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Simple and Pseudo Quadratic Leibniz Superalgebras

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Abstract

Compactness of subspaces of a $\mathbb{Z}_2$-graded vector space is introduced and used to study simple Leibniz superalgebras. We introduce left and right super-invariance of bilinear forms over superalgebras. Pseudo-quadratic Leibniz superalgebras are Leibniz superalgebras endowed with a non degenerate, supersymmetric and super-invariant bilinear form. In this paper, we show that every nondegenerate, supersymmetric and super-invariant bilinear form over a Leibniz superalgebra induce a Lie superalgebra over the underlying vector space. Then by using double extension extended to Leibniz superalgebras, we study pseudo-quadratic Leibniz superalgebras and the induced Lie superalgebras. In particular, we generalize some results on Leibniz algebras to Leibniz superalgebras.

Keywords

Leibniz Superalgebras, Lie Superalgebras, Double Extension, Left (Resp. Right) Super-Invariant Bilinear Form

1. Introduction

A left (resp. right) Leibniz superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathcal{L} := \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ endowed with an even bilinear map $[\,,\,]: \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ such that $[p,[q,r]] = [[p,q],r] + (-1)^{|p||q|}[q,[p,r]]$ (resp. $[p,[q,r]] = [[p,q],r] - (-1)^{|q||r|}[[p,r],q]$) for all $p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}$ and $r \in \mathcal{L}_{|r|}$. The Leibniz superalgebra $(\mathcal{L},[\,,\,])$ is said to be symmetric if it is simultaneously a left and right Leibniz superalgebra. Leibniz superalgebras can be seen as $\mathbb{Z}_2$-graded Leibniz algebras, and therefore as generalizations of Leibniz algebras, Lie superalgebras and Lie algebras. Then, it seems natural to extend some properties and known results established on Lie (super) algebras and on Leibniz algebras to Leibniz superalgebras.

In this purpose, many results about quadratic Lie algebras (that is Lie algebras endowed with a nondegenerate, symmetric and invariant bilinear form) have been extended to Lie superalgebras and Leibniz (super)algebras (see for example, [1] [2] [3] [4] [5] [6]). For instance, in [7] the authors generalize the notion of double extension and describe quadratic Lie superalgebras; quadratic Leibniz algebras are also studied in [1] and in [8]. The authors in [9] investigate odd quadratic Leibniz superalgebras, in particular they proved that all quadratic Leibniz superalgebras are symmetric and they gave an inductive description of quadratic Leibniz superalgebra. In all these study of quadratic structure, we noticed that the invariance property or the associativity of the bilinear form B (that is $B([p,q],r) = B(p,[q,r])$) played an important role.

But unlike in the case of Lie superalgebras, the bracket of Leibniz superalgebra is not necessary super-anticommutative. This allows us to define a new type of invariance for a bilinear form B over a Leibniz superalgebra $(\mathcal{L},[\,,])$, that is different from the associativity. In fact, for a bilinear form $B:\mathcal{L}\otimes\mathcal{L}\to\mathbb{K}$, we said that B is left (resp. right) super-invariant if $B([p,q],r) = -(-1)^{|p||q|}B(q,[p,r])$ (resp. $B([p,q],r) = -(-1)^{|q||r|}B(p,[r,q])$) for all $p\in\mathcal{L}_{|p|}, q\in\mathcal{L}_{|q|}$ and $r\in\mathcal{L}_{|r|}$. A Leibniz superalgebra endowed with a nondegenerate, supersymmetric and left (resp. right) super-invariant bilinear form is called a left (resp. right) pseudo-quadratic Leibniz superalgebra. For simplicity, we will use LPQ (resp. RPQ) Leibniz superalgebra for left (resp. right) pseudo-quadratic Leibniz supaeralgebra.

The aim of this paper is to investigate simple Leibniz superalgebras and pseudo-quadratic Leibniz superalgebras. In fact, we introduce the notion of compactness with respect to the graduation of $\mathcal{L} := \mathcal{L}_{\bar{0}}\oplus\mathcal{L}_{\bar{1}}$ for a vector subspace of $\mathcal{L}$ and characterize simple Leibniz superalgebras through this notion. We show that for any LPQ symmetric Leibniz superalgebra $(\mathcal{L},[\,,])$, there exists a Lie superalgebra $(\mathcal{L},\mu)$ induced by B over the underlying vector space $\mathcal{L}$ *i.e.* the product μ is super-anticommutative over $\mathcal{L}$. By using the notion of central extension and representation of Leibniz superalgebras, we introduce a new type of double extension for LPQ Leibniz superalgebras. We show that any non-Lie LPQ Leibniz superalgebra is isomorphic to a double extension of a symmetric Leibniz superalgebra. Finally, we study relation between the double extension of $(\mathcal{L},[\,,])$ and its associated Lie superalgebra $(\mathcal{L},\mu)$.

This paper is organized as follows: the first section is devoted to state fondamental definitions and elementary properties necessary for the understanding of this paper. In Section 2 we give some properties of Leibniz superalgebras and characterize simple Leibniz superalgebras. In Section 3 we define left (resp. right) super-invariant bilinear form and study LPQ Leibniz superalgebras. Finally, Section 4 deals with the method of double extension for LPQ Leibniz superalgebras and theirs induced Lie superalgebras.

The study of RPQ Leibniz superalgebras is similary and leads to analogous results. Then we will deal only with LPQ Leibniz superalgebras.

Notation

Throughout this paper all vector spaces considered are finite dimensional and we will use the same notation as in [7].

2. Preliminaries

Let $\mathcal{L} := \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space and $[\,,]: \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ a bilinear map over $\mathcal{L}$ such that $[\mathcal{L}_\alpha, \mathcal{L}_\beta] \subseteq \mathcal{L}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2$. Then $(\mathcal{L}, [\,,])$ is said to be left(resp. right) Leibniz superalgebra if

$$[p,[q,r]] = [[p,q],r] + (-1)^{|p||q|}[q,[p,r]] \quad \forall p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}, r \in \mathcal{L}_{|r|} \tag{2.1}$$

(resp.

$$[p,[q,r]] = [[p,q],r] - (-1)^{|q||r|}[[p,r],q] \quad \forall p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}, r \in \mathcal{L}_{|r|} \tag{2.2}$$

The Leibniz superalgebra $(\mathcal{L}, [\,,])$ is said to be symmetric if it is simultaneously a left and right Leibniz superalgebra. From [9], a Leibniz superalgebra $(\mathcal{L}, [\,,])$ is symmetric if and only if

$$[p,[q,r]] = -(-1)^{|p|(|q|+|r|)}[[q,r],p] \quad \forall p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}, r \in \mathcal{L}_{|r|} \tag{2.3}$$

A homomorphism $f: \mathcal{L} \to \mathcal{W}$ between two $\mathbb{Z}_2$-graded vector spaces is said to be homogeneous of degree $\alpha \in \mathbb{Z}_2$ if $f(\mathcal{L}_\beta) \subseteq \mathcal{W}_{\alpha+\beta}$ for all $\beta \in \mathbb{Z}_2$. Given three $\mathbb{Z}_2$-graded vector spaces $\mathcal{L}, \mathcal{W}$ and $\mathcal{H}$, a bilinear map $g: \mathcal{L} \otimes \mathcal{W} \to \mathcal{H}$ is said to be homogeneous of degree $\alpha \in \mathbb{Z}_2$ if $g(\mathcal{L}_\beta, \mathcal{W}_\gamma) \subseteq \mathcal{H}_{\alpha+\beta+\gamma}$ for all $\beta, \gamma \in \mathbb{Z}_2$. The degree of a homogeneous linear or bilinear map f is denoted by $|f|$ and f is said to be an even (resp. odd) map if $|f| = \bar{0}$ (resp. $|f| = \bar{1}$).

