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Dynamic Analysis of a Predator-Prey Model with Holling-II Functional Response

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Abstract

A predator-prey model with linear capture term Holling-II functional response was studied by using differential equation theory. The existence and the stabilities of non-negative equilibrium points of the model were discussed. The results show that under certain limited conditions, these two groups can maintain a balanced position, which provides a theoretical reference for relevant departments to make decisions on ecological protection.

Keywords

Predator-Prey Model, Holling-II Functional Response, Equilibrium Point, Stability

1. Introduction

Predator-prey system refers to the relationship in which one species is preyed upon by another species in a community of organisms. This relationship is an important part of the biome and can affect the number, distribution and composition of species, thereby altering the stability and integrity of the environment.

The Holling-II model (also known as the Lotka-Volterra model or food chain model) is a classic dynamical model used to study predator-prey interactions. The model is based on the assumption that as the number of prey increases, so does the number of predators; however, as an increase in predators, the likelihood of prey being preyed on increases, leading to a decrease in the number of prey. In turn, the decrease in prey populations inhibits the growth of predator populations. From the assumption Holling-II model, it can be seen that the quantitative relationship between predator and prey is nonlinear. In recent years, many researchers have studied predator-prey dynamics model from different angles (see [1]-[11]). In the literature [10], the authors studied a Holling-II predatory functional response in a predator-prey model as follows

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x\left(a - x - \dfrac{y}{1+x}\right), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\left(\dfrac{bx}{1+x} - d_1 - d_2 y\right). \end{cases} \tag{1}$$

They investigated the stabilities of the equilibrium point and the existence of limit cycle. In literature [11], the authors considered a Leslie-Gower predator-prey model with time delay and linear harvesting term

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = (1 - x_1)(x_1 - m) - q x_1 x_2 + h_1 x_1, \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = S x_2 \left(1 - \dfrac{x_2(t-\tau_2)}{x_1(t-\tau_1)}\right) + h_2 x_2 \end{cases} \tag{2}$$

where $h_1 x_1$ and $h_2 x_2$ denote the linear capture terms caused by external interference, and $h_1 < 0$ and $h_2 < 0$. They studied the positive steady-state properties of the model (2) and gave the conditions for Hopf bifurcation near the positive equilibrium point. Mainly inspired by literatures [10] and [11], combining models (1) and (2), we propose the following predator-prey model with Holling-II type functional response and a linear capture term

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u\left(a - u - \dfrac{v}{1+u}\right) + h_1 u, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(\dfrac{bu}{1+u} - d_1 - d_2 v\right) + h_2 v \end{cases} \tag{3}$$

where u and v represent the densities of prey and predator populations at time t respectively, a, b, d_1 and d_2 are positive parameters, and h_1 and h_2 are negative parameters. We use differential equation theory to discuss the existence and the stabilities of non-negative equilibrium points of the model (3). The results show that under certain limited conditions, these two populations can maintain balance.

2. The Equilibria of the System Model

In view of the practical significance of the system, we only consider the case where u and v satisfy $E = \{(u,v) | u \ge 0, v \ge 0\}$. Define

$$\begin{cases} Q(u,v) := u\left(a - u - \dfrac{v}{1+u}\right) + h_1 u, \\ R(u,v) := v\left(\dfrac{bu}{1+u} - d_1 - d_2 v\right) + h_2 v. \end{cases}$$

Let

$$Q(u,v) = 0, \quad R(u,v) = 0 \tag{4}$$

Then, we have three equilibrium points:

$$P_1(0,0),\ P_2(a+h_1,0),\ P_3(u^*,v^*)$$

where u^*,v^* satisfy

$$\begin{cases} l_1: a-u^*-\dfrac{v^*}{1+u^*}+h_1=0, \\ l_2: \dfrac{bu^*}{1+u^*}-d_1-d_2v^*+h_2=0. \end{cases}$$

We easily know that $u^*>0$ and $v^*>0$ when

$$0<\frac{d_1-h_2}{b-d_1+h_2}<a+h_1.$$

3. Stabilities of the Equilibrium Points

The linear approximation at any point $P(u_0,v_0)$ of the model (3) is as follows

