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Conservative Vector Fields and the Intersect Rule

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Abstract

This paper covers the concept of a conservative vector field, and its application in vector physics and Newtonian mechanics. Conservative vector fields are defined as the gradient of a scalar-valued potential function. Gradient fields are irrotational, as in the curl in all conservative vector fields is zero, by Clairaut's Theorem. Additionally, line integrals in conservative vector fields are path-independent, and line integrals over closed paths are always equal to zero, properties proved by the Gradient Theorem of multivariable calculus. Gradient fields represent conservative forces, and the associated potential function is analogous to potential energy associated with said conservative forces. The Intersect Rule provides a new, unique shortcut for determining if a vector field is conservative and deriving potential functions, by treating the indefinite integral as a set of infinitely many functions which satisfy the integral.

Keywords

Vector Physics, Vector Calculus, Multivariable Calculus, Gradient Fields, Vector Fields, Conservative Vector Fields, Newtonian Mechanics

1. Introduction to Conservative Vector Fields

Conservative vector fields, also referred to as gradient fields, are defined as vector fields which represent the gradient of a particular multivariable, scalar-valued function. The function for which a conservative vector field is the gradient is referred to as the potential function. The notion of a potential function roots from the representation of conservative forces with gradient fields, and the corresponding potential function as the associated potential energy of said conservative force (see Section IV: Potential Energy and Potential Functions). To formally define conservative vector fields:

$$\text{Let } \mathbf{F} \text{ be a vector field in } \mathbb{R}^n$$
$$\mathbf{F} \text{ is conservative } \Leftrightarrow \exists f \mid \mathbf{F} = \nabla f$$

To dissect the formal definition, **F** is defined as a vector field in nth-dimensional real space. In order to classify **F** as conservative, there must exist some scalar-valued potential function *f*, such that **F** is the gradient field of *f*. The gradient is defined as a vector-valued function that contains all of the first partial derivatives of a scalar-valued function *f*. Thus, conservative vector field **F** is a vector-valued function, containing the first partial derivatives of *f* as each corresponding component.

Conservative vector fields are utilized in vector physics to represent particular conservative forces, as each vector in the field portrays said force acting upon a particle in the space. Vectors in gradient fields are oftentimes employed to illustrate fluid movement in a conservative force field, and the manners in which said conservative force interacts with the fluid particles by enacting work. Moreover, gradient fields in vector physics provide a correlation between a conservative force and its associated potential energy, as discussed in Section IV.

This paper covers theoretical aspects of conservative vector fields, particularly, properties of gradient fields and the derivation of potential functions through the Intersect Rule. Additionally, this paper discusses applications of conservative vector fields in the context of Newtonian mechanics and vector physics in real spaces.

2. Properties of Conservative Vector Fields

By the formal definition of conservative vector fields, various properties of gradient fields arise. To prove the below properties, it is crucial to recall Clairaut's Theorem of mixed partial derivatives, which states that: "If the second partial derivatives of a function are continuous, then the order of differentiation is immaterial," (Clairaut 1740). To formally convey this Theorem, define scalar-valued function *f*:

$$f : \mathbb{R}^2 \to \mathbb{R}$$

Clairaut's Theorem states that, if the second, mixed partial derivatives of *f* exist and are continuous, this implies that:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \tag{1}$$

To prove Clairaut's Theorem, define the first partial derivatives of *f* at point (x_0, y_0) as a limit:

$$f_x(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

$$f_y(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

By the limit definition of partial derivatives, derive the mixed partial derivatives of *f* at (x_0, y_0) as limits, in terms of *f*'s first partial derivatives:

$$f_{xy}(x_0,y_0)=\lim_{y\to y_0}\left(\frac{f_x(x_0,y)-f_x(x_0,y_0)}{y-y_0}\right)$$

$$f_{yx}(x_0,y_0)=\lim_{x\to x_0}\left(\frac{f_y(x,y_0)-f_y(x_0,y_0)}{x-x_0}\right)$$

Express the first partial derivatives as limits:

$$f_{xy}(x_0,y_0)=\lim_{y\to y_0}\left(\frac{\lim\limits_{x\to x_0}\dfrac{f(x,y)-f(x_0,y)}{x-x_0}-\lim\limits_{x\to x_0}\dfrac{f(x,y_0)-f(x_0,y_0)}{x-x_0}}{y-y_0}\right)$$

$$f_{yx}(x_0,y_0)=\lim_{x\to x_0}\left(\frac{\lim\limits_{y\to y_0}\dfrac{f(x,y)-f(x,y_0)}{y-y_0}-\lim\limits_{y\to y_0}\dfrac{f(x_0,y)-f(x_0,y_0)}{y-y_0}}{x-x_0}\right)$$

