Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2023, 11, 2904-2912
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

2D Curves in Conformal Hyperquaternion Algebras $\mathbb{H}^{\otimes 2m}$

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How to cite this paper: Panga, G.L. (2023) 2D Curves in Conformal Hyperquaternion Algebras $\mathbb{H}^{\otimes 2m}$. *Journal of Applied Mathematics and Physics*, **11**, 2904-2912.
https://doi.org/10.4236/jamp.2023.1110191

Received: August 28, 2023
Accepted: October 20, 2023
Published: October 23, 2023

Abstract

The aim of this paper is to outline the conditions of a conformal hyperquaternion algebra $\mathbb{H}^{\otimes 2m}$ in which a higher order plane curve can be described by generalizing the well-known cases of conics and cubic curves in 2D. In other words, the determination of the order of a plane curve through n points and its conformal hyperquaternion algebra $\mathbb{H}^{\otimes 2m}$ is the object of this work.

Keywords

Clifford Algebra, Hyperquaternion Algebra, Conformal Hyperquaternion Algebra

1. Introduction

The definition of hyperquaternion algebra and the development of the hyperquaternion formalism (product, multivector calculus, hyperconjugation concept, ...) have been given in [1] [2] [3] [4] and [5]. In [6], the authors studied the hyperquaternion conformal groups and concluded this paper by inviting researches to explore potential applications of hyperquaternions in conformal field theory, computer graphics and conformal geometry. This last declaration inspired us to embark on the study of some concepts from conformal geometry in the realm of hyperquaternion algebras.

It is well-known that a curve in Euclidean space can be represented in conformal geometric algebra (CGA). In the framework of geometric algebra (Clifford algebra), the conformal geometric algebra $Cl_{3,1}$ has been described as the standard conformal geometric algebra CCA, the conformal conics geometric algebra CCGA ($Cl_{5,3}$) has been developed respectively by Perwass in [7], Hitzer *et al.* in [8] and Hrdina *et al.* in [9]. The conformal cubic curves geometric algebra ($Cl_{9,7}$) has been studied Hitzer and Hildenbrand in [10].

The isomorphism between Clifford algebras and hyperquaternion algebras $Cl_{2m+1,2m-1} \simeq \mathbb{H}^{\otimes 2m}$ for m integer ($m \geq 1$) established in [5] by Girard *et al.*, allows to provide the isomorphisms $Cl_{3,1} \simeq \mathbb{H}^{\otimes 2}$ (tetraquaternion algebra), $Cl_{5,3} \simeq \mathbb{H}^{\otimes 4}$ and $Cl_{9,7} \simeq \mathbb{H}^{\otimes 8}$.

In this work, we present briefly the analogous for a 2D curves of order two and order three in the context of conformal hyperquaternion algebras and we extend the process to higher order 2D curves in the conformal hyperquaternonion algebras of type $\mathbb{H}^{\otimes 2m}$. We show how to determine conformal hyperquaternion algebra $\mathbb{H}^{\otimes 2m}$ for 2D curve through n given points.

This paper is structured as follows:

In the introduction, we briefly present some works relating 2D curves in the realm of geometric algebra. The second section provides some basic results concerning the conformal hyperquaternion algebras $\mathbb{H}^{\otimes 2m}$ for $m=1, m=2$ and $m=4$. In the third section, we investigate upon the order of 2D curve through n points which have $\mathbb{H}^{\otimes 2m}$ as conformal hyperquaternion algebra. In the last section which is the conclusion, we present the central result of this paper.

2 Background: Conformal Hyperquaternion Algebras $\mathbb{H}^{\otimes 2}$, $\mathbb{H}^{\otimes 4}$ and $\mathbb{H}^{\otimes 8}$

2.1. Conformal Hyperquaternion Algebra $\mathbb{H}^{\otimes 2}$

Let (e_1, e_2, e_3, e_4) be an orthonormal basis of the vector space $\mathbb{R}^{3,1}$, and q be a quadratic form defined by $q(x) = q(x^i e_i) = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2$, for any $x \in \mathbb{R}^{3,1}$.