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}=Q'_u(u_0,v_0)\cdot(u-u_0)+Q'_v(u_0,v_0)\cdot(v-v_0), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t}=R'_u(u_0,v_0)\cdot(u-u_0)+R'_v(u_0,v_0)\cdot(v-v_0) \end{cases} \tag{5}$$

where

$$Q'_u(u_0,v_0)=a-u_0-\frac{v_0}{1+u_0}+u_0\left[-1+\frac{v_0}{(1+u_0)^2}\right]+h_1,\ Q'_v=-\frac{u_0}{1+u_0},$$

$$R'_u=\frac{v_0b}{(1+u_0)^2},\ R'_v(u_0,v_0)=\frac{bu_0}{1+u_0}-2d_2v_0-d_1+h_2.$$

3.1. Stability of Equilibrium Point $P_1(0,0)$

In this case, we have

$$Q'_u(0,0)\cdot(u-0)=(a+h_1)u,\ Q'_v(0,0)\cdot(v-0)=0,$$

$$R'_u(0,0)\cdot(u-0)=0,\ R'_v(0,0)\cdot(v-0)=(-d_1+h_2)v.$$

So the linear approximation Equation (5) is as follows:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t}=(a+h_1)u, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t}=(h_2-d_1)v. \end{cases} \tag{6}$$

The coefficient matrix of Equation (5) is $B:=\begin{pmatrix} a+h_1 & 0 \\ 0 & h_2-d_1 \end{pmatrix}$. Then the characteristicroots are $\lambda_1=a+h_1$ and $\lambda_2=h_2-d_1$. Because $\lambda_2<0$, the following conclusion is obvious.

Theorem 1. If $a+h_1>0$, then $P_1(0,0)$ is an unstable saddle point. If $a+h_1<0$, then $P_1(0,0)$ is an asymptotically stable node.

3.2. Stability of Equilibrium Point $P_2(a+h_1,0)$

In this case, we have

$$Q_u'(a+h_1,0) = -(a+h_1),\quad Q_v'(a+h_1,0) = -\frac{a+h_1}{1+a+h_1},$$

$$R_u'(a+h_1,0) = 0,\quad R_v'(a+h_1,0) = \frac{b(a+h_1)}{1+a+h_1} - d_1 + h_2.$$

So the linear approximation Equation (5) is:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = (-a-h_1)u - \dfrac{a+h_1}{1+a+h_1}\cdot v + (a+h_1)^2, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \left[\dfrac{b(a+h_1)}{1+a+h_1} - d_1 + h_2\right]v. \end{cases} \tag{7}$$

The coefficient matrix of Equation (7) is

$$B := \begin{pmatrix} -a-h_1 & -\dfrac{a+h_1}{1+a+h_1} \\ 0 & \dfrac{b(a+h_1)}{1+a+h_1} - d_1 + h_2 \end{pmatrix}.$$

The characteristic roots are

$$\lambda_1 = -a-h_1 = -(a+h_1) < 0,\quad \lambda_2 = \frac{(a+h_1)(b-d_1+h_2)-(d_1-h_2)}{1+a+h_1}.$$

Therefore, the following conclusion is obtained:

Theorem 2. If $(a+h_1)(b-d_1+h_2) > (d_1-h_2)$, then $\lambda_2 > 0$, so $P_2(a+h_1,0)$ is an unstable saddle point. If $(a+h_1)(b-d_1+h_2) < (d_1-h_2)$, then $\lambda_2 < 0$, so $P_2(a+h_1,0)$ is an asymptotically stable node.

3.3. Stability of Plane Equilibrium Point $P_3(u^*,v^*)$

From

$$v^* = (1+u^*)\cdot(a-u^*+h_1) = \frac{bu^*}{d_2(1+u^*)} - \frac{d_1-h_2}{d_2},$$

we have that

$$Q_u'(u^*,v^*) = u^*\left(\frac{a-u^*+h_1}{1+u^*} - 1\right),\quad Q_v'(u^*,v^*) = -\frac{u^*}{1+u^*},$$

$$R_u'(u^*,v^*) = \frac{b(a-u^*+h_1)}{1+u^*},\quad R_v'(u^*,v^*) = \frac{-bu^*}{1+u^*} + d_1 - h_2.$$

