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Quantization of Time Independent Damping Systems Using WKB Approximation

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Abstract

In this work time independent damping systems are studied using Lagrangian and Hamiltonian for time independent damping, which are present through the factor $e^{\lambda q}$. The Hamilton Jacobi equation is formulated to find the Hamilton Jacobi function S using separation of variables technique. We can form this function in compact form of two parts the first part as a function of coordinate q, and the second part as a function of time t. Finally, we find the ability of these systems to quantize through an illustrative example.

Keywords

Quantization, Hamilton Jacobi Equation, Hamilton Jacobi Function, Momentum

1. Introduction

The quantization of constrained systems has been started by Dirac [1]. The quantization of constrained systems has been studied using the WKB approximation [2] [3] [4], where the WKB approximation is semiclassical approximation and it is a basic technique for obtaining an approximate solution to Schrodinger's equation.

The quantization of constrained systems has been studied for first order singular Lagrangians using the WKB approximation [5]. The Hamilton Jacobi functions in configuration space have been obtained by solving the Hamilton Jacobi partial differential equations. This has led to another approach for solving mechanical problems for singular systems in the same manner as for regular systems. By calculating the Hamilton Jacobi function [6] the researchers have constructed the wave function of constrained systems for which the constraints became conditions on it in the semiclassical limit; or, equivalently, they have quantized constrained systems using the WKB approximation.

The WKB approximation has been studied for systems containing fractional derivatives using the canonical method [7] [8]. More recently, a powerful approach, the canonical method, has been developed for dissipative systems [9]. In this approach, the equations of motion are written as total differential equations and the formulation leads to a set of Hamilton Jacobi partial differential equations which are familiar to regular systems. Recently, constrained systems have been studied using fractional WKB approximation by [10]. More recently, quantization of damped systems using fractional WKB approximation has been investigated by [11].

In this paper, we wish to find the ability of the time independent damping systems to quantize using WKB approximation.

This paper is organized as follows: In Section **2**, quantization of time independent damping systems using WKB approximation is discussed. In Section **3**, one illustrative example is studied in detail. The work closes with some concluding remarks in Section **4**.

2. Quantization of Time Independent Damping Systems Using WKB Approximation

The Lagrangian function for time independent damping systems is given by:

$$L = L(q, \dot{q}, t)\mathrm{e}^{\lambda q} \tag{1}$$

And the conjugate momentum is [12]:

$$p_q = \frac{\partial L}{\partial \dot{q}} \tag{2}$$

Using the Lagrangian and the momentum to find the Hamiltonian of damping systems which is written as:

$$H_0 = p\dot{q} - L \tag{3}$$

From this Hamiltonian and by using the p_0, we can find the Hamilton Jacobi equation as:

$$H = p_0 + H_0 = 0 \tag{4}$$

where;

$$p_0 = \frac{\partial S}{\partial t} \tag{5}$$

Then, the Hamilton Jacobi equation takes this final form:

$$H = \frac{\partial S}{\partial t} + H_0 = 0 \tag{6}$$

Now, we will use separation of variables method to find the Hamilton Jacobi function S

$$S(q, E, t) = W(q, E) - f(t) \tag{7}$$

where $W(q, E)$ is a function of coordinate q, and $f(t)$ is a function of time t.

Also we can write;

$$\frac{\partial f}{\partial t}=\frac{\partial S}{\partial t} \tag{8}$$

And

$$\frac{\partial f}{\partial t}=E \tag{9}$$

From the Hamilton Jacobi function, we can find the equation of motion in the following form:

$$\frac{\partial S}{\partial E}=\beta \tag{10}$$

And also the conjugate momentum:

$$\frac{\partial S}{\partial q}=p \tag{11}$$

The momenta are defined as operators in this form:

$$\hat{p}_q=\frac{\hbar}{i}\frac{\partial}{\partial q} \tag{12}$$

Then,

$$\hat{p}_0=\frac{\hbar}{i}\frac{\partial}{\partial t} \tag{13}$$

Now, in quantization area the wave function can be formulated as:

$$\psi(q,E,t)=\psi_0(q)\exp\left[\frac{i}{\hbar}S(q,E,t)\right] \tag{14}$$

where;

$$\psi_0(q)=\frac{1}{\sqrt{p}} \tag{15}$$

So that that the wave function becomes:

$$\psi(q,E,t)=\frac{1}{\sqrt{p}}\exp\left[\frac{i}{\hbar}S(q,E,t)\right] \tag{16}$$