The generators of the Clifford algebra $Cl_{3,1}$ satisfy the following relations:

$$e_i^2 = 1 (i \in \{1,2,3\}), e_i^2 = -1 (i = 4), \tag{1}$$

and

$$e_i e_j + e_j e_i = 0, (i \neq j). \tag{2}$$

The result $\mathbb{H}^{\otimes 2m} \simeq Cl_{2m+1,2m-1}$, expressed in [5] (p. 6), provides the isomorphism between the Clifford algebra $Cl_{3,1}$ and the hyperquaternion algebra $\mathbb{H}^{\otimes 2}$.

Consider the quaternionic systems (i, j, k) and (I, J, K), we can define a basis of the hyperquaternion algbebra $\mathbb{H}^{\otimes 2}$ as follows

$$(1, i, j, k) \otimes (1, I, J, K) \tag{3}$$

We choice the generators of the hyperquaternion algbebra $\mathbb{H}^{\otimes 2}$ as follows

$$e_1 = k \otimes I = kI, e_2 = k \otimes J = kJ, e_3 = k \otimes K = kK, e_4 = j. \tag{4}$$

From the basis $e_1 = k \otimes I = kI$, $e_2 = k \otimes J = kJ$, $e_3 = k \otimes K = kK$, $e_4 = j$ of $\mathbb{R}^{3,1}$, we define a new basis as follows

$$\left(e_1 = kI, e_2 = kJ, e_\infty = \frac{1}{\sqrt{2}}(kK + j), e_0 = \frac{1}{\sqrt{2}}(-kK + j) \right), \tag{5}$$

such that $e_\infty^2 = 0, e_0^2 = 0$ and $e_\infty e_0 = -1$.

Let $\varphi:\mathbb{R}^2\to\mathbb{R}^{3,1}\subset\mathbb{H}^{\otimes 2}$ be the conformal embedding defined as follows

$$\varphi(X)=xkI+ykJ+\frac{1}{2\sqrt{2}}x^2(kK+j)+\frac{1}{\sqrt{2}}(-kK+j). \tag{6}$$

where $X=xkI+ykJ\in\mathbb{R}^2$.

For a given point X in the Euclidean plane $\mathbb{R}^2$, $\varphi(X)$ defined in (6) is called point in the conformal hyperquaternion algebra $\mathbb{H}^{\otimes 2}$. Note that each point $\varphi(X)$ in $\mathbb{H}^{\otimes 2}$ is a null vector (isotropic) and can be seen as a linear combination of $1,x,y,x^2$.

Let $V=akI+bkJ+\frac{d}{\sqrt{2}}(kK+j)+\frac{c}{\sqrt{2}}(-kK+j)\in\mathbb{H}^{\otimes 2}$, the inner product null space and the outer product null space of *V*, denoted respectively $IPNS(V)$ and $OPNS(V)$, are defined as follows

$$IPNS(V)=\left\{X\in\mathbb{R}^2:\varphi(X)\cdot V=0\right\} \tag{7}$$

and

$$OPNS(V)=\left\{X\in\mathbb{R}^2:\varphi(X)\wedge V^*=0\right\} \tag{8}$$

where $V^*=VI^{-1}$ and I is the pseudoscalar blade.

Hence the inner product space of V is

$$IPNS(V)=\left\{X\in\mathbb{R}^2:ax+by-\frac{c}{2}x^2-d=0\right\} \tag{9}$$

By laying $A=a,B=b,C=-\frac{c}{2}$ and $D=-d$ in (9), we obtain the equation

$$Ax+By+Cx^2+D=0 \tag{10}$$

which is the equation of a parabola.

We note that $IPNS(V)=\left\{X\in\mathbb{R}^2:\varphi(X)\cdot V=\varphi(X)\wedge V^*=0\right\}$.

2.2. Conformal Hyperquaternion Algebra $\mathbb{H}^{\otimes 4}$

The conformal hyperquaternion algebra $\mathbb{H}^{\otimes 4}$ has been described in [11], its multivector structure and a representation of conic sections in the hyperquaternionic context have be developed in details.