Therefore, the coefficient matrix of Equation (5) for $P_3(u^*,v^*)$ is

$$B := \begin{pmatrix} u^*\left(\dfrac{a-u^*+h_1}{1+u^*} - 1\right) & -\dfrac{u^*}{1+u^*} \\ \dfrac{b(a-u^*+h_1)}{1+u^*} & \dfrac{-bu^*}{1+u^*} + d_1 - h_2 \end{pmatrix}$$

and its characteristic equation is

$$\left|B-\lambda E\right|=\left[u^*\left(\frac{a-u^*+h_1}{1+u^*}-1\right)-\lambda\right]\cdot\left(\frac{-bu^*}{1+u^*}+d_1-h_2-\lambda\right)+\frac{bu^*\left(a-u^*+h_1\right)}{\left(1+u^*\right)^2}=0.$$

Let

$$a_1=\frac{a-u^*+h_1}{1+u^*},\quad b_1=\frac{bu^*}{1+u^*}, \tag{8}$$

then we have

$$\lambda^2+\left(\left(1-a_1\right)u^*+b_1-d_1+h_2\right)\lambda+u^*\left(a_1-1\right)\left(-b_1+d_1-h_2\right)+a_1b_1=0. \tag{9}$$

Let

$$p=\left(1-a_1\right)u^*+b_1-d_1+h_2,\quad q=u^*\left(a_1-1\right)\cdot\left(-b_1+d_1-h_2\right)+a_1b_1. \tag{10}$$

Then the characteristic roots of Equation (9) satisfy

$$\lambda_1+\lambda_2=-p,\quad \lambda_1\lambda_2=q.$$

From (8) and (10), we have

$$\begin{aligned}p&=\left(1-a_1\right)u^*+b_1-d_1+h_2=\left(1-\frac{a-u^*+h_1}{1+u^*}\right)u^*+\frac{bu^*}{1+u^*}-d_1+h_2\\&=\frac{2u^{*2}+\left(1-a-h_1+b-d_1+h_2\right)u^*-d_1+h_2}{1+u^*}.\end{aligned}$$

Define

$$f\left(s\right)=2s^2+\left(1-a-h_1+b-d_1+h_2\right)s-d_1+h_2,$$

then, its discriminant is

$$\Delta_1=\left(1-a-h_1+b-d_1+h_2\right)^2+8\left(d_1-h_2\right)>0.$$

Thus, equation $f\left(s\right)=0$ has two roots:

$$s_{1,2}=\frac{-\left(1-a-h_1+b-d_1+h_2\right)\mp\sqrt{\Delta_1}}{4}.$$

Obviously, we have

$$s_1<0,\quad s_2>0.$$

Therefore, we can get

(i) as $0<u^*<s_2$, we have $f\left(u^*\right)<0$, which implies $p<0$;

(ii) as $u^*>s_2$, we have $f\left(u^*\right)>0$, which implies $p>0$;

(iii) as $u^*=s_2$, we have $f\left(u^*\right)=0$, which implies $p=0$.

From (8) and (10), we also have

$$\begin{aligned}q=&\frac{u^*}{\left(1+u^*\right)^2}\Big(-2\left(-b+d_1-h_2\right)u^{*2}+\left[\left(a+h_1\right)\left(-b+d_1-h_2\right)-3\left(d_1-h_2\right)\right]u^*\\&+\left(a+h_1\right)\left(d_1-h_2+b\right)-d_1+h_2\Big).\end{aligned}$$

Define

$$g(s) = -2(-b+d_1-h_2)s^2 + \left[(a+h_1)(-b+d_1-h_2)-3(d_1-h_2)\right]s$$
$$+(a+h_1)(d_1-h_2+b)-d_1+h_2.$$

then, its discriminant is

$$\Delta_2 = \left[(a+h_1)(-b+d_1-h_2)-3(d_1-h_2)\right]^2$$
$$+8(-b+d_1-h_2)\cdot\left[(a+h_1)(d_1-h_2+b)-d_1+h_2\right].$$

From condition of $0<\dfrac{d_1-h_2}{b-d_1+h_2}<a+h_1$, we can know that

$$-b+d_1-h_2<0,\ (a+h_1)(b-d_1+h_2)>d_1-h_2.$$

Therefore, the following relationships between q and Δ_2 hold,

(a) If $\Delta_2<0$, then $g(s)>0$ for all $s\in R$, and thus $q>0$;

