Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2023, 11, 2644-2651
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Differential Calculus the Study of the Growth and Decay of an Entity's Population

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How to cite this paper: Toure, L. and Ndiaye, M. (2023) Differential Calculus the Study of the Growth and Decay of an Entity's Population. *Journal of Applied Mathematics and Physics*, **11**, 2644-2651.
https://doi.org/10.4236/jamp.2023.119173

Received: August 6, 2023
Accepted: September 24, 2023
Published: September 27, 2023

Abstract

Population Growth and Decay study of the growth or the decrease of a population of a given entity, is carried out according to the environment. In an infinite environment, *i.e.* when the resources are unlimited, a population P believes according to the following differential equation $P' = KP$, with the application of the differential calculus we obtasin an exponential function of the variable time (t). The function of which we can predict approximately a population according to the signs of k and time (t). If $k > 0$, we speak of the Malthusian croissant. On the other hand, in a finite environment *i.e.* when resources are limited, the population cannot exceed a certain value. and it satisfies the logistic equation proposed by the economist Francois Verhulst: $P' = P(1-P)$.

Keywords

Growth, Decay, an Exponential Function, Logistic Equation, Differential Equation, Graph of the Differential Equation

1. Differential Systems

1.1. Introduction General

In this research report, it is the study of the growth or decay of a population according to the resources. To do so, we need mathematical modeling to express the problem in mathematical terms. And afterwards, this led us to the application of differential equations. In Section 1, we have briefly examined the differential equations. In Section 2, we studied a population according to unlimited resources, which resulted in obtaining an exponential law of the form $P' = KP$ and at the end of the last and Section 3, in the case where the attempts are we had to study a logistic function of the form $P' = kP(1-\alpha P)$

In a study by Pac [1], *the concept of differential systems was explored*:

A dynamic system is a mechanical, physical, economic, environmental or any other domain whose state evolves as a function of time.

The study of the evolution of a system therefore requires knowledge:

Its initial state that is to say its state at time t_0;

Its law of evolution.

A system can be in continuous time or in discrete time, it can also be autonomous, if its law of evolution does not depict time, in this case, its law is said to be stationary, and in the end it can be also non autonomous, in this case, its law of evolution depends on time.

1.2. Modelization

Modeling involves formalizing or constructing a problem in physics, chemistry, biology, economics or any other system in mathematical terms. For example: In physics the fundamental laws of physics have mathematical expressions and others like; in economics the equations of the demand or the supply of a good, the calculation of the elasticity of the demand in relation to the price of a good or in relation to the income of the consumer in demography the growth or decay of the population according to a finite or infinite environment.

The description of this system is at least numerical so it is a question of studying evolution over time.

1.3. What Is a Differential Equation

In [2] [3] [4], we learned that a differential equation is an equation that contains one or more derivatives of an unknown function.

The order of a differential equation is the order of the highest derivative that it contains. A differential equation is the ordinary differential equation if it involved an unknown function of only one variable. The simplest differential equations are first order equations of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

or equivalently,

$$y' = f(x) \tag{1}$$

where f is a known function of x. we already know calculus how to find functions that satisfy this kind of equation. For example, if $y' = x^2$, then

$$\int \mathrm{d}y = \int x^2 \mathrm{d}x = \frac{x^3}{3} + c$$

where c is the arbitrary constant. If $n > 1$ we can find functions y that satisfy equations of the form

$$y^{(n)} = f(x) \tag{2}$$

by repeated integration. We can also write

$$y^{(n)} = f\left(x, y, y', \cdots, y^{(n-1)}\right). \quad (3)$$

1.4. Solutions of Differential Equations

A solution of a differential equation is a function that satisfies the differential equation on some open interval; thus, y is a solution of 3 if y is n times differentiable and $y^{(n)} = f\left(x, y, y', \cdots, y^{(n-1)}\right)$ for all x in some open interval (a,b).

1.5. The Graph

The graph of a solution of a differential equation is a solution curve. More generally, a curve C is said to be an integral curve of a differential equation if every function $y = y(x)$ whose graph is a segment of C is a solution of the differential equation.