We recall the eight generators of $\mathbb{H}^{\otimes 4}$

$$e_1=kI,e_2=kJ,e_{\infty 1}=\frac{1}{\sqrt{2}}(kKnL+kKm),e_{\infty 2}=\frac{1}{\sqrt{2}}(kKnM+kKl),$$

$$e_{\infty 3}=\frac{1}{\sqrt{2}}(kKnN+j),e_{01}=\frac{1}{\sqrt{2}}(-kKnL+kKm),e_{02}=\frac{1}{\sqrt{2}}(-kKnM+kKl),$$

and

$$e_{03}=\frac{1}{\sqrt{2}}(-kKnN+j). \tag{11}$$

satisfy the following conditions $e_{\infty i}^2=0,e_{0i}^2=0$ and $e_{\infty i}e_{0i}=-1$ for any $i\in\{1,2,3\}$.

A point in conic conformal hyperquaternion algebra is expressed as follows

$$\begin{aligned}\phi(X) &= xkI + ykJ + \frac{1}{2\sqrt{2}}x^2(kKnL + kKm) + \frac{1}{2\sqrt{2}}y^2(kKnM + kKl) \\ &\quad + \frac{1}{\sqrt{2}}xy(kKnN + j) + \frac{1}{\sqrt{2}}(-kKnL + kKm) + \frac{1}{\sqrt{2}}(-kKnM + kKl)\end{aligned} \tag{12}$$

where $X = xkI + ykJ \in \mathbb{R}^2$ and $\phi : \mathbb{R}^2 \to \mathbb{R}^{5,3} \subset \mathbb{H}^{\otimes 4}$.

The inner product null space of a vector $V \in \mathbb{H}^{\otimes 4}$ is the set of points $X = xkI + ykJ \in \mathbb{R}^2$ such that

$$Ax^2 + Bx^2 + Cxy + Dx + Ey + F = 0 \tag{13}$$

which is the equation of a conic section.

2.3. Conformal Hyperquaternion Algebra $\mathbb{H}^{\otimes 8}$

2.3.1. Isomorphism $Cl_{9,7} \simeq \mathbb{H}^{\otimes 8}$

Consider $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_a, e_b, e_c, e_d, e_e, e_f, e_g)$ be an orthonormal basis of $\mathbb{R}^{9,7}$ and q be a quadratic form defined by $q(x) = q(x^i e_i) = \sum_{i=1}^{9}(x^i) - (x^a)^2 - (x^b)^2 - (x^c)^2 - (x^d)^2 - (x^e)^2 - (x^f)^2 - (x^g)^2$, for any $x \in \mathbb{R}^{9,7}$.

The Clifford algebra $Cl_{9,7}$ is spanned by theses basis vectors fulfilling the following relations:

$$e_i^2 = 1\left(i \in \{1,2,3,4,5,6,7,8,9\}\right), e_i^2 = -1\left(i \in \{a,b,c,d,e,f,g\}\right), \tag{14}$$

and

$$e_i e_j + e_j e_i = 0, (i \neq j). \tag{15}$$

Since the isomorphism between the algebras $\mathbb{H}^{\otimes 2m}$ and $Cl_{2m+1,2m-1}$ described in [5] (p. 6), one obtains $\mathbb{H}^{\otimes 8} \simeq Cl_{9,7}$.

From the eight quaternionic systems (i, j, k), (I, J, K), (l, m, n), (L, M, N), (p, q, r), (P, Q, R), (s, t, u) and (S, T, U), we set out a basis of the hyperquaternion algbebra $\mathbb{H}^{\otimes 8}$ as follows

$$\begin{aligned}&(1, i, j, k) \otimes (1, I, J, K) \otimes (1, l, m, n) \otimes (1, L, M, N) \otimes (1, p, q, r) \\ &\otimes (1, P, Q, R) \otimes (1, s, t, u) \otimes (1, S, T, U).\end{aligned} \tag{16}$$

We opt for the multivector structure of the hyperquaternion algbebra $\mathbb{H}^{\otimes 8}$ obtained for the following fixing sixteen generators

$$\begin{aligned}&e_1 = k \otimes I \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 = kI, e_2 = k \otimes J \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 = kJ, \\ &e_3 = kKnNrRuS, e_4 = kKnNrRuT, e_5 = kKnNrRuU, e_6 = kKnNrP, \\ &e_7 = kKnNrQ, e_8 = kKnL, e_9 = kKnM, e_a = kKnNrRs, e_b = kKnNrRt, \\ &e_c = kKnNp, e_d = kKnNq, e_e = kKl, e_f = kKm, e_g = j.\end{aligned} \tag{17}$$