For any left Leibniz superalgebra $(\mathcal{L}, [\,,])$, the left and the right multiplication L and R defined by $L_p(q) := [p,q]$ and $R_p(q) := (-1)^{|p||q|}[q,p]$ satisfy the following relations

Lemma 2.1.

1) $L_{[p,q]} = L_p \cdot L_q - (-1)^{|p||q|} L_q \cdot L_p$
2) $R_{[p,q]} = L_p \cdot R_q - (-1)^{|p||q|} R_q \cdot L_p$
3) $R_{[p,q]} = L_p \cdot R_q + (-1)^{|p||q|} R_q \cdot R_p$
4) $R_q \cdot L_p = -R_q \cdot R_p$

Proof. Straightforward computation. □

Let $\mathcal{L}$ be a left Leibniz superalgebra. We define the left and the right centre of $\mathcal{L}$ by $Z^l(\mathcal{L}) = \{x \in \mathcal{L}, [x, \mathcal{L}] = 0\}$ and $Z^r(\mathcal{L}) = \{x \in \mathcal{L}, [\mathcal{L}, x] = 0\}$. We also define the so called super-Leibniz kernel which is the subspace generated by elements of the form $[p,q] + (-1)^{|p||q|}[q,p]$ and it is denoted by $\ker(\mathcal{L})$.

Remark 2.2. *The relation 4) of the above Lemma implies that for all* $p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}$ *and* $r \in \mathcal{L}$ *we have*

$$[[p,q],r] = -(-1)^{|p||q|}[[q,p],r] \tag{2.4}$$

which implies that $\ker(\mathcal{L}) \subseteq Z^l(\mathcal{L})$. *For a left Leibniz superalgebra* $\mathcal{L}$, *the left centre* $Z^l(\mathcal{L})$ *is a two-sided ideal and the right centre is a Leibniz sub-superalgebra.*

Definition 2.3. *Let* $\mathcal{L}$ *be a left Leibniz superalgebra and H a Leibniz sub-superalgebra of L. We say that H is a left (resp. right) ideal of* $\mathcal{L}$ *if* $[H,\mathcal{L}] \subseteq H$ *(resp.* $[\mathcal{L},H] \subseteq H$ *).*

Definition 2.4. *Let* $(\mathcal{L},[,])$ *be a Leibniz superalgebra. The bilinear map* $\psi : \mathcal{L} \otimes \mathcal{L} \to \mathcal{H}$ *is called scalar bi-cocycle if we have*

$$\psi\left(p,[q,r]\right) - \psi\left([p,q],r\right) - (-1)^{|p||q|}\psi\left(q,[p,r]\right) = 0 \tag{2.5}$$

for all $p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}$ *and* $r \in \mathcal{L}_{|r|}$.

3. Simple Leibniz Superalgebras

The goal of this section is to study and characterize simple Leibniz superalgebra. For that, we introduce the notion of compactness of a subspace of a $\mathbb{Z}_2$-graded vector space and use this notion to give some characterizations of simple Leibniz superalgebras.

In general, we refer to simple algebraic object by an object that admits no ideal except {0} and itself. But in the case of Leibniz superalgebra, since for any left Leibniz superalgebra $(\mathcal{L},[,])$ we have $\ker(\mathcal{L})$ is a two-sided ideal then the definition of simple Leibniz superalgebra has to take this situation into account. This motivates the following defnition

Definition 3.1. *A Leibniz superalgebra* $(\mathcal{L},[,])$ *is said to be simple if it admits no ideal different from* $\{0\}, \ker(\mathcal{L})$ *and* $\mathcal{L}$.

Lemma 3.2. *Let* $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ *be a Leibniz superalgebra such that* $\mathcal{L}_{\bar{0}} = [\mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{1}}]$ *and* $Z(\mathcal{L}) = 0$. *If I is an ideal of* $\mathcal{L}$ *such that* $I \subset \mathcal{L}_{\bar{0}}$ *then* $I = 0$.

Proof. Let us assume that there exists an ideal $I \neq \{0\}$ such that $I \subseteq \mathcal{L}_{\bar{0}}$. Since I is an ideal contained in $\mathcal{L}_{\bar{0}}$ then
$[I,\mathcal{L}_{\bar{1}}] + [\mathcal{L}_{\bar{1}}, I] \subseteq I \cap \mathcal{L}_{\bar{1}} \subset \mathcal{L}_{\bar{0}} \cap \mathcal{L}_{\bar{1}} = \{0\}$. Which implies that
$[I,\mathcal{L}_{\bar{0}}] + [\mathcal{L}_{\bar{0}}, I] \neq 0$ because otherwise $I \subseteq Z(\mathcal{L}) = \{0\}$ that contradicts the fact that $I \neq \{0\}$, hence $[I,\mathcal{L}_{\bar{0}}] \neq 0$ or $[\mathcal{L}_{\bar{0}}, I] \neq 0$. Since $\mathcal{L}_{\bar{0}} = [\mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{1}}]$ we have $[I,[\mathcal{L}_{\bar{1}},\mathcal{L}_{\bar{1}}]] \neq 0$ or $[[\mathcal{L}_{\bar{1}},\mathcal{L}_{\bar{1}}], I] \neq 0$. If $[I,[\mathcal{L}_{\bar{1}},\mathcal{L}_{\bar{1}}]] \neq 0$, by using the fact that $\mathcal{L}$ is a Leibniz superalgebra we obtain according to relation (2.1) that $[[I,\mathcal{L}_{\bar{1}}],\mathcal{L}_{\bar{1}}] + [\mathcal{L}_{\bar{1}},[I,\mathcal{L}_{\bar{1}}]] \neq 0$ hence $0 \neq [I,\mathcal{L}_{\bar{1}}] \subseteq \mathcal{L}_{\bar{0}} \cap \mathcal{L}_{\bar{1}} = \{0\}$ which is impossible. If $[[\mathcal{L}_{\bar{1}},\mathcal{L}_{\bar{1}}], I] \neq 0$, with a similar process, we show that $[\mathcal{L}_1, I] \neq 0$ that is not thrue. Therefore $I = \{0\}$. □

Definition 3.3. *Let I be an ideal of* $\mathcal{L}$. *I is said to be compact with respect to the graduation of* $\mathcal{L}$ *if for all* $\alpha \in \mathbb{Z}_2$ *such that* $\mathcal{L}_\alpha \cap I \neq 0$ *we have either* $\mathcal{L}_\alpha \subsetneq I$ *or* $I \subsetneq \mathcal{L}_\alpha$.