2. Exponential Law

Population Growth and Decay According to Trench [5], although the number of members of a population (people in a given country, bacteria in a laboratory culture, wildflowers in a forest, etc.) at any given time t is necessarily an integer, models that use differential equations to describe the growth and decay of populations usually rest on the simplifying assumption that the number of members of the population can be regarded as a differentiable function $P = P(t)$. In most models it is assumed that the differential equation takes the form

$$P'(t) = k(P)P(t), \quad (4)$$

where k is a continuous function of P that represents the rate of change of population per unit time per individual. In the Malthusian model, it is assumed that $k(P)$ is a constant, so (1) becomes

$$P'(t) = kP(t)$$

2.1. Modeling of the Problem

In this application we consider a population consisting of P individuals and evolving as a function of time t: P is a function of time t (human population in the case of Malthus, bacterial population or others.) We consider P as a real number, in fact the populations concerned are numerous and if we count for example P in billion of individuals, P will take a priori all the decimal values with 9 digits after the decimal point and we can thus assimilate P with a real number. The basic hypothesis is that during a small time interval delta t, the ΔP variation of the number of individuals in the population is proportional to P and Δt. If $P(t)$ is the population at time t, the population at time $t + \Delta t$, is

$$P(t + \Delta t) = P(t) + kP(t)\Delta t \quad (5)$$

where

$$\Delta P(t) = P(t + \Delta t) - P(t).$$

k is a positive constant if the population grows and negative if the population decreases.

2.2. Setting: Differential Growth Relationship

The population at the moment $(t+\Delta t)$ where Δt is small, is equal

$$P(t+\Delta t) = P(t) + kN(t)\Delta t$$

Dividing by Δt and passing to the limit when Δt tends to 0, we obtain by $P'(t)$ the derivative of $P(t)$ with respect to the variable *t*:

$$P'(t) = kP(t) \tag{6}$$

This model assumes that the numbers of birth and deaths per unit time are both proportional to the population. The constant of proportionality are the birth rate (birth per unit time per individual) and the death rate (death per unit per individual); a is the birth rate minus the death rate. You learned in calculus that if c is any constant then

$$P = ce^{kt} \tag{7}$$

To select the solution of the specific problem that we're considering, we must know the population P_0 at an initial time, say $t=0$. Setting

$$t=0 \Rightarrow c = P(0) = P_0$$

so the applicable solution is

$$P(t) = P_0 e^{kt} \tag{8}$$

This implies that

$$\lim_{t\to\infty} P(t) = \begin{cases} \infty & \text{if } k>0 \\ 0 & \text{if } k<0 \end{cases}$$

Note: That is the population approaches infinity if the birth rate exceeds the death rate, or zero if the death rate exceeds the bith rate. To see the limitations of the Malthusian model, suppose we're modeling the population of a country, starting from a time $t=0$ when the birth rate exceeds the death rate (so $k>0$), and the country's resources in terms of space, food supply, and other necessities of life can support the existing population. Then the prediction $P = P_0 e^{kt}$ may be reasonably accurate as long as it remains within limits that the country's resources can support. However, the model must inevitably lose validity when the prediction exceeds these limits (if nothing else, eventually there won't be enough space for the prediction population).

2.3. Graphic Representation

In this sub-section, we will graphically represent the evolution of the population according to the two cases;

2.3.1. Growth in the Population *P*(*t*) of an Entity as a Function of Time *t*

Let $P(0)=1000$ represent the population of an entity (for example the total number of Muslim living in Chengdu) at $t=0$ and for reasons of freedom and the high level of living conditions, we can estimate the value of $k = 0.75 > 0$.

$$P(t) = 1000\exp(0.75t)$$

Figure 1 presents the evolution of this population as a function of time *t*. This function is increasing.

2.3.2. Decrease in the Population *P*(*t*) of an Entity as a Function of Time *t*

Let $P(0) = 1000$ represent the population of an entity (for example the total number of Muslim living in Chengdu) at $t = 0$ and for reasons of war, epidemics and precarious living conditions, we can estimate the value of $k = -0.75 < 0$ (Figure 2).

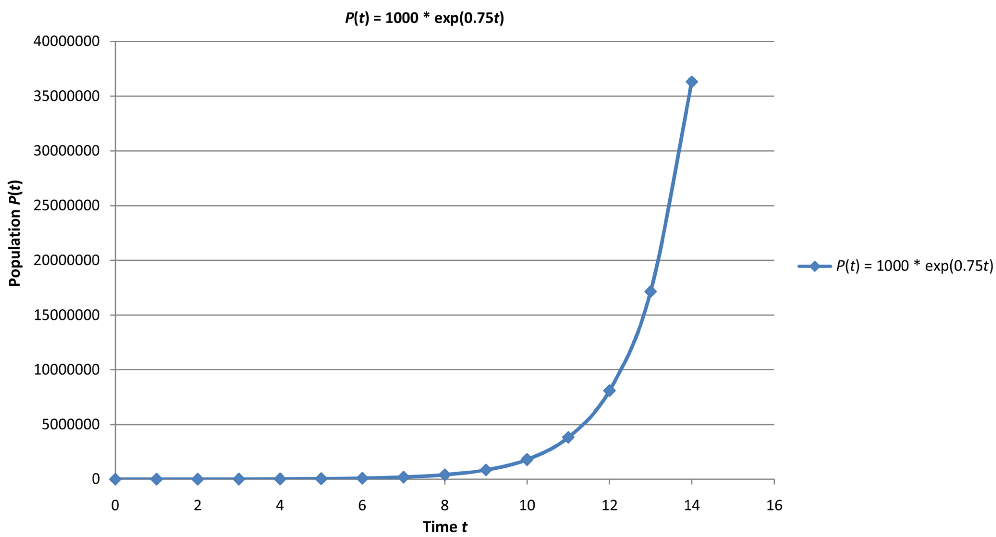

Figure 1. Growth in the population *P*(*t*) of an entity as a function of time *t*.