The basis $(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_a, e_b, e_c, e_d, e_e, e_f, e_g)$ of the vector space $\mathbb{R}^{9,7}$ allows to build a new basis as follows:

$(e_1 = kI, e_2 = kJ)$ a basis of the Euclidean space $\mathbb{R}^2$,

$$e_{\infty 1} = \frac{1}{\sqrt{2}}(kKnNrRuS + kKnNrRs), e_{\infty 2} = \frac{1}{\sqrt{2}}(kKnNrRuT + kKnNrRt),$$

$$e_{\infty 3}=\frac{1}{\sqrt{2}}\left(kKnNrRuU+kKnNp\right),e_{\infty 4}=\frac{1}{\sqrt{2}}\left(kKnNrP+kKnNq\right),$$

$$e_{\infty 5}=\frac{1}{\sqrt{2}}\left(kKnNrQ+kKl\right),e_{\infty 6}=\frac{1}{\sqrt{2}}\left(kKnL+kKm\right),e_{\infty 7}=\frac{1}{\sqrt{2}}\left(kKnM+j\right) \quad (18)$$

and

$$e_{01}=\frac{1}{\sqrt{2}}\left(-kKnNrRuS+kKnNrRs\right),e_{02}=\frac{1}{\sqrt{2}}\left(-kKnNrRuT+kKnNrRt\right),$$

$$e_{03}=\frac{1}{\sqrt{2}}\left(-kKnNrRuU+kKnNp\right),e_{04}=\frac{1}{\sqrt{2}}\left(-kKnNrP+kKnNq\right),$$

$$e_{05}=\frac{1}{\sqrt{2}}\left(-kKnNrQ+kKl\right),e_{06}=\frac{1}{\sqrt{2}}\left(-kKnL+kKm\right),e_{07}=\frac{1}{\sqrt{2}}\left(kKnM+j\right) \quad (19)$$

such that $e_{\infty i}^2=0\left(1\le i\le 9\right)$ and $e_{0i}^2=0,\left(i\in\left\{a,b,c,d,e,f,g\right\}\right)$.

2.3.2. Cubic Curves in Conformal Hyperquaternion Algebra $\mathbb{H}^{\otimes 8}$

Before we describe the cubic curves in 2D using the conformal hyperquaternion algebra $\mathbb{H}^{\otimes 8}$, we firstly define a point in cubic curves conformal hyperquaternion algebra.

Let $\varphi:\mathbb{R}^2\to\mathbb{R}^{9,7}\subset\mathbb{H}^{\otimes 8}$ be the conformal embedding defined as follows

$$\begin{aligned}\varphi\left(X\right)=&\,xkI+ykJ+\frac{1}{2\sqrt{2}}x^2\left(kKnNrRuS+kKnNrRs\right)\\&+\frac{1}{2\sqrt{2}}y^2\left(kKnNrRuT+kKnNrRt\right)+\frac{1}{\sqrt{2}}xy\left(kKnNrRuU+kKnNp\right)\\&+\frac{1}{\sqrt{2}}x^3\left(kKnNrP+kKnNq\right)+\frac{1}{\sqrt{2}}x^2y\left(kKnNrQ+kKl\right)\\&+\frac{1}{\sqrt{2}}xy^2\left(kKnNrRuU+kKnNp\right)+\frac{1}{\sqrt{2}}y^3\left(KkNl+KkM\right)\\&+\frac{1}{\sqrt{2}}\left(-kKnNrRuS+kKnNrRs\right)+\frac{1}{\sqrt{2}}\left(-kKnNrRuT+kKnNrRt\right)\end{aligned} \quad (20)$$

where $X=xkI+ykJ\in\mathbb{R}^2$.

A point in the conformal hyperquaternion algebra $\mathbb{H}^{\otimes 8}$ is the vector $\varphi\left(X\right)$ defined in (18).