Lemma 3.4. *If I is an ideal of* $\mathcal{L}$ *such that* $I \subset \mathcal{L}_{\bar{1}}$ *then* $[I,\mathcal{L}_{\bar{1}}] = [\mathcal{L}_{\bar{1}}, I] = 0$.

Proof. It is clear that $[I,\mathcal{L}_{\bar{1}}] \subset [\mathcal{L}_{\bar{1}},\mathcal{L}_{\bar{1}}] \subset \mathcal{L}_{\bar{0}}$ because $[\mathcal{L}_\alpha,\mathcal{L}_\beta] \subseteq \mathcal{L}_{\alpha+\beta}$. On the other hand the fact that I is an ideal implies that $[I,\mathcal{L}_{\bar{1}}] \subseteq I \subseteq \mathcal{L}_{\bar{1}}$. Hence $[I,\mathcal{L}_{\bar{1}}] \subseteq \mathcal{L}_{\bar{0}} \cap \mathcal{L}_{\bar{1}} = \{0\}$. With an analogous way, we show that $[\mathcal{L}_{\bar{1}}, I] = 0$. □

Lemma 3.5. *Let $\mathcal{L}$ be Leibniz superalgebra. If $I \neq \{0\}$ is a graded ideal compact with respect to the graduation of $\mathcal{L}$, then $I = \mathcal{L}$.*

Proof. Since I is compact with respect to the graduation of $\mathcal{L}$ and $I \cap \mathcal{L}_{\bar{0}} \neq 0$ and $I \cap \mathcal{L}_{\bar{1}} \neq 0$ then we have $I \subseteq \mathcal{L}_{\bar{0}}$ or $\mathcal{L}_{\bar{0}} \subseteq I$ and $I \subseteq \mathcal{L}_{\bar{1}}$ or $\mathcal{L}_{\bar{1}} \subseteq I$. And gives us four cases to consider

1) If $I \subseteq \mathcal{L}_{\bar{0}}$ and $I \subseteq \mathcal{L}_{\bar{1}}$. In this case $I \subseteq \mathcal{L}_{\bar{0}} \cap \mathcal{L}_{\bar{1}} = \{0\}$. Hence $I = \{0\}$ which is impossible.

2) If $\mathcal{L}_{\bar{0}} \subseteq I$ and $\mathcal{L}_{\bar{1}} \subseteq I$. Then $I = \mathcal{L}$.

3) If $\mathcal{L}_{\bar{0}} \subseteq I$ and $I \subseteq \mathcal{L}_{\bar{1}}$. Then $\mathcal{L}_{\bar{0}} \subseteq \mathcal{L}_{\bar{1}}$ which is impossible because otherwise $\mathcal{L}_{\bar{0}} = \{0\}$.

4) If $I \subseteq \mathcal{L}_{\bar{0}}$ and $\mathcal{L}_{\bar{1}} \subseteq I$. Then $\mathcal{L}_{\bar{1}} = \{0\}$.

Therefore $I = \mathcal{L}$. □

Lemma 3.6. *Let $\mathcal{L}$ be a Leibniz superalgebra such that $[\mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{1}}] \neq 0$ and $Z_{\mathcal{L}}(\mathcal{L}_{\bar{1}}) = 0$. If $I \neq \{0\}$ is a compact graded ideal with respect to the graduation of $\mathcal{L}$ such that $I \cap \mathcal{L}_{\bar{1}} \neq 0$ then $I = \mathcal{L}$.*

Proof. Since $I \cap \mathcal{L}_{\bar{1}} \neq 0$ and I is a compact graded ideal with respect to the graduation of $\mathcal{L}$ then $I \subseteq \mathcal{L}_{\bar{1}}$ or $\mathcal{L}_{\bar{1}} \subseteq I$. If $I \subseteq \mathcal{L}_{\bar{1}}$ then according to the Lemma 3.4 we have $I \subseteq Z_{\mathcal{L}}(\mathcal{L}_{\bar{1}}) = \{0\}$ which contradicts the fact that $I \neq \{0\}$ therefore $\mathcal{L}_{\bar{1}} \subseteq I$. Hence the fact that I is an ideal and $[\mathcal{L}_{\alpha}, \mathcal{L}_{\beta}] \subseteq \mathcal{L}_{\alpha+\beta}$ imply $0 \neq [\mathcal{L}_{\bar{1}}, \mathcal{L}_{\bar{1}}] \subseteq \mathcal{L}_{\bar{0}} \cap I$. Then according to the Lemma 3.5 we have $I = \mathcal{L}$. □

The above results give us a characterization of simple Leibniz superalgebras.

Theorem 3.7. *Let $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ be a Leibniz superalgebra such that $Z^{l}(\mathcal{L}) = \{0\}$. Then $\mathcal{L}$ is simple if and only if all graded ideals are compact with respect to the graduation of $\mathcal{L}$.*

Proof. Let us assume that $\mathcal{L}$ is simple. Let I be a graded ideal of $\mathcal{L}$. Then $I = \{0\}$ or $I = \ker(\mathcal{L})$ or $I = \mathcal{L}$. Then the fact that $\ker(\mathcal{L}) \subseteq Z^{l}(\mathcal{L})$ and $Z(\mathcal{L}) = \{0\}$ we have $I = \{0\}$ or $I = \mathcal{L}$. In both cases, one can easily sees that I is compact with respect to the graduation of $\mathcal{L}$. The converse is obvious from Lemma 3.5. □

4. LPQ Leibniz Superalgebras

In this section, we define left and right pseudo quadratic Leibniz superalgebras. We establish some properties of those superalgebras and show that every pseudo quadratic Leibniz superalgebra $(\mathcal{L}, [,])$ induce a structure of Lie superalgebra over the vector space $\mathcal{L}$.

Definition 4.1. *Let $(\mathcal{L}, [,])$ be a Leibniz superalgebra and B a bilinear form over $\mathcal{L}$. Then, B is said to be*

- *Invariant if* $B([p,q],r) = B(p,[q,r])$.
- *Left super-invariant if* $B([p,q],r) = -(-1)^{|p||q|} B(q,[p,r])$.
- *Right super-invariant if* $B([p,q],r) = -(-1)^{|q||r|} B(p,[r,q])$.
- *Supersymmetric if* $B(p,q) = (-1)^{|p||q|} B(q,p) \quad \forall p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}$.
- *Nondegenerate if* $B(p,q) = 0$ *for all* $q \in \mathcal{L}$ *implies* $p = 0$.

for all $p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}$ *and* $r \in \mathcal{L}_{|r|}$.