Assuming the partial derivatives are continuous and the limits exist, express the limits in the numerators as a single limit:

$$f_{xy}(x_0,y_0)=\lim_{y\to y_0}\left(\frac{\lim\limits_{x\to x_0}\dfrac{f(x,y)-f(x_0,y)-f(x,y_0)+f(x_0,y_0)}{x-x_0}}{y-y_0}\right)$$

$$f_{yx}(x_0,y_0)=\lim_{x\to x_0}\left(\frac{\lim\limits_{y\to y_0}\dfrac{f(x,y)-f(x,y_0)-f(x_0,y)+f(x_0,y_0)}{y-y_0}}{x-x_0}\right)$$

In the first limit, multiply the numerator and denominator by $x-x_0$. In the second limit, multiply the numerator and denominator by $y-y_0$:

$$f_{xy}(x_0,y_0)=\lim_{y\to y_0}\lim_{x\to x_0}\left(\frac{f(x,y)-f(x_0,y)-f(x,y_0)+f(x_0,y_0)}{(y-y_0)(x-x_0)}\right)$$

$$f_{yx}(x_0,y_0)=\lim_{x\to x_0}\lim_{y\to y_0}\left(\frac{f(x,y)-f(x_0,y)-f(x,y_0)+f(x_0,y_0)}{(y-y_0)(x-x_0)}\right)$$

The above limits approach the same value and contain equivalent expressions, thus, the order of the limits may be changed:

$$\begin{aligned}f_{xy}(x_0,y_0)&=\lim_{y\to y_0}\lim_{x\to x_0}\left(\frac{f(x,y)-f(x_0,y)-f(x,y_0)+f(x_0,y_0)}{(y-y_0)(x-x_0)}\right)\\ f_{yx}(x_0,y_0)&=\lim_{y\to y_0}\lim_{x\to x_0}\left(\frac{f(x,y)-f(x_0,y)-f(x,y_0)+f(x_0,y_0)}{(y-y_0)(x-x_0)}\right)\end{aligned}\tag{2}$$

Therefore, it is evident that:

$$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)\tag{3}$$

Clairaut's Theorem holds true for all functions with continuous second derivatives, and is crucial for the proof of certain properties of conservative vector fields, particularly, the irrotational property, as seen below.

2.1. The Path Independence Property

The first notable property is entitled the path independence property. This property essentially states that line integrals over conservative fields that start and end at the same point are always equivalent, regardless of the path taken.

To visualize this property, see **Figure 1**, which portrays conservative vector field **F** in $\mathbb{R}^2$. Paths C_1 and C_2 both start at point A and end at point B.

Constructing the line integrals over both paths:

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}}$$

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}}$$

To interpret the definite integrals above, consider a fluid particle flowing through this conservative field, along path C_1, whilst a second particle flows along path C_2 [1]. The above integrals model the total work done by this conservative force on each molecule, where **F** represents the force vector, and d**r** represents a displacement vector with an infinitely miniscule magnitude. Thus, integrating along the paths yields the total work done on each particle flowing through conservative vector field **F**.

By the formal definition of conservative vector fields, **F** must be the gradient of some scalar valued potential function, f. Thus, the line integrals can be rewritten as:

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_A^B \nabla f \cdot \mathrm{d}\vec{\mathbf{r}}$$

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_A^B \nabla f \cdot \mathrm{d}\vec{\mathbf{r}}$$

Considering that the integrals are expressed as gradients of scalar-valued functions, recall the gradient theorem to evaluate the definite integrals [2]:

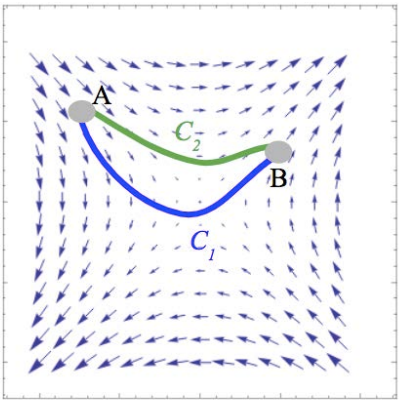

Figure 1. Conservative vector field F with paths C1 and C2 to convey the path independence property.

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_A^B \nabla f \cdot \mathrm{d}\vec{\mathbf{r}} = f(B) - f(A)$$

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_A^B \nabla f \cdot \mathrm{d}\vec{\mathbf{r}} = f(B) - f(A)$$

Thus:

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}} = f(B) - f(A) \tag{4}$$

Therefore, if **F** is defined as a conservative vector field, all line integrals are dependent solely on the starting and ending points, rather than the path taken.