(b) If $\Delta_2=0$, then $g(s)=0$ has two equal roots:

$$s_3=s_4=\frac{-\left[(a+h_1)(-b+d_1-h_2)-3(d_1-h_2)\right]}{-4(-b+d_1-h_2)},$$

and

$$s_3=s_4>0.$$

Therefore, $g(s)>0$ for either $0<s<s_3$ or $s>s_3$, and thus $q>0$;

(c) If $\Delta_2>0$, Then $g(s)=0$ has two real roots:

$$u_{5,6}=\frac{-\left[(a+h_1)(-b+d_1-h_2)-3(d_1-h_2)\right]\mp\sqrt{\Delta_2}}{-4(-b+d_1-h_2)}$$

and

$$s_5>0,\ s_6>0.$$

Therefore, as $s_5<u^*<s_6$, we have $g(u^*)<0$, which implies $q<0$; as $0<u^*<s_5$ or $s_6<u^*$, we have $g(u^*)>0$, which implies $q>0$.

We give some assumptions as the follows.

(H1) $\begin{aligned}&\left((a+h_1)(-b+d_1-h_2)-3(d_1-h_2)\right)^2\\&>8(b-d_1+h_2)\left((a+h_1)(d_1-h_2+b)-d_1+h_2\right)\end{aligned}$;

(H2) $\begin{aligned}&\left((a+h_1)(-b+d_1-h_2)-3(d_1-h_2)\right)^2\\&=8(b-d_1+h_2)\left((a+h_1)(d_1-h_2+b)-d_1+h_2\right)\end{aligned}$;

(H3) $\begin{aligned}&\left((a+h_1)(-b+d_1-h_2)-3(d_1-h_2)\right)^2\\&<8(b-d_1+h_2)\left((a+h_1)(d_1-h_2+b)-d_1+h_2\right)\end{aligned}$.

Theorem 3. Suppose conditions (H1) and $s_5<u^*<s_6$ hold, then $P_3(u^*,v^*)$ is an unstable saddle point.

Proof If condition (H1) holds, we know $\Delta_2>0$. Combined with condition $s_5<u^*<s_6$ and above discussion, we get $q<0$. It implies that Equation (9) has two real roots which have different signs, thus $P_3(u^*,v^*)$ is an unstable saddle point.

Theorem 4. Suppose conditions (H1) holds, we have

(A) if $u^* < \min\{s_2, s_5\}$ or $s_6 < u^* < s_2$, then $P_3(u^*, v^*)$ is an unstable node or unstable focus;

(B) if $s_2 < u^* < s_5$ or $\max\{s_2, s_6\} < u^*$, then $P_3(u^*, v^*)$ is a stable node or stable focus;

(C) if $u^* = s_2$, then $P_3(u^*, v^*)$ is a center.

Theorem 5. Suppose conditions (H2) holds, we have

(A) if $u^* < s_2$ and $u^* \neq s_3$, then $P_3(u^*, v^*)$ is an unstable node or unstable focus;

(B) if $u^* > s_2$ and $u^* \neq s_3$, then $P_3(u^*, v^*)$ is a stable node or stable focus;

(C) if $u^* = s_2$, then $P_3(u^*, v^*)$ is a center.

Theorem 6. Suppose conditions (H3) holds, we have

(A) if $u^* < s_2$, then $P_3(u^*, v^*)$ is an unstable node or unstable focus;

(B) if $u^* > s_2$, then $P_3(u^*, v^*)$ is a stable node or stable focus;

(C) if $u^* = s_2$, then $P_3(u^*, v^*)$ is a center.

4. Conclusion

This paper investigates a Holling-II functional response predator-prey model with linear capture terms. The existence of non-negative equilibrium point of the model was discussed, and the stability of the equilibrium point was analyzed. It is clear from this paper that human capture intensity can maintain an equilibrium position for both populations under certain constraints. However, when the intensity of human capture exceeds a certain limit, the two populations will lose equilibrium, which will have a negative impact on species diversity and environmental stability. In other words, human capture behavior, within certain limits, can not only maintain human needs but also achieve sustainable development of ecological resources. This paper provides a theoretical reference for the decision-making of relevant departments.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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