Proposition 4.2. *Let* $(\mathcal{L},[,])$ *be a Leibniz superalgebra and B a supersymmetric and nondegenerate bilinear form over* $\mathcal{L}$. *Then, if B satisfies at least two type of invariance, then* $\mathcal{L}$ *is a Lie superalgebra.*

Proof. A Leibniz superalgebra is a Lie superalgebra if and only if $[p,q] = -(-1)^{|p||q|}[q,p]$ for all $p \in \mathcal{L}_{|p|}$ and $q \in \mathcal{L}_{|q|}$. Let us assume that B is invariant and left super-invariant. Let $p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}$ and $r \in \mathcal{L}_{|r|}$. We have:

$$\begin{aligned}
&B\left([p,q]+(-1)^{|p||q|}[q,p],r\right)\\
&= B([p,q],r)+(-1)^{|p||q|}B([q,p],r)\\
&= B([p,q],r)-B(p,[q,r]) \quad \text{left super-invariance of } B\\
&= B([p,q],r)-B([p,q],r)=0 \quad \text{invariance of } B
\end{aligned}$$

and since B is non degenerate, $[p,q] = -(-1)^{|p||q|}[q,p]$, hence $\mathcal{L}$ is a Lie superalgebra. By proceeding in the same way, we show that $\mathcal{L}$ is a Lie superalgebra if B is invariant and right super-invariant.

Now let us assume that B is left and right super-invariant. Let $p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}$ and $r \in \mathcal{L}_{|r|}$. We have:

$$\begin{aligned}
&B([p,q],r)+(-1)^{|p||q|}B([q,p],r)\\
&= B([p,q],r)-B(p,[q,r]) \quad \text{left super-invariance of } B\\
&= B([p,q],r)+(-1)^{|q||r|}B([p,r],q) \quad \text{right super-invariance of } B\\
&= B([p,q],r)-(-1)^{|q||r|+|p||r|}B(r,[p,q]) \quad \text{left super-invariance of } B\\
&= B([p,q],r)-B([p,q],r)=0
\end{aligned}$$

the fact that B is non degenerate implies that $\mathcal{L}$ is a Lie superalgebra. □

Definition 4.3. *A Leibniz superalgebra* $(\mathcal{L},[,])$ *endowed with a non degenerate, supersymmetric and left (resp. right) super-invariant bilinear form B is called left (resp. right) pseudo-quadratic Leibniz superalgebra. We shall use the abreviation LPQ (resp. RPQ) Leibniz superalgebras for left (resp. right) pseudo quadratic Leibniz superalgebras.*

For a LPQ Leibniz superalgebra $(\mathcal{L},[,],B)$, the bilinear form B induces a bilinear map $\mu : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ defined by

$$B(p,\mu(q,r)) = B([p,q],r) \quad \forall p,q,r \in \mathcal{L}.$$

The following resul gives some properties of the bilinear map μ

Lemma 4.4. *For all* $p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}$ *and* $r \in \mathcal{L}_{|r|}$,

1) $\mu(p,q) = -(-1)^{|p||q|}\mu(q,p)$
2) $\mu(p,\mu(q,r)) = [p,\mu(q,r)]$
3) $[p,\mu(q,r)] = \mu([p,q],r)+(-1)^{|p||q|}\mu(q,[p,r])$
4) *Moreover, if* $\mathcal{L}$ *is symmetric then we have*:

$$\mu(p,[q,r]) = \mu([p,q],r)+(-1)^{|p||q|}\mu(q,[p,r]).$$

Proof. Let $p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}, r \in \mathcal{L}_{|r|}$ and $t \in \mathcal{L}_{|t|}$. We have

$$B\left(t,\mu(p,q)+(-1)^{|p||q|}\mu(q,p)\right)=B\left(t,\mu(p,q)\right)+(-1)^{|p||q|}B\left(t,\mu(q,p)\right)$$
$$=B\left([t,p],q\right)+(-1)^{|p||q|}B\left([t,q],p\right)$$
$$=B\left([t,p],q\right)-(-1)^{|p||q|+|t||q|}B\left(q,[t,p]\right)$$
$$=B\left([t,p],q\right)-B\left([t,p],q\right)=0$$

since B is nondegenerate, then 1) holds.

For relation 2 we have

$$B\left(t,\mu\left(p,\mu(q,r)\right)-\left[p,\mu(q,r)\right]\right)$$
$$=B\left([t,p],\mu(q,r)\right)+(-1)^{|t||p|}B\left([p,t],\mu(q,r)\right)$$
$$=B\left(\left[[t,p],q\right],r\right)+(-1)^{|t||p|}B\left(\left[[p,t],q\right],r\right)$$
$$=B\left(\left[[t,p]+(-1)^{|p||t|}[p,t],q\right],r\right)=0$$

since B is nondegenerate, we have relation 2).

$$B\left(t,\left[p,\mu(q,r)\right]-\mu\left([p,q],r\right)-(-1)^{|p||q|}\mu\left(q,[p,q]\right)\right)$$
$$=-(-1)^{|p||t|}B\left([p,t],\mu(q,r)\right)-B\left(t,\mu\left([p,q],r\right)\right)-(-1)^{|p||q|}B\left([t,q],[p,r]\right)$$
$$=-(-1)^{|p||t|}B\left(\left[[p,t],q\right],r\right)-B\left(\left[t,[p,q]\right],r\right)+(-1)^{|p||t|}B\left(\left[p,[t,q]\right],r\right)$$
$$=B\left(\left[[t,p],q\right]-\left[t,[p,q]\right]+(-1)^{|p||t|}\left[p,[t,q]\right],r\right)=0$$

therefore, we have relation 3) because B is nondegenerate. By a similar proof, we establish relation 4). □

Remark 4.5. *According to relation 2) and 1), the product* μ *and* $[\,,]$ *coincide over the vector space* $\mu(\mathcal{L},\mathcal{L})$ *and* $\left(\mu(\mathcal{L},\mathcal{L}),\mu\right)$ *is a Lie superalgebra.*

Let $p\in\mathcal{L}$ and define $L_{\mu,p}:\mathcal{L}\to\mathcal{L}$ by $L_{\mu,p}(q):=\mu(p,q)$ for all $q\in\mathcal{L}$. This linear map satifies the following property:

Lemma 4.6. *For all* $p\in\mathcal{L}$, *we have*

$$L_{\mu,p}^{n}=L_{p}^{n-1}\circ L_{\mu,p}\quad \forall n\in\mathbb{N}^{*}$$

where L_p *define the left multiplication by p that is* $L_p(q):=[p,q]$.