2.2. The Closed Line Integral Property

The second notable property of gradient fields is that all line integrals over simply closed loops are always equal to zero, regardless of the orientation of the loop.

Reverting to the prior scenario, see Figure 2, which portrays the previous gradient field, however, paths C_1 and C_2 are simply closed at point A. It is crucial to note that path C_1 is positively oriented whilst path C_2 is negatively oriented.

When constructing the line integrals, it is pivotal to recall that for all line integrals over the same closed loop, negative orientation flips the sign of the line integral. Additionally, recall the path independence property:

$$\oint_{C_1} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}} = -\oint_{C_2} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}}$$

By the gradient theorem:

$$\oint_{C_1} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}} = -\oint_{C_2} \mathbf{F} \cdot \mathrm{d}\vec{\mathbf{r}} \Rightarrow \oint_A^A \nabla f \cdot \mathrm{d}\vec{\mathbf{r}} = -\oint_A^A \nabla f \cdot \mathrm{d}\vec{\mathbf{r}}$$

Evaluating the integrals:

$$f(A) - f(A) = -\big(f(A) - f(A)\big) = 0 \tag{5}$$

Therefore, if **F** is a conservative field, line integrals over simply closed loops are always equal to zero, regardless of the orientation.

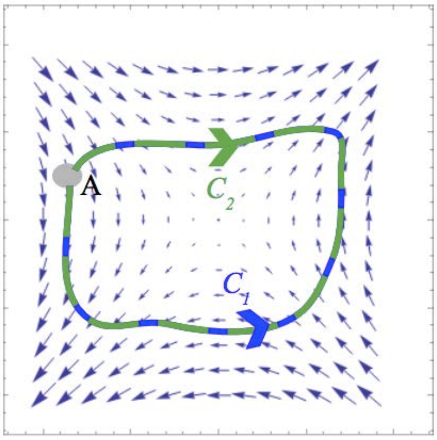

Figure 2. Vector field F with closed path C1 and negatively oriented, closed path C2 to illustrate closed line integral property.

2.3. The Irrotational Property

The third and final significant property of gradient fields is the irrotational property, which states that the curl of all conservative vector fields is always zero. To prove this property, consider conservative vector field F in $\mathbb{R}^2$:

$$\mathbf{F}(x,y)=P(x,y)\hat{\mathbf{i}}+Q(x,y)\hat{\mathbf{j}}$$
$$\mathbf{F}=\nabla f$$

F can be redefined in terms of *f*, as **F** is the gradient of scalar-valued function *f*:

$$\mathbf{F}(x,y)=\frac{\partial f}{\partial x}\hat{\mathbf{i}}+\frac{\partial f}{\partial y}\hat{\mathbf{j}}$$

Calculating the two-dimensional curl of **F**:

$$\text{Curl}(\mathbf{F})=\nabla\times\nabla f=\det\begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial f}{\partial x}\\ \frac{\partial}{\partial y} & \frac{\partial f}{\partial y}\end{bmatrix}=\frac{\partial^2 f}{\partial y\partial x}-\frac{\partial^2 f}{\partial x\partial y}$$

Recall Clairaut's Theorem, assuming that the second partial derivatives of *f* are continuous:

$$\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial x\partial y}=0$$
$$\Downarrow \tag{6}$$
$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=0\Rightarrow\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$$

Thus, the curl of a conservative vector field in two-dimensions is always zero. In three dimensions, however, it is crucial to clarify that the three-dimensional curl of a conservative vector field produces the zero vector, as opposed to the two-dimensional curl yielding the scalar zero. Therefore, in three-dimensions:

$$\text{Curl}(\mathbf{F})=\vec{\mathbf{0}} \tag{7}$$

To prove this property, redefine vector field **F** as a three-dimensional gradient field, expressed as follows:

$$\mathbf{F}(x,y)=\frac{\partial f}{\partial x}\hat{\mathbf{i}}+\frac{\partial f}{\partial y}\hat{\mathbf{j}}+\frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

where, *f* is a scalar-valued function with a three-dimensional input space. Calculating the curl of **F**:

$$\text{Curl}(\mathbf{F})=\nabla\times\nabla f=\det\begin{bmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\end{bmatrix}$$
$$=\left(f_{yz}-f_{zy}\right)\hat{\mathbf{i}}-\left(f_{xz}-f_{zx}\right)\hat{\mathbf{j}}+\left(f_{xy}-f_{yx}\right)\hat{\mathbf{k}}$$

By Clairaut's Theorem:

$$\left(f_{yz}-f_{yz}\right)\hat{\mathbf{i}}-\left(f_{xz}-f_{xz}\right)\hat{\mathbf{j}}+\left(f_{xy}-f_{xy}\right)\hat{\mathbf{k}}=0\hat{\mathbf{i}}-0\hat{\mathbf{j}}+0\hat{\mathbf{k}}=\vec{\mathbf{0}} \tag{8}$$

The curl operator primarily exists in two and three dimensions, although, this property may be generalized for all conservative vector fields in nth-dimensional real space. The above properties are directly analogous to the properties of conservative forces in Newtonian mechanics, as demonstrated in the following sections [3].