The inner product null space of a vector V is

$$\begin{aligned}IPNS\left(V\right)=\{&X=xkI+ykJ\in\mathbb{R}^2:Ax^3+By^3+Cx^2y+Dxy^2+Ex^2\\&+Fy^2+Gxy+Hx+Ly+My+N=0\}\end{aligned}$$

and the equation.

$$Ax^3+By^3+Cx^2y+Dxy^2+Ex^2+Fy^2+Gxy+Hx+Ly+My+N=0 \quad (21)$$

represents a 2D cubic curves in $\mathbb{H}^{\otimes 8}$.

3. Plane Curves of Higher Order in Conformal Hyperquaternion Algebra $\mathbb{H}^{\otimes 2m}$

In this section, we relate the integer number $m\left(m>1\right)$ and the order d of a 2D curves will be outlined in the conformal hyperquaternion algebra $\mathbb{H}^{\otimes 2m}$.

It is well known that:

1) The 2D curve passes through three points $\varphi(X_1), \varphi(X_2)$ and $\varphi(X_3)$ is a parabola (order 2) in the conformal tetraquartenion algebra $\mathbb{H}^{\otimes 2}$ where $X_i = x_i kI + y_i kJ, 1 \le i \le 3$. The primal form of this parabola is the 3-blade ${}_{i=1}^{3}\varphi(X_i)$.

2) The 2D curve passes through five points $\varphi(X_1), \varphi(X_2), \varphi(X_3), \varphi(X_4)$ and $\varphi(X_5)$ is a conic (order 2) in the conformal hyperquartenion algebra $\mathbb{H}^{\otimes 4}$ where $X_i = x_i kI + y_i kJ, 1 \le i \le 5$. The 5-blade ${}_{i=1}^{5}\varphi(X_i)$ primal form of this conic [7].

3) The primal form of the 2D cubic curve (order 3) passes through nine $\varphi(X_1), \varphi(X_2), \varphi(X_3), \varphi(X_4), \varphi(X_5), \varphi(X_6), \varphi(X_7), \varphi(X_8)$ and $\varphi(X_9)$ is the 9-blade ${}_{i=1}^{9}\varphi(X_i)$ and this 2D cubic curve is in the conformal hyperquartenion algebra $\mathbb{H}^{\otimes 8}$ where $X_i = x_i kI + y_i kJ, 1 \le i \le 9$, see [10].

It is obvious that the order d of 2D curve in the conformal huyerquaternion algebra $\mathbb{H}^{\otimes 2m}$ can be expressed as follows $d = 2 + 4k$ or $d = 3 + 4k$ with $k \in \mathbb{N}$.

Note that:

For $d = 2$, the five non constant terms of a quadratic polynomial are the terms in x, y, x^2, xy and y^2. The sum of two terms of order 1 and three terms of order 2 is $5 = 2 + 3$.

And for $d = 3$, the nine non constant terms of a cubic polynomial are the terms in $x, y, x^2, xy, y^2, x^3, x^2y, xy^2$ and y^3. This number corresponds to the sum of two terms of order 1, three terms of order 2 and four terms of order 4 *i.e.* $9 = 2 + 3 + 4$.

According to the above two cases $d = 2$ and $d = 3$, we see that for any order d the number of the non constant terms of a d-polynomial is

$$2 + 3 + 4 + \cdots + d + (d+1) = \sum_{2}^{d+1} i = \frac{2 + (d+1)}{2} d. \tag{22}$$

Proposition 3.1. *Let* $\mathbb{H}$ *be the quaternion algebra, a curve in a plane Euclidean space* $\mathbb{R}^2$ *of order* $d = 2 + 4k\,(k \in \mathbb{N})$ *is in the conformal hyerquaternion algebra* $\mathbb{H}^{\otimes 8k^2 + 14k + 4}$.

Proof. Firstly, we recall the algebra isomorphism $\mathbb{H}^{\otimes 2m} \simeq Cl_{2m+1,2m-1}$ $(m > 1)$. As the order of the plane curve is $d = 2 + 4k$ with $k \in \mathbb{N}$, it is easy to show that the number of the non constant terms is

$$\frac{2 + (d+1)}{2} d = \frac{2 + (2 + 4k + 1)}{2}(2 + 4k) = 8k^2 + 14k + 5 \tag{23}$$

and is equal to $2m + 1$.