Proof. we shall proceed by induction over $n\in\mathbb{N}^{*}$. If $n=2$, then according to relation 2 of Lemma 4.4, we have

$$L_{\mu,p}^{2}(q)=\mu\left(p,\mu(p,q)\right)=\left[p,\mu(p,q)\right]=L_p\circ L_{\mu,p}(q)$$

Now let us assume that the result is true for *n*. By using again relation 2) of Lemma 4.4 and the induction hypothesis, we obtain

$$L_{\mu,p}^{n+1}(q)=L_{\mu,p}^{n}\circ L_{\mu,p}(q)=L_{p}^{n-1}\circ L_{\mu,p}\left(\mu(p,q)\right)$$
$$=L_{p}^{n-1}\left(\mu\left(p,\mu(p,q)\right)\right)=L_{p}^{n-1}\left(\left[p,\mu(p,q)\right]\right)=L_{p}^{n}\circ L_{\mu,p}(q)$$

this proves the lemma. □

For any LPQ symmetric Leibniz superalgebra, there exists an associated Lie superalgebra. Indeed, we have the following result

Lemma 4.7. *Let* $\left(\mathcal{L},[\,,],B\right)$ *be a LPQ symmetric Leibniz superalgebra. Then*

$(\mathcal{L},\mu)$ *is a Lie superalgebra.*

Proof. Let $t\in\mathcal{L}_{|t|}$,

$$\begin{aligned}
&B\Big(t,(-1)^{|p||r|}\mu(p,\mu(q,r))+(-1)^{|p||q|}\mu(q,\mu(r,p))+(-1)^{|q||r|}\mu(r,\mu(p,q))\Big)\\
&=(-1)^{|p||r|}B\big([[t,p],q],r\big)+(-1)^{|p||q|}B\big([[t,q],r],p\big)+(-1)^{|q||r|}B\big([[t,r],p],q\big)\\
&=(-1)^{|p||r|}B\big([[t,p],q],r\big)-(-1)^{|p||q|+|r|(|t|+|q|)}B\big(r,[[t,q],p]\big)+(-1)^{|q||r|}B\big([[t,r],p],q\big)\\
&=(-1)^{|p||r|}B\big([[t,p],q],r\big)-(-1)^{|r|(|q|+|r|)}B\big([[t,q],p],r\big)+(-1)^{|q||r|}B\big([[t,r],p],q\big)\\
&=(-1)^{|p||r|}B\big([[t,p],q]-(-1)^{|p||q|}[[t,q],p],r\big)+(-1)^{|q||r|}B\big([[t,r],p],q\big)\\
&=(-1)^{|p||r|}B\big([t,[p,q]],r\big)+(-1)^{|q||r|}B\big([[t,r],p],q\big)\\
&=-(-1)^{|p||r|+|t|(|p|+|q|)}B\big([p,q],[t,r]\big)+(-1)^{|q||r|}B\big([[t,r],p],q\big)\\
&=(-1)^{|p|(|q|+|r|)+|t|(|p|+|q|)}B\big(q,[p,[t,r]]\big)+(-1)^{|q||r|}B\big([[t,r],p],q\big)\\
&=(-1)^{|p|(|r|+|t|)+|q||r|}B\big([p,[t,r]],q\big)+(-1)^{|q||r|}B\big([[t,r],p],q\big)\\
&=-(-1)^{|q||r|}B\big([[t,r],p],q\big)+(-1)^{|q||r|}B\big([[t,r],p],q\big)=0
\end{aligned}$$

and the fact that B is nondegenerate implies that

$$(-1)^{|p||r|}\mu(p,\mu(q,r))+(-1)^{|p||q|}\mu(q,\mu(r,p))+(-1)^{|q||r|}\mu(r,\mu(p,q))=0.$$

Hence, μ satisfies the super Jacobi identity, and according to relation 1) of Lemma 4.4 we have $(\mathcal{L},\mu)$ is a Lie superalgebra. □

Let $(\mathcal{L},[,],B)$ be a symmetric LPQ Leibniz superalgebra, the product μ is given by the following result.

Lemma 4.8. *For all* $p\in\mathcal{L}_{|p|}$ *and* $q\in\mathcal{L}_{|q|}$, *we have*

$$\mu(p,q)=\frac{1}{2}\Big([p,q]-(-1)^{|p||q|}[q,p]\Big)+\psi(p,q)$$

where $\psi:\mathcal{L}\otimes\mathcal{L}\to Z(\mathcal{L})$ *is a bi-cocycle.*

Proof. Define $\varphi:\mathcal{L}\otimes\mathcal{L}\to\mathcal{L}$ by $\varphi(p,q)=[p,q]-\mu(p,q)$. Since for all $p\in\mathcal{L}_{|p|}$ and $q\in\mathcal{L}_{|q|}$ we have $\mu(p,q)=-(-1)^{|p||q|}\mu(q,p)$, then

$$\mu(p,q)=\frac{1}{2}\Big([p,q]-(-1)^{|p||q|}[q,p]\Big)-\frac{1}{2}\Big(\varphi(p,q)-(-1)^{|p||q|}\varphi(q,p)\Big).$$

Set $\psi(p,q)=-\frac{1}{2}\Big(\varphi(p,q)-(-1)^{|p||q|}\varphi(q,p)\Big)$, we have $\psi(p,q)\in Z(\mathcal{L})$. In fact, let $r\in\mathcal{L}_{|r|}$ and $t\in\mathcal{L}_{|t|}$, by the left super-invariance of B and the fact that $\mathcal{L}$ is symmetric we have

$$\begin{aligned}
B\big(t,[r,\varphi(p,q)]\big)&=B\big(t,[r,[p,q]]\big)-B\big(t,[r,\mu(p,q)]\big)\\
&=B\big(t,[r,[p,q]]\big)+(-1)^{|t||r|}B\big([[r,t],p],q\big)\\
&=-(-1)^{|t||r|}B\big([r,t],[p,q]\big)+(-1)^{|t||r|}B\big([[r,t],p],q\big)\\
&=(-1)^{|t||r|+|p|(|r|+|t|)}B\big([r,[r,t]],q\big)+(-1)^{|t||r|}B\big([[r,t],p],q\big)\\
&=-(-1)^{|t||r|}B\big([[r,t],p],q\big)+(-1)^{|t||r|}B\big([[r,t],p],q\big)=0
\end{aligned}$$

since B is nondegenerate, then $\varphi(p,q) \in Z(\mathcal{L})$. Hence $\psi(p,q) \in Z(\mathcal{L})$ as required. And by using the Leibniz super identity of $(\mathcal{L},[,])$ and the fact that $(\mathcal{L},\mu)$ is a Lie superalgebra, one can see that ψ is a bicocycle. □

5. Double Extension

In this section, we construct a double extension for LPQ Leibniz superalgebras and for Lie superalgebras by using the central extension and representation of Leibniz and Lie superalgebras over a $\mathbb{Z}_2$-graded vector space. For more details about the method of double extension see [1] [2] [3].