3. Conservative Vector Fields as Conservative Force

As briefly mentioned in section II, the line integral within a conservative vector field may be interpreted as the total work done by a conservative force, represented by vector field **F**, on a particle flowing through the field [4].

Vector fields depict force fields, in which each point is associated with a vector that represents a particular force acting upon a particle at that point in space. Consider **Figure 3**, which portrays an arbitrary three-dimensional vector field, $\mathbf{F}_1$. At each point in the input space, a corresponding three-dimensional vector is output. If $\mathbf{F}_1$ represents a particular force field, each vector conveys the force acting upon an object in the force field. This interpretation holds true for all vector fields.

With regards to conservative vector fields, each vector in the output space describes a particular conservative force acting upon an object in the force field. Thus, conservative vector fields are utilized to represent conservative forces.

Concerning line integrals within conservative vector fields, the path independence property holds true with the interpretation of a gradient field as a conservative force field. Conservative forces are defined as forces for which the work done on an object by that force is independent of the path taken, and exclusively depends on the start and end points of the path. This is directly analogous to the path independence property of gradient fields, as line integrals in gradient fields are equivalent regardless of the path taken, and solely based on the start and end points.

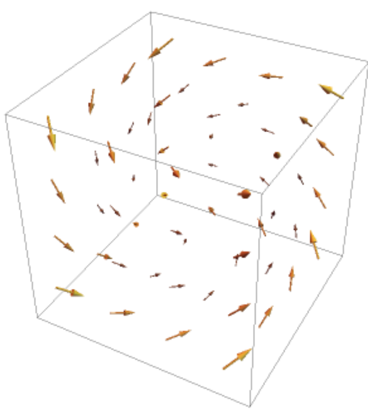

Figure 3. Ordinary three dimensional vector field to represent an arbitrary force interacting with a particle in space.

Physical Application of Conservative Vector Fields

To contextualize this notion, redefine vector field $\mathbf{F}_1$ as a gravitational force field. Gravitational force is conservative, as the work done on an object in motion within a gravitational force field solely depends on the start and end points of the path, rather than the path taken. Consider arbitrary paths Ω_1 and Ω_2, within vector field $\mathbf{F}_1$, both of which start at point i_0 and end at point f_0. The formula for work is given by:

$$W = \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}$$

where, $\mathbf{F}_1$ is the force vector, and $\mathbf{d}$ is the displacement vector. Suppose paths Ω_1 and Ω_2 are non-linear. To measure the total work done by gravitational force on an arbitrary fluid particle flowing through field $\mathbf{F}_1$ along paths Ω_1 and Ω_2, construct the below line integrals:

$$W_{\Omega_1} = \int_{\Omega_1} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s$$
$$W_{\Omega_2} = \int_{\Omega_2} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s$$

where $\mathbf{d}s$ is a displacement vector of infinitely miniscule magnitude. Integrating along the paths essentially sums up the dot products of all miniscule displacement vectors and the force vector, which yields the total work done by gravity on an arbitrary particle within real space.

Considering gravitational force is conservative, the above integrals may be expressed in terms of the start and end points of the paths taken:

$$W_{\Omega_1} = \int_{i_0}^{f_0} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s$$
$$W_{\Omega_2} = \int_{i_0}^{f_0} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s$$

This conveys that the total work done by gravity is path-independent, by the path independence property of conservative vector fields, considering gravity is a conservative force.