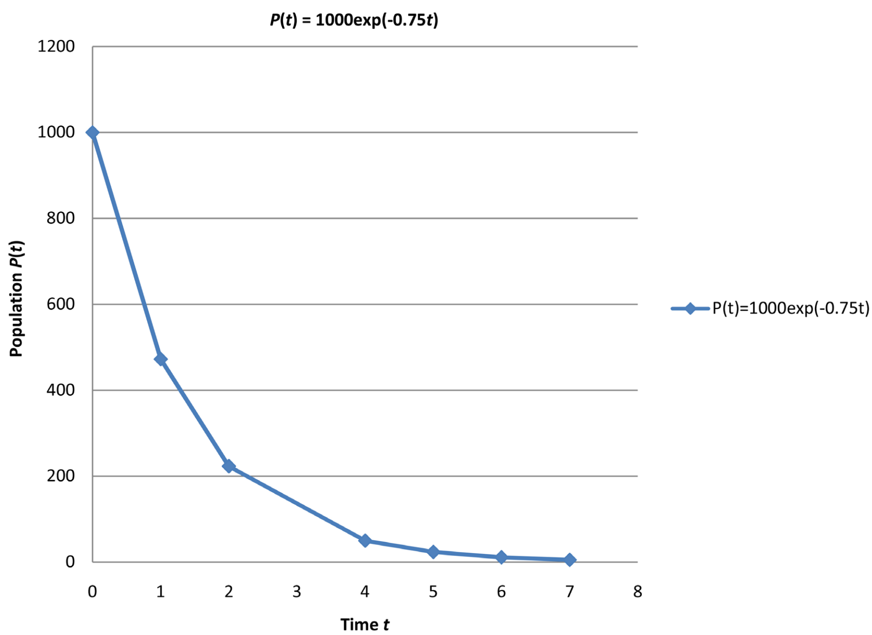

Figure 2. Decrease in the population *P*(*t*) of an entity as a function of time *t*.

This is a decreasing function. Let $P(t)=1000\exp(-0.75t)$ (exponential law)

3. The Logistic Function

Population growth and decay.

3.1. The Law or the Logistic Function

This flaw in the *Malhusian model* suggests the need for a model that accounts for limitations of space and resources that tend to oppose the rate of population growth as the population increases. Perhaps the most famous model of kind is the *Verhulst model*, where 4 is replaced by

$$P'=kP(1-\alpha P), \tag{9}$$

see [5], where α is a positive constant. As long as P is small compared to $1/\alpha$, the ratio P'/P is approximately equal to k. Therefore the growth is approximately exponential; however, as P, the ration P'/P decreases as opposing factors become significant. Equation (9) is the *logistic equation*

$$P'=aP(1-\alpha P)\Leftrightarrow\int\frac{\mathrm{d}P}{aP(1-\alpha P)}=\int\mathrm{d}t$$

$$\Leftrightarrow\ln\left|\frac{P-\frac{1}{\alpha}}{P}\right|=-at+C$$

(where C is a constant)

$$P=\frac{1}{\alpha}\left(\frac{1}{1-k\mathrm{e}^{-at}}\right)$$

and $t=0\Leftrightarrow P(0)=P_0$

$$\Leftrightarrow k=\frac{P_0\alpha-1}{P_0\alpha}$$

$$\Leftrightarrow P(t)=\frac{P_0}{\alpha P_0+(1-P_0\alpha)\mathrm{e}^{-at}}$$

The solution

$$P(t)=\frac{P_0}{\alpha P_0+(1-P_0\alpha)\mathrm{e}^{-at}}$$

where $P_0=P(0)>0$. Therefore

$$\lim_{t\to\infty}P(t)=1/\alpha$$

independent of P_0.

Figure 3 shows typical graphs of P versus t for various values of P_0.