5.1. Double Extension of LPQ Leibniz Superalgebras

Lemma 5.2. *Let $(\mathcal{L},[,])$ be a Leibniz superalgebra, H a vector space and $\psi : \mathcal{L} \otimes \mathcal{L} \to H^*$ a bi-cocycle. Then, the space $\tilde{\mathcal{L}} := \mathcal{L} \oplus H^*$ endowed with the bracket*

$$[p+f, q+g] = [p,q] + \psi(p,q) \quad \forall p+f \in \tilde{\mathcal{L}}_{|p|}, q+g \in \tilde{\mathcal{L}}_{|q|} \tag{5.1}$$

is a Leibniz superalgebra.

Proof. Straightforward calculation. □

The Leibniz superalgebra $\tilde{\mathcal{L}}$ constructed in the above lemma is called central extension of $\mathcal{L}$ by means of ψ.

Definition 5.3. *Let $\mathcal{L}$ be a left Leibniz superalgebra and $V := V_{\bar{0}} \oplus V_{\bar{1}}$ a $\mathbb{Z}_2$-graded vector space. A representation of $\mathcal{L}$ over V is given by a couple (ϕ,γ) where $\phi,\gamma : \mathcal{L} \to End(V)$ are even morphisms such that*

$$\phi_{[p,q]} = \phi_p \cdot \phi_q - (-1)^{|p||q|} \phi_q \cdot \phi_p$$

$$\gamma_{[p,q]} = \phi_p \cdot \gamma_q + (-1)^{|p||q|} \gamma_q \cdot \gamma_p$$

$$\gamma_{[p,q]} = \phi_p \cdot \gamma_q - (-1)^{|p||q|} \gamma_q \cdot \phi_p$$

for all $p \in \mathcal{L}_{|p|}$ and $q \in \mathcal{L}_{|q|}$. The set of all representations of $\mathcal{L}$ over a given $\mathbb{Z}_2$-graded vector space V will be denoted by $Rep_V^{\mathcal{L}}$.

Lemma 5.4. *Let $(\mathcal{L},[,])$ be a Leibniz superalgebra and $(\phi,\gamma) \in Rep_V^{\mathcal{L}}$. Then, $\mathcal{L}_1 := \mathcal{L} \oplus V$ endowed with the bracket*

$$[p+u, q+v] = [p,q] + \phi_p(v) + (-1)^{|p||q|} \gamma_q(u) \quad \forall p+u \in (\mathcal{L}_1)_{|p|}, q+v \in (\mathcal{L}_1)_{|q|} \tag{5.2}$$

is a Leibniz superalgebra.

Proof. Straightforward calculation. □

Theorem 5.5. *Let $(\mathcal{L},[,],B)$ be a LPQ Leibniz superalgebra, $H = \mathbb{K}e$ an one dimensional vector space, $(\phi,\gamma) \in Rep_H^{\mathcal{L}}$ and $\Gamma \in End(H)$ such that*

$$\Gamma([p,q]) = [p,\Gamma q] - (-1)^{|p||q|}[q,\Gamma p] \quad \forall p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}. \tag{5.3}$$

Then, $\hat{\mathcal{L}} := H \oplus \mathcal{L} \oplus H^*$ endowed with the bracket

$$[h+p+f, h'+q+g] = [p,q] + B(\Gamma p,q)e^* + \phi_p(h') + (-1)^{|p||q|}\gamma_q(h) \tag{5.4}$$

is a Leibniz superalgebra.

Proof. Set $\tilde{\mathcal{L}} := \mathcal{L} \oplus H^*$. Let $\psi : \mathcal{L} \otimes \mathcal{L} \to H^*$ defined by $\psi(p,q) = B(\Gamma p, q)e^*$. The relation 5.3 implies that ψ is a bi-cocycle of $\mathcal{L}$, then according to Lemma 5.2, the space $\tilde{\mathcal{L}}$ endowed with the bracket

$$[p+f, q+g]_{\tilde{\mathcal{L}}} = [p,q] + B(\Gamma p, q)e^* \tag{5.5}$$

is a Leibniz superalgebra. Define $\tilde{\phi}, \tilde{\gamma} : \tilde{\mathcal{L}} \to End(H)$ by $\tilde{\phi}(p+f) = \phi_p$ and $\tilde{\gamma}(p+f) = \gamma_p$ for all $p+f \in \tilde{\mathcal{L}}$. Since (ϕ,γ) is a representation of $\mathcal{L}$ over *H*, then $(\tilde{\phi},\tilde{\gamma})$ define a representation of $\tilde{\mathcal{L}}$ over *H*. Hence, by Lemma 5.4, we have $\mathcal{L}_1 := \tilde{\mathcal{L}} \oplus H$ endowed with the bracket

$$[p+f+h, q+g+h'] = [p+f, q+g]_{\tilde{\mathcal{L}}} + \tilde{\phi}_{p+f}(h') + (-1)^{|p||q|}\tilde{\gamma}_{q+g}(h) \tag{5.6}$$

is a Leibniz superalgebra. Therefore, $\hat{\mathcal{L}}$ endowed with the bracket 5.4 is a Leibniz superalgebra. □

The Leibniz superalgebra $\hat{\mathcal{L}}$ obtained in the above theorem is called Leibniz double extension of $\mathcal{L}$ by *H* by means of (ϕ,γ,Γ). The triplet (ϕ,γ,Γ) is called context of Leibniz double extension.

For a given non-Lie odd LPQ symmetric Leibniz superalgebra $(\mathcal{L},[\,,],B)$, we have

Theorem 5.6. $\mathcal{L}$ *is isomorph to a double extension of a symmetric Leibniz superalgebra.*

Proof. Since $(\mathcal{L},[\,,])$ is a non-Lie Leibniz superalgebra, then there exists a non zero element $e \in Ker(\mathcal{L})_{|e|}$. The fact that *B* is odd and nondegenerate implies the existence of a non zero element $d \in \mathcal{L}_{|e|+\bar{1}}$ such that $B(e,d) = 1$. Set $H = \mathbb{K}e, \mathcal{V} = \mathbb{K}d$ and $\mathcal{E} = (H \oplus \mathcal{V})^{\perp}$. Since $B_{|H\oplus\mathcal{V}}$ is non degenerate, then $\mathcal{L} = \mathcal{E} \oplus \mathcal{E}^{\perp} = H \oplus \mathcal{E} \oplus \mathcal{V}$ and $H^{\perp} = H \oplus \mathcal{E}$.

It's clear that *H* is an ideal of $\mathcal{L}$, and by using the left super-invariance of *B*, we see that $H^{\perp}$ is also an ideal. Therefore, there exists a bilinear map $\psi : \mathcal{E} \otimes \mathcal{E} \to \mathbb{K}$ and $[\,,]_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ such that $[p,q] = [p,q]_{\mathcal{E}} + \psi(p,q)$ for all $p,q \in \mathcal{E}$. With a straighforward calculation and the help of the Leibniz super-identity, we obtain $(\mathcal{E},[\,,]_{\mathcal{E}})$ is a symmetric Leibniz superalgebra and ψ is a bi-cocycle.