Additionally, consider the sum of the work done by gravity on a particle moving along path Ω_1 and the reversed orientation of path Ω_2:

$$W = \int_{\Omega_1} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s + \int_{\tilde{\Omega}_2} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s$$

where $\tilde{\Omega}_2$ is the reversed orientation of path Ω_2. Express the line integrals in terms of the start and end points of the paths:

$$W = \int_{i_0}^{f_0} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s + \int_{f_0}^{i_0} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s$$

The above line integrals form a closed loop, as the first line integral's upper bound is equivalent to the lower bound of the second line integral. Thus, the work may be expressed as:

$$W = \oint_{\Omega} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s \mid \Omega = \Omega_1 \cup \tilde{\Omega}_2$$

Recall that gravity is a conservative force, which implies that the work done by gravity on the arbitrary particle is dependent exclusively on the start and end points. However, considering path Ω is a closed loop, this implies that the start and end points are equivalent. Thus, the work simplifies to:

$$W = \oint_{i_0}^{i_0} \vec{\mathbf{F}}_1 \cdot \vec{\mathbf{d}}s = 0 \tag{9}$$

This outcome is directly analogous to the closed line integral property of gradient fields, considering the formation of a closed loop due to the union of the paths. Moreover, this outcome is also comparable to the reversible property of conservative forces, as the work done by a conservative force in a closed system is reversible without disturbance. The work enacted on a particle by a conservative force in a closed system is always reversible. Put simply, the closed system and its surrounding environment may return to its initial state by reversing the path taken. In the above example, by reversing path Ω_2, the system reverted to its zero position, without disturbing or altering the environment, due to the conservative force's reversible nature.

Hence, conservative forces possess the path independence property and the closed line integral property of conservative vector fields. Thus, if a force is conservative, the associated force vector field must also be conservative.

Conservative vector fields are crucial in the context of vector physics and fluid mechanics for engineering, as gradient fields portray the interaction of fluid particles with certain atmospheric conservative forces. In the above scenario, the gradient field depicts the interaction between fluid particles in real space and the conservative force of gravity, and the manner in which gravitational force enacts work upon said particles.

4. Potential Energy and Potential Functions

All conservative forces hold an associated potential energy [5]. Potential energy is defined as stored energy which exists due to an object's position with respect to the particular zero position, and possesses the potential to release in the form of kinetic energy upon a conservative force acting on the object. Essentially, when a conservative force acts upon an object, the potential energy associated with that conservative force converts to kinetic energy, as the object falls in motion.

Conservative forces acting upon an object have the tendency to shift the object into a state of lesser potential energy. By acting on the object, a conservative force releases potential energy in the form of kinetic energy on the object, as net work is done by the force. Put simply, as potential energy is released as kinetic energy, potential energy decreases, as kinetic energy consequently increases.

To model this relationship, consider what is depicted by a gradient field. Let **F** be the gradient field of scalar-valued function f. Gradient fields associate vectors with all points in the input space. The vectors within **F** point in the direction of greatest increase of function f, also referred to as the direction of steepest ascent. Essentially, each vector indicates the direction in which function f increases the greatest, at each particular point.

To contextualize this, consider **Figure 4**, which depicts the three-dimensional graph of $f(x,y) = -x^2 - y^2$. The function possesses a global maximum at point $(0,0,0)$. On the gradient field of *f*, as depicted in **Figure 5**, this translates to the zero vector, as, at this particular point, there does not exist a direction in which the function increases, as this particular point is a global maximum. Essentially, in all other directions, *f* decreases, thus, the gradient vector at this particular point is the zero vector.

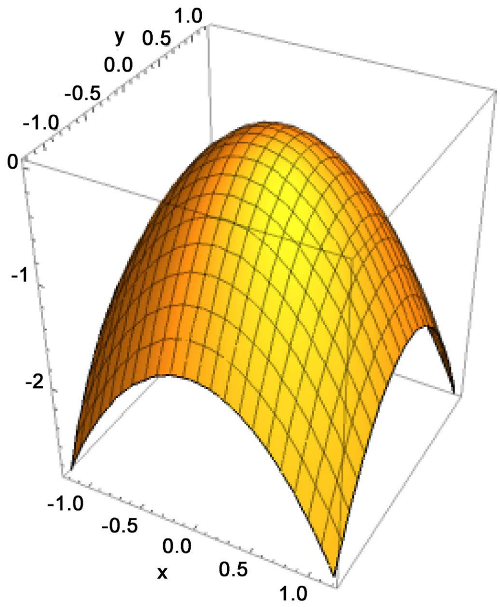

Figure 4. Elliptic paraboloid $f(x,y) = -x^2 - y^2$ to convey global maximum at the point (0, 0, 0).

