\theta} \tag{16}$$

For small θ, we have

$$\cos\theta=1-\frac{\theta^2}{2} \tag{17}$$

Then Equation (16) reads:

$$L=\left[\frac{1}{2}ml^2\dot{\theta}^2-\frac{1}{2}mgl\theta^2\right]\mathrm{e}^{\lambda\theta} \tag{18}$$

From Equation (2) and using Equation (18), the linear momentum is:

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=ml^2\dot{\theta}\mathrm{e}^{\lambda\theta} \tag{19}$$

Then,

$$\dot{\theta} = \frac{p_\theta}{ml^2}\mathrm{e}^{-\lambda\theta} \tag{20}$$

Squaring of Equation (20), we get

$$\dot{\theta}^2 = \left(\frac{p_\theta}{ml^2}\right)^2 \mathrm{e}^{-2\lambda\theta} \tag{21}$$

Substituting Equation (18) and Equation (20) into Equation (3), the Hamiltonian is:

$$H_0 = \frac{p_\theta^2}{2ml^2}\mathrm{e}^{-\lambda\theta} + \frac{1}{2}mgl\theta^2\mathrm{e}^{\lambda\theta} \tag{22}$$

Making use of Equation (6) and Equation (22), the Hamilton Jacobi equation written as:

$$H = \frac{\partial S}{\partial t} + \frac{1}{2ml^2}\left(\frac{\partial S}{\partial \theta}\right)^2 \mathrm{e}^{-\lambda\theta} + \frac{1}{2}mgl\theta^2\mathrm{e}^{\lambda\theta} = 0 \tag{23}$$

From Equation (7), the Hamilton Jacobi function is:

$$S(\theta, E, t) = W(\theta, E) - Et \tag{24}$$

Then,

$$\frac{\partial S}{\partial t} = -E \tag{25}$$

Thus,

$$\frac{\partial S}{\partial \theta} = \frac{\partial W}{\partial \theta} \tag{26}$$

Inserting Equation (25) and Equation (26) into Equation (23), we obtain

$$-E + \frac{1}{2ml^2}\left(\frac{\partial W}{\partial \theta}\right)^2 \mathrm{e}^{-\lambda\theta} + \frac{1}{2}mgl\theta^2\mathrm{e}^{\lambda\theta} = 0 \tag{27}$$

Thus,

$$\sqrt{2ml^2\left(E\mathrm{e}^{\lambda\theta} - \frac{1}{2}mgl\theta^2\mathrm{e}^{2\lambda\theta}\right)} = \frac{\partial W}{\partial \theta} \tag{28}$$

Then,

$$\sqrt{2ml^2\left(E\mathrm{e}^{\lambda\theta} - \frac{1}{2}mgl\theta^2\mathrm{e}^{2\lambda\theta}\right)}\mathrm{d}\theta = \mathrm{d}W \tag{29}$$

Integrating of Equation (29), we have

$$W = \int \sqrt{2ml^2\left(E\mathrm{e}^{\lambda\theta} - \frac{1}{2}mgl\theta^2\mathrm{e}^{2\lambda\theta}\right)}\mathrm{d}\theta \tag{30}$$

Substituting of Equation (30) into Equation (24), the Hamilton Jacobi function takes this form:

$$S = -Et + \int \sqrt{2ml^2\left(E\mathrm{e}^{\lambda\theta} - \frac{1}{2}mgl\theta^2\mathrm{e}^{2\lambda\theta}\right)}\mathrm{d}\theta \tag{31}$$

Form Equation (10), the equation of motion is:

$$\frac{\partial S}{\partial E}=\beta=-t+\int\frac{ml^2\mathrm{e}^{\lambda\theta}}{\sqrt{2ml^2\left(E\mathrm{e}^{\lambda\theta}-\frac{1}{2}mgl\theta^2\mathrm{e}^{2\lambda\theta}\right)}}\mathrm{d}\theta \tag{32}$$

And the conjugate momentum is:

$$p=\frac{\partial S}{\partial\theta}=\sqrt{2ml^2\left(E\mathrm{e}^{\lambda\theta}-\frac{1}{2}mgl\theta^2\mathrm{e}^{2\lambda\theta}\right)} \tag{33}$$

Now, we can write the Hamilton Jacobi function, by substituting Equation (33) into Equation (31) then;

$$S=-Et+\int p\mathrm{d}\theta \tag{34}$$

Thus,

$$S=p\theta-Et \tag{35}$$

Then, the path integral representation may be written as

$$D(\theta,p)=\int\left[\exp\left(i\left(p\theta-Et\right)\right)\right]dpd\theta \tag{36}$$

Finally;

$$D(\theta,p)=\int\frac{1}{i\theta}\left[\exp\left(i\left(p\theta-Et\right)\right)\right]d\theta \tag{37}$$

4. Conclusion

Path integral quantization is presented in details for **non-natural** Lagrangian. We use Hamilton Jacobi equation to find the Hamilton Jacobi function S which helps us to formulate the exponential function, then by integration over the space that contains the generalized coordinate q and the generalized momentum p; we reach to path integral quantization.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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