The fact that $\mathcal{L} = H \oplus \mathcal{E} \oplus \mathcal{V}$ and $H^{\perp} = H \oplus \mathcal{E}$ is an ideal of $\mathcal{L}$ implies that the product over $\mathcal{L}$ is given by:

$$[p,d] = \alpha_p^{\varphi} e + \xi(p) \quad \text{with } \alpha_p^{\varphi} \in \mathbb{K}, \xi \in End(\mathcal{E})$$

$$[d,p] = \alpha_p^{\phi} e + \delta(p) \quad \text{with } \alpha_p^{\phi} \in \mathbb{K}, \delta \in End(\mathcal{E})$$

$$[d,d] = \alpha e + p_0 + \lambda d \quad \text{where } \alpha,\lambda \in \mathbb{K}, p_0 \in \mathcal{E}$$

we define two applications $\varphi,\phi : \mathcal{E} \to End(H)$ by $\varphi_p(e) = \alpha_p^{\varphi} e$ and $\phi_p(e) = \alpha_p^{\phi} e$ for all $p \in \mathcal{E}_{|p|}$. Let us show that $(\varphi,\phi) \in Rep_H^{\mathcal{E}}$. Let $p \in \mathcal{E}_{|p|}$.

$$B([p,d],d) = B(\alpha_p^{\varphi} e + \xi(p), d) = \alpha_p^{\varphi} B(e,d) + B(\xi(p),d) = \alpha_p^{\varphi}$$

and

$$B([d,p],d) = B(\alpha_p^\phi e + \delta(p), d) = \alpha_p^\phi B(e,d) + B(\delta p, d) = \alpha_p^\phi$$

Thus $\alpha_p^\varphi = B([p,d],d)$ and $\alpha_p^\phi = B([d,p],d)$. Since the bilinear form B is odd, then if $p \in \mathcal{E}_{\bar{0}}$ we have $\alpha_p^\varphi = \alpha_p^\phi = 0$. Now if $p \in \mathcal{E}_{\bar{1}}$, then $\varphi_p(e) = \alpha_p^\varphi = 0$ and $\phi_p(e) = \alpha_p^\phi = 0$ because φ_p and ϕ_p change the degree of *e*. Hence, (φ,ϕ) is a trivial representation of $\mathcal{E}$ over *H*.

Let $p \in \mathcal{E}_{|p|}$ and $q \in \mathcal{E}_{|q|}$, we have

$$B([p,q],d) = \psi(p,q) B(e,d) + B([p,q]_{\mathcal{E}}, d) = \psi(p,q)$$

On the other hand, by using the left super-invariance of *B*, we have

$$\begin{aligned} B([p,q],d) &= -(-1)^{|p||q|} B(q,[p,d]) = -(-1)^{|p||q|} B(q, \alpha_p^\varphi e + \xi(p)) \\ &= -(-1)^{|p||q|} B(q,\xi(p)) = -B(\xi(p),q) \end{aligned}$$

thus, $\psi(p,q) = -B(\xi(p),q)$. Therefore $(\varphi,\phi,-\xi)$ is a context of Leibniz double extension of $\mathcal{E}$ by *H*. Hence we obtain a Leibniz double extension $\mathcal{E}_1 = H \oplus \mathcal{E} \oplus H^*$ of $(\mathcal{E},[\,,]_{\mathcal{E},B_{\mathcal{E}}})$ where $B_{\mathcal{E}} := B_{|\mathcal{E}\otimes\mathcal{E}}$, by *H* by means of $(\varphi,\phi,-\xi)$. One can easily see that $\mathcal{E}_1$ is isomorphic to $\mathcal{L} = H \oplus \mathcal{E} \oplus \mathcal{V}$. This proves the theorem. □

5.2. The Induced Lie Superalgebra $(\mathcal{L},\mu)$

In the previous section, we showed that every LPQ Leibniz superalgebra $(\mathcal{L},[\,,],B)$ is isomorphic to a double extension of $(\mathcal{E},[\,,]_{\mathcal{E},B_{\mathcal{E}}})$ (see Theorem 5.6). And we also showed that any LPQ Leibniz superalgebra $(\mathcal{L},[\,,])$ induce a Lie superalgebra $(\mathcal{L},\mu)$. In this section, we study the relationship between the Lie superalgebra $(\mathcal{L},\mu)$ induced by $(\mathcal{L},[\,,],B)$ and the Lie superalgebra $(\mathcal{E},\mu_{\mathcal{E}})$.

Definition 5.8. *Let* $(\mathcal{L},\mu)$ *be a Lie superalgebra and V a* $\mathbb{Z}_2$*-graded vector space. A representation of* $\mathcal{L}$ *over V is given by a linear map* $\varphi : \mathcal{L} \to End(V)$ *such that*

$$\varphi_{\mu(p,q)} = \varphi_p \circ \varphi_q - (-1)^{|p||q|} \varphi_q \circ \varphi_p \quad \forall p \in \mathcal{L}_{|p|}, q \in \mathcal{L}_{|q|}$$

Lemma 5.9. *Let* $(\mathcal{L},\mu)$ *be a Lie superalgebra, H a vector space and* $\psi : \mathcal{L}\otimes\mathcal{L} \to H^*$ *a bi-cocyle of* $\mathcal{L}$ *such that* $\psi(p,q) = -(-1)^{|p||q|}\psi(q,p)$ *for all* $p \in \mathcal{L}_{|p|}$ *and* $q \in \mathcal{L}_{|q|}$ *Then* $\mathcal{L} \oplus H^*$ *endowed with the bracket*

$$[p+f, q+g] = \mu(p,q) + \psi(p,q) \quad p+f, q+g \in \mathcal{L} \oplus H^* \tag{5.7}$$

is a Lie superalgebra.

Proof. Straightforward calculation □

Lemma 5.10. *Let* $(\mathcal{L},\mu)$ *be a Lie superalgebra and* $\varphi \in Rep_V^{\mathcal{L}}$. *Then* $\mathcal{L} \oplus V$ *endowed with the product*

$$[p+u, q+v] = \mu(p,q) + \varphi_p(v) - (-1)^{|p||q|} \varphi_q(u) \tag{5.8}$$

is a Lie superalgebra.

By using Lemma 5.9 and 5.10 and following the same path of the proof of

Theorem 5.5 we obtain the following Lie double extension

Theorem 5.11. *Let* $(\mathcal{L},[\,,],B)$ *be a LPQ symmetric Leibniz superalgebra,* $H=\mathbb{K}e$ *an one dimensional vector space,* $(\mathcal{L},\mu)$ *the induced Lie superalgebra,* $\varphi\in Rep_H^{(\mathcal{L},\mu)}$ *and* $\Gamma\in End(\mathcal{L})$ *such that*

$$\Gamma\circ\mu(p,q)=\mu(p,\Gamma q)+\mu(\Gamma p,q)\quad p\in\mathcal{L}_{|p|},q\in\mathcal{L}_{|q|} \tag{5.9}$$

Then $\tilde{\mathcal{L}}:=H\oplus\mathcal{L}\oplus H^*$ *endowed with the product*

$$\tilde{\mu}(h+p+f,h'+q+g)=\mu(p,q)+B(\Gamma p,q)e^*+\varphi_p(h')-(-1)^{|p||q|}\varphi_q(h)$$

is a Lie superalgebra.