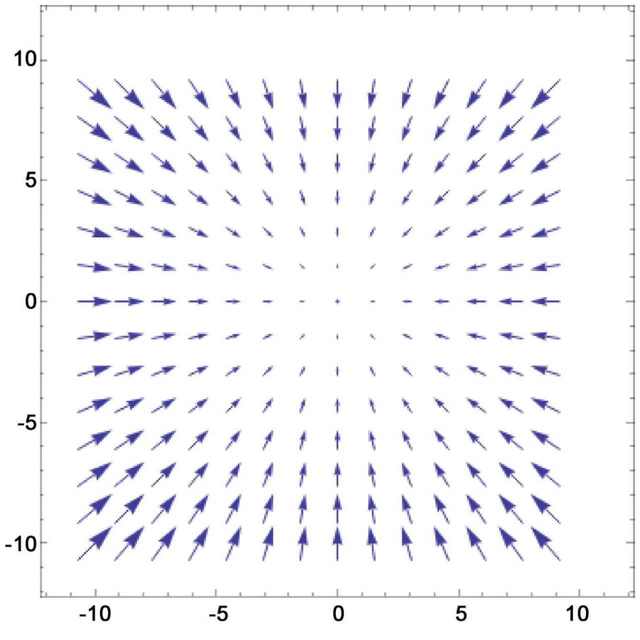

Figure 5. Gradient field of *f*, portraying the effect of a global maximum point on the corresponding gradient vector.

With respect to conservative forces and potential energy, as previously mentioned, conservative forces tend to shift objects acted upon into a state of lesser potential energy, and, as a result, greater kinetic energy. To model the relationship between a conservative force field and its associated potential energy, contemplate the relationship between a function and its gradient field. The vectors of a gradient field point in the direction of greatest increase, whilst conservative forces attempt to shift the acted upon object in the direction of greatest decrease in potential energy, to maximize the kinetic energy.

Thus, to model the relationship, consider conservative vector field **F**, which models an arbitrary conservative force. Let P_F be a scalar-valued function which represents the associated potential energy with conservative force field **F**. By this, the following formula can be derived:

$$\mathbf{F} = -\nabla P_F \tag{10}$$

The incentive behind the negative sign on the gradient of the potential energy is that the vectors in a conservative force field point in the direction of greatest decrease, at all given input points. Hence, the negative sign abides by the tendency of conservative forces to minimize potential energy, and maximize kinetic energy.

To express this relationship in a form analogous to the formal definition of gradient fields, a scalar-valued function, E_P can be constructed, such that:

$$E_P = -P_F$$

The gradient of the above functions:

$$\nabla E_P = \begin{bmatrix} \frac{\partial E_P}{\partial x_1} \\ \vdots \\ \frac{\partial E_P}{\partial x_n} \end{bmatrix}, \ \nabla P_F = \begin{bmatrix} \frac{\partial P_F}{\partial x_1} \\ \vdots \\ \frac{\partial P_F}{\partial x_n} \end{bmatrix}$$

Substitute $-P_F$ for E_P:

$$\nabla E_P = \nabla\left(-P_F\right) = \begin{bmatrix} \frac{\partial\left(-P_F\right)}{\partial x_1} \\ \vdots \\ \frac{\partial\left(-P_F\right)}{\partial x_n} \end{bmatrix} = -\begin{bmatrix} \frac{\partial P_F}{\partial x_1} \\ \vdots \\ \frac{\partial P_F}{\partial x_n} \end{bmatrix} = -\nabla P_F$$

Thus, the relationship between a conservative force field **F** and its associated potential energy function, P_F can be expressed as:

$$\mathbf{F} = \nabla E_P \tag{11}$$

where, E_P is the negative potential energy function of **F**.

By the notion of conservative force fields being the gradient of the negative associated potential energy function emerges the concept of a conservative field being the gradient of an associated potential function.

5. Deriving Potential Functions and the Intersect Rule

To restate the formal definition of a conservative vector field:

$$\text{Let } \mathbf{F} \text{ be a vector field in } \mathbb{R}^n$$
$$\mathbf{F} \text{ is conservative } \Leftrightarrow \exists f \mid \mathbf{F} = \nabla f$$

The function for which conservative field **F** is the gradient is referred to as the potential function, rooting from the notion of potential energy, as discussed above. When deriving potential functions, it is crucial to revert to the formal definition of gradient fields. As a contextualization of this concept, consider conservative vector field **F** in $\mathbb{R}^2$:

$$\mathbf{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$$

It has been established that **F** is conservative, thus, **F** may be expressed in terms of scalar-valued potential function f:

$$\mathbf{F}(x,y) = \begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

By this definition emerges a system of equations:

$$\begin{cases} P(x,y) = \frac{\partial f}{\partial x} \\ Q(x,y) = \frac{\partial f}{\partial y} \end{cases}$$

Although the following step lacks mathematical rigor, it provides a general intuition behind the derivation of potential functions. Essentially, consider multiplying both sides of the first equation by the partial-x operator, whilst also multiplying both sides of the second equation by the partial-y operator.