It follows that $2m = 8k^2 + 14k + 4$ hence the conformal hyerquaternion algebra in concerned is $\mathbb{H}^{\otimes 8k^2 + 14k + 4}$. ■

Proposition 3.2. *Let* $\mathbb{H}$ *be the quaternion algebra, a curve in a plane Euclidean space* $\mathbb{R}^2$ *of order* $d = 3 + 4k\,(k \in \mathbb{N})$ *is in the conformal hyerquaternion algebra* $\mathbb{H}^{\otimes 8k^2 + 18k + 8}$.

Proof. By hypothesis, the order of a 2D curve is $d = 3 + 4k$ with $k \in \mathbb{N}$. It follows that the number of the non constant terms is $\frac{2+(d+1)}{2}d=\frac{2+(3+4k+1)}{2}(2+4k)=8k^2+18k+9=2m+1$. Hence the 2D curve lives in the conformal hyerquaternion algebra $\mathbb{H}^{\otimes 8k^2+18k+8}$. ■

To summarize the result

$$\begin{vmatrix} d:\text{order} & \text{points}:2m+1 & \text{algebra}:\mathbb{H}^{\otimes 2m} \\ 2+4k & 8k^2+14k+5 & \mathbb{H}^{\otimes 8k^2+14k+4} \\ 3+4k & 8k^2+18k+9 & \mathbb{H}^{\otimes 8k^2+18k+8} \end{vmatrix}$$

we present a few 2D curves in conformal hyperquaternion $\mathbb{H}^{\otimes 2m}$ in **Table 1** and **Table 2**.

Examples: 1) The 2D curve through 119 points is a curve of order 14 in the conformal hyperquaternion algebra $\mathbb{H}^{\otimes 118}$.

2) The 2D curve through 35 points is a curve of order 7 in the conformal hyperquaternion algebra $\mathbb{H}^{\otimes 34}$.

4. Conclusions

In this paper, we derive the conformal hyperquaternion algebras by using classical techniques of conformal geometric algebras (conformal Clifford algebras). After the construction of the conformal hyperquaternion algebras $\mathbb{H}^{\otimes 2}$, $\mathbb{H}^{\otimes 4}$ and $\mathbb{H}^{\otimes 8}$ as well as the representation of plane curves in these algebras, we

Table 1. 2D curves of order $d=2+4k\ (k\in\mathbb{N})$ through $2m+1$ points.

k	d: order	points: $2m+1$	algebra: $\mathbb{H}^{\otimes 2m}$
0	2	5	$\mathbb{H}^{\otimes 4}$
1	6	27	$\mathbb{H}^{\otimes 26}$
2	10	65	$\mathbb{H}^{\otimes 64}$
3	14	119	$\mathbb{H}^{\otimes 118}$
4	18	189	$\mathbb{H}^{\otimes 188}$
5	22	275	$\mathbb{H}^{\otimes 274}$
⋮	⋮	⋮	⋮

Table 2. 2D curves of order $d=3+4k\ (k\in\mathbb{N})$ through $2m+1$ points.

k	d: order	points: $2m+1$	algebra: $\mathbb{H}^{\otimes 2m}$
0	3	9	$\mathbb{H}^{\otimes 8}$
1	7	35	$\mathbb{H}^{\otimes 34}$
2	11	77	$\mathbb{H}^{\otimes 76}$
3	15	135	$\mathbb{H}^{\otimes 134}$
4	19	209	$\mathbb{H}^{\otimes 208}$
5	23	299	$\mathbb{H}^{\otimes 298}$
⋮	⋮	⋮	⋮

provide a generalization of plane curves in $\mathbb{H}^{\otimes 2m}, m \geq 5$.

The connection between the Clifford algebra $Cl_{2m+1,2m-1}$ and the hyperquaternion algebra $\mathbb{H}^{\otimes 2m}$ highlights an important relation regarding the order of 2D curves through k points in $\mathbb{H}^{\otimes 2m}$.

In our paper in preparation, we especially investigate on the study of 3D curves through k points in conformal hyperquaternion algebras $\mathbb{C} \otimes \mathbb{H}^{\otimes(2m-1)}$ and the analogous of nD curves in conformal hyperquaternion algebras $\mathbb{H}^{\otimes(2m-1)}$ and $\mathbb{C} \otimes \mathbb{H}^{\otimes(2m-2)}$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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