Graphic Representation

See **Figure 3**.

t	P1 = 120/(2.4 - 1.4exp(-0.75t))	P2 = 100/ (2-1exp(-0.75t))	P3 = 90/(1.8 - 0.8exp(-0.75t))	P4 = 60/(1.2 - 0.2exp(-0.75t))	P5 = 50	P6 = 45/(0.9 + 0.1 * exp(-0.75 * t))	P7 = 35 * (0.7 + 0.3exp(-0.75t))	P8 = 20/ (0.1 + 0.9exp(-0.75t))
0	120.00	100.00	90.00	60.00	50.00	45.00	35.00	20.00
1	69.02	65.46	63.29	54.27	50.00	47.51	41.58	29.26
2	57.48	56.28	55.50	51.93	50.00	48.79	45.64	37.46
3	53.28	52.78	52.46	50.89	50.00	49.42	47.84	43.17
4	51.50	51 .28	51.13	50.42	50.00	49.72	48.96	46.53
5	50.70	50.59	50.53	50.20	50.00	49.87	49.50	48.30
6	50.33	50.28	50.25	50.09	50.00	49.94	49.76	49.18
7	50.15	50.13	50.12	50.04	50.00	49.97	49.89	49.61
8	50.07	50.06	50.06	50.02	50.00	49.99	49.95	49.81
9	50.03	50.03	50.03	50.01	50.00	49.99	49.97	49.91
10	50.02	50.01	50.01	50.00	50.00	50.00	49.99	49.96
11	50.01	50.01	50.01	50.00	50.00	50.00	49.99	49.98
12	50.00	50.0031	50.00	50.00	50.00	50.00	50.00	49.99

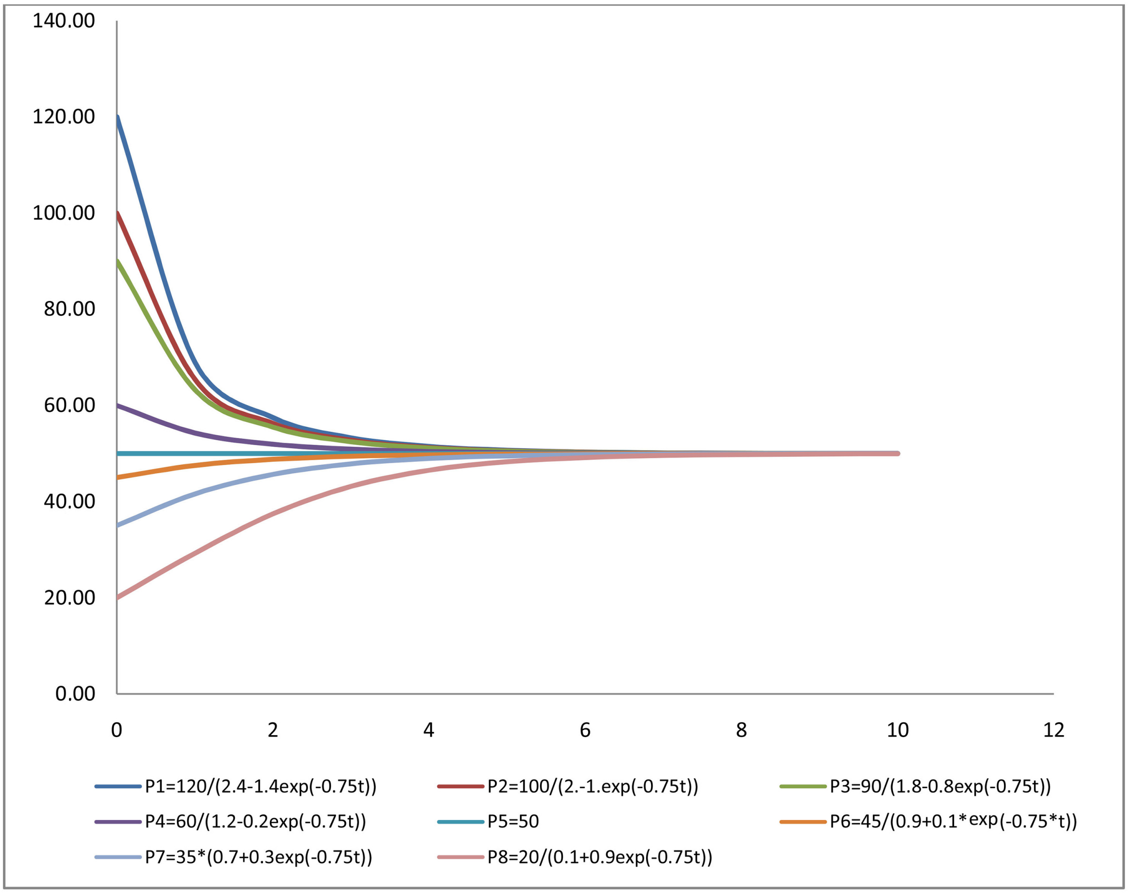

Figure 3. Population evolution with the logistic model.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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