$$\begin{cases} P(x,y)\partial x = \partial f \\ Q(x,y)\partial y = \partial f \end{cases}$$

To solve the system, integrate the above equations:

$$\begin{cases} \int P(x,y)\mathrm{d}x = \int \mathrm{d}f \\ \int Q(x,y)\mathrm{d}y = \int \mathrm{d}f \end{cases}$$

The indefinite integrals above may be expressed as sets of infinitely many elements. Particularly, define set Λ_1 and set Λ_2 as follows:

$$\Lambda_1 = \left\{ \rho_1(x,y) \mid \rho_1(x,y) = \int P(x,y)\mathrm{d}x \right\}$$
$$\Lambda_2 = \left\{ \rho_2(x,y) \mid \rho_2(x,y) = \int Q(x,y)\mathrm{d}y \right\}$$

Essentially, the above sets consist of all scalar valued functions, ρ_1 and ρ_2, which satisfy the first and second indefinite integrals, respectively. Put simply, set Λ_1 consists of all the multivariable, scalar-valued functions that satisfy the first indefinite integral, whilst set Λ_2 contains all of the scalar-valued functions that satisfy the second indefinite integral.

The notion of the above sets roots from the existence of infinitely many functions which satisfy the corresponding indefinite integrals. As in, there exist infinitely many multivariable, scalar-valued functions that, upon differentiating with respect to a particular variable, yield the corresponding component of gradient field F. Consequently, upon integrating with respect to that same particular variable, there exist infinitely many scalar-valued functions that satisfy the indefinite integral.

Thus, the above sets consist of all the possible potential functions of conservative vector field **F**, however, to derive the true potential function of **F**, there must exist an element shared by both sets, which satisfy both indefinite integrals within the above system.

Note that this single particular element is shared by both sets, thus, in order to locate this element, first derive the intersection of the sets:

$$\Lambda_1 \cap \Lambda_2$$
$$\Downarrow$$
$$\left\{\rho_1(x,y) \mid \rho_1(x,y) = \int P(x,y)\,\mathrm{d}x\right\} \cap \left\{\rho_2(x,y) \mid \rho_2(x,y) = \int Q(x,y)\,\mathrm{d}y\right\}$$

As a result, the potential function of **F** can be expressed as the intersection of set Λ_1 and set Λ_2, considering the above sets will exclusively intersect at the element which satisfies both indefinite integrals. Thus:

$$f(x,y) = \left\{\rho_1(x,y) \mid \rho_1(x,y) = \int P(x,y)\,\mathrm{d}x\right\} \cap \left\{\rho_2(x,y) \mid \rho_2(x,y) = \int Q(x,y)\,\mathrm{d}y\right\} \tag{12}$$

Therefore, by interpreting the indefinite integral as a set, a potential function of gradient field **F** was derived.

5.1. Generalization in Nth Dimensional Real Space

This notion may be generalized for conservative vector fields in $\mathbb{R}^n$:

$$\mathbf{F}(x_1, x_2, \cdots, x_n) = \begin{bmatrix} \gamma_1(x_1, x_2, \cdots, x_n) \\ \gamma_2(x_1, x_2, \cdots, x_n) \\ \vdots \\ \gamma_n(x_1, x_2, \cdots, x_n) \end{bmatrix}$$

By the above principle, the potential function of vector field **F** is defined as:

$$f = \bigcap_{i=1}^{n} \int \gamma_i \,\mathrm{d}x_i$$

To prove this result, construct a system of equations based on the formal definition of gradient fields:

$$\begin{cases} \gamma_1 = f_{x_1} \\ \gamma_2 = f_{x_2} \\ \vdots \\ \gamma_n = f_{x_n} \end{cases}$$

Manipulate the system such that:

$$\begin{cases} \int \gamma_1 \mathrm{d}x_1 = \int \mathrm{d}f \\ \int \gamma_2 \mathrm{d}x_2 = \int \mathrm{d}f \\ \vdots \\ \int \gamma_n \mathrm{d}x_n = \int \mathrm{d}f \end{cases}$$

Express the indefinite integrals in the system as sets:

$$\Lambda_1 = \left\{ \rho_1 \left(x_1, x_2, \cdots, x_n \right) \mid \rho_1 \left(x_1, x_2, \cdots, x_n \right) = \int \gamma_1 \mathrm{d}x_1 \right\}$$
$$\Lambda_2 = \left\{ \rho_2 \left(x_1, x_2, \cdots, x_n \right) \mid \rho_2 \left(x_1, x_2, \cdots, x_n \right) = \int \gamma_2 \mathrm{d}x_2 \right\}$$
$$\vdots$$
$$\Lambda_n = \left\{ \rho_n \left(x_1, x_2, \cdots, x_n \right) \mid \rho_n \left(x_1, x_2, \cdots, x_n \right) = \int \gamma_n \mathrm{d}x_n \right\}$$