Proof. Straightforward. □

The Lie superalgebra $(\tilde{\mathcal{L}},\tilde{\mu})$ obtained in the above thereom is called Lie double extension of $(\mathcal{L},\mu)$ by H by means the so-called Lie context of double extension (φ,Γ).

Now for a non-Lie LPQ symmetric Leibniz superalgebra $(\mathcal{L},[\,,],B)$, there exists an associated Lie superalgebra $(\mathcal{L},\mu)$. According to Theorem 5.6, $\mathcal{L}$ is isomorph to a certain double extension $\mathcal{E}_1:=H\oplus\mathcal{E}\oplus H^*$ where $(\mathcal{E},[\,,]_{\mathcal{E}},B_{|\mathcal{E}\times\mathcal{E}})$ is a LPQ Leibniz superalgebra. Thus $B_{|\mathcal{E}\times\mathcal{E}}$ induces over $\mathcal{E}$ a Lie superalgebra $(\mathcal{E},\mu_{\mathcal{E}})$.

Therefore a natural question arises: what is the link between $(\mathcal{L},\mu)$ and $(\mathcal{E},\mu_{\mathcal{E}})$? The following result answer to this question.

Theorem 5.12. $(\mathcal{L},\mu)$ *is a Lie double extension of* $(\mathcal{E},\mu_{\mathcal{E}})$.

Proof. Since $(\mathcal{L},[\,,])$ is a non-Lie Leibniz superalgebra, there exists a non zero element $e\in Ker(\mathcal{L})_{|e|}$. The fact that B is odd and non degenerate implies that there exists $0\neq d\in\mathcal{L}_{|e|+\bar{1}}$ such that $B(e,d)=1$. Set $H=\mathbb{K}e$, $\mathcal{V}=kd$ and $\mathcal{E}=(H\oplus\mathcal{V})^{\perp}$. We have $\mathcal{L}=H\oplus\mathcal{E}\oplus\mathcal{V}$ and $H^{\perp}=\mathcal{E}\oplus H$ (see the proof of theorem 5.6). $H^{\perp}$ is an ideal of $(\mathcal{L},\mu)$. In fact, let $a\in Ker(\mathcal{L})$ and $p,q\in\mathcal{L}$ we have $B(q,\mu(a,p))=B([q,a],p)=0$. Since B is non degenerate, then $\mu(a,p)=0$, hence $Ker(\mathcal{L})\subseteq Z_\mu(\mathcal{L})$ where $Z_\mu(\mathcal{L})$ is the centre of $(\mathcal{L},\mu)$. Therefore, H is an ideal of $(\mathcal{L},\mu)$ because $H\subseteq Ker(\mathcal{L})$. By using the property of Left super-invariance of B, we conclude that $H^{\perp}$ is also an ideal of $(\mathcal{L},\mu)$. Consequently, there exists $\mu_{\mathcal{E}}:\mathcal{E}\otimes\mathcal{E}\to\mathcal{E}$ and $\psi_{\mathcal{E}}:\mathcal{E}\otimes\mathcal{E}\to\mathbb{K}$ such that:

$$\mu(p,q)=-(-1)^{|p||q|}\mu(q,p)=\mu_{\mathcal{E}}(p,q)+\psi_{\mathcal{E}}(p,q)e$$

$$\mu(p,d)=-(-1)^{|p||d|}\mu(d,p)=\pi(p)+\lambda(p)e,\quad \pi\in End(\mathcal{E}),\lambda\in\mathcal{E}^*$$

$$\mu(d,d)=0\quad\text{if } d\in\mathcal{L}_{\bar{0}}$$

$$\mu(d,d)=\alpha e+p_0+\beta d;\quad \alpha,\beta\in\mathbb{K},\ p_0\in\mathcal{E}\quad\text{if } d\in\mathcal{L}_{\bar{1}}$$

for all $p\in\mathcal{E}_{|p|}$ and $q\in\mathcal{E}_{|q|}$.

From the Jacobi super identity of $(\mathcal{L},\mu)$, we obtain that $(\mathcal{E},\mu_{\mathcal{E}})$ is a Lie superalgebra and $\psi_{\mathcal{E}}$ is a bi-cocycle of $(\mathcal{E},\mu_{\mathcal{E}})$. The Lie superalgebra $(\mathcal{E},\mu_{\mathcal{E}})$ is the Lie superalgebra induced by $B_{|\mathcal{E}\times\mathcal{E}}$ over $(\mathcal{E},[\,,]_{\mathcal{E}})$. Indeed, let $p\in\mathcal{E}_{|p|},q\in\mathcal{E}_{|q|}$ and $r\in\mathcal{E}_{|r|}$, we have

$$
\begin{aligned}
B\left([p,q]_{\mathcal{E}},r\right) &= B\left([p,q]-\psi(p,q)e,r\right)=B\left([p,q],r\right)-\psi(p,q)B(e,r) \\
&= B\left([p,q],r\right)=B\left(p,\mu(q,r)\right) \\
&= B\left(p,\mu_{\mathcal{E}}(q,r)+\psi_{\mathcal{E}}(q,r)e\right)=B\left(p,\mu_{\mathcal{E}}(q,r)\right)
\end{aligned}
$$

Now consider the linear map $\varphi:(\mathcal{E},\mu_{\mathcal{E}})\to End(H)$ defined by $\varphi_p(e)=\lambda(p)e$. It's clear that if $p\in\mathcal{E}_{\bar{1}}$ then $\lambda(p)=0$ because φ_p change the degree of *e*. If $p\in\mathcal{E}_{\bar{0}}$ the fact that B is odd implies that

$$0=B\left(\mu(p,d),d\right)=B\left(\pi(p)+\lambda(p)e,d\right)=\lambda(p)$$

Hence, φ is a trivial representation of $(\mathcal{E},\mu_{\mathcal{E}})$ over *H*. Therefore $(\varphi,\mu_{\mathcal{E}})$ is a context Lie double extension of $(\mathcal{E},\mu_{\mathcal{E}})$ by *H*. Consequently, we obtain a Lie double extension $\hat{\mathcal{E}}:=H\oplus\mathcal{E}\oplus H^*$ of $(\mathcal{E},\mu_{\mathcal{E}})$ by H by means of $(\varphi,\psi_{\mathcal{E}})$, which is isomorphic to $(\mathcal{L},\mu)$. This proves the theorem. □

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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