Locate the intersection of the n sets:

$$\Lambda_1 \cap \Lambda_2 \cap \cdots \cap \Lambda_n = \int \gamma_1 \mathrm{d}x_1 \cap \int \gamma_2 \mathrm{d}x_2 \cap \cdots \cap \int \gamma_n \mathrm{d}x_n$$

Assuming **F** is conservative, the intersection of the n sets yields a set with one element, which is the potential function of **F**. To, simplify the above:

$$f = \bigcap_{i=1}^{n} \Lambda_i = \bigcap_{i=1}^{n} \int \gamma_i \mathrm{d}x_i \tag{13}$$

5.2. Feasibility of the Intersect Rule

This method of deriving potential functions is entitled the Intersect Rule. The proof behind this rule lies in simple mathematical deduction, considering the intersection of the sets yields a single element that satisfies all n indefinite integrals. Essentially, the Intersect Rule derives a potential function that adheres to the components of the parent gradient field.

By this definition, another principle emerges. If the Intersect Rule yields the null set, as in, the set of indefinite integrals never intersect, this implies that the vector field is not conservative, as there does not exist a potential function for which the vector field is the gradient. To formally convey this principle:

$$\text{If } \bigcap_{i=1}^{n} \int \gamma_i \mathrm{d}x_i = \varnothing \Leftrightarrow \nexists f \mid \mathbf{F} = \nabla f \tag{14}$$

Due to this notion, the Intersect Rule may be utilized to immediately determine whether there exists a potential function for the vector field, also determining if the vector field is conservative. As in, if the intersection does not exist, this directly implies that the vector field is not conservative, as there is no function which satisfies all the indefinite integrals, simultaneously.

The Intersect Rule treats the indefinite integral as a set of infinitely many elements, each of which is scalar valued functions which satisfy the indefinite integral. This rule is directly analogous to the notion of the integral being the anti-derivative in single variable calculus. In multivariable calculus, the gradient is directly comparable to the single variable derivative, as the gradient yields all of the first partial derivatives of a scalar-valued function. Thus, the Intersect

Rule may be interpreted as the anti-gradient of a vector-valued function, as the rule intakes a vector field, and yields the function for which the vector field is the gradient, if there exists such a function.

This derivation of this rule arises from the notion of pure functions eliminated due to partial derivatives. As in, when applying the partial derivative operator with respect to a particular variable, x_0, all pure functions that do not act as coefficients on functions of x_0 are lost. Due to this, when integrating, arbitrary pure functions of other members of the input space must be included, considering said functions may have been lost due to the partial derivative operator.

This is analogous to the notion of adding an arbitrary real constant, C, when evaluating single variable indefinite integrals, as all constants may be lost due to the derivative operator.

In multivariable calculus, the partial derivative operator with respect to x_0 treats all other pure functions as constants, consequently zeroing out the non-x_0 pure functions. Therefore, the Intersect Rule acts as a method of retrieving said lost pure functions, and finding which particular combination of functions satisfy all n indefinite integrals.

By taking the intersection of the sets containing all functions which satisfy each integral, a particular function is derived, shared by all n sets. This element is, essentially, the anti-gradient of vector field $\mathbf{F}$, as it is the function for which the partial derivative with respect to each input variable is equivalent to each corresponding component of vector field $\mathbf{F}$.

This method of deriving potential functions and determining conservative vector fields proves feasible and intuitive, as it connects the integrals of each partial derivative of a scalar valued function, in a set of a single element.

6. Discussion

Conservative vector fields represent conservative force fields, and are the gradient of a particular potential function. Various properties of gradient fields are directly analogous to properties of conservative forces, particularly, the path independence property. By interpreting line integrals in a gradient field as a measure of the total work done on an object within the field, another notion emerges. Essentially, if an arbitrary force is conservative, this implies that the associated force vector field must also be conservative. The intersect rule provides not only a unique shortcut for deriving potential functions of conservative vector fields in all nth-dimensional real space, however, a method for determining if a vector field is conservative. Moreover, the Intersect Rule acts as a direct analogy to the anti-derivative nature of the single variable integral, and proves a beneficial tool in the context of vector physics and Newtonian mechanics. This innovation develops a unique interpretation of the indefinite integral, and correlates operators on sets with concepts in multivariable and vector calculus.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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