Now, using the momentum as operators from Equation (12) and Equation (13), the Schrodinger equation will be written as

$$\hat{H}\psi=\left[\frac{\hbar}{i}\frac{\partial}{\partial t}\psi-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}-V(q)\right]\psi=0 \tag{17}$$

Now we can show that in the limit $\hbar\to 0$, $\hat{H}\psi=0$. This satisfies the quantization condition.

3. Example

Let us consider one dimensional Lagrangian of a free particle of mass m in the presence of damping [13].

$$L=\frac{1}{2}m\dot{q}^2\mathrm{e}^{\lambda q} \tag{18}$$

From Equation (2) and using Equation (18) the conjugate momentum is:

$$p_q = \frac{\partial L}{\partial \dot{q}} = m\dot{q}\mathrm{e}^{\lambda q} \tag{19}$$

Thus, we can write the coordinate time derivative as

$$\dot{q} = \frac{p}{m}\mathrm{e}^{-\lambda q} \tag{20}$$

Square of Equation (20)

$$\dot{q}^2 = \frac{p^2}{m^2}\mathrm{e}^{-2\lambda q} \tag{21}$$

Making use of Equation (3) and substituting Equation (20) and Equation (18) in it, the Hamiltonian constructed as

$$H_0 = \frac{p^2}{2m}\mathrm{e}^{-\lambda q} \tag{22}$$

Using Equations (6), (11) and (22), Hamilton Jacobi equation will be in the following form:

$$H = \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 \mathrm{e}^{-\lambda q} = 0 \tag{23}$$

Then, the Hamilton Jacobi function is

$$S(q,E,t) = W(q,E) - Et \tag{24}$$

From the previous equation we get;

$$\frac{\partial S}{\partial t} = -E \tag{25}$$

and

$$\frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} \tag{26}$$

Putting Equation (25) and Equation (26) into Equation (23) we obtain:

$$-E + \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 \mathrm{e}^{-\lambda q} = 0 \tag{27}$$

Thus,

$$E = \frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 \mathrm{e}^{-\lambda q} \tag{28}$$

Then,

$$\sqrt{2mE\mathrm{e}^{\lambda q}} = \frac{\partial W}{\partial q} \tag{29}$$

Finally;

$$\int \sqrt{2mE\mathrm{e}^{\lambda q}}\,\mathrm{d}q = W \tag{30}$$

Substituting Equation (30) in to Equation (24) we have,

$$S = \int \sqrt{2mE\mathrm{e}^{\lambda q}}\,\mathrm{d}q - Et \tag{31}$$

From Equation (10) and by using Equation (31) the equation of motion will be

$$\frac{\partial S}{\partial E} = \beta = -t + \int \frac{m\mathrm{e}^{\lambda q}}{\sqrt{2mE\mathrm{e}^{\lambda q}}}\mathrm{d}q \tag{32}$$

Also, by using Equation (11) and Equation (31) the generalized momentum is:

$$\frac{\partial S}{\partial q} = p = \sqrt{2mE\mathrm{e}^{\lambda q}} \tag{33}$$

The example now is ready to quantize;

From Equations (14), (15), (16), (31) and (33), the wave function takes this final form:

$$\psi(q,t) = \left(2mE\mathrm{e}^{\lambda q}\right)^{-1/4} \exp\left[\frac{i}{\hbar}\left[\int \sqrt{2mE\mathrm{e}^{\lambda q}}\,\mathrm{d}q - Et\right]\right] \tag{34}$$

The Hamilton Jacobi Equation (23) using operators takes the following form

$$\hat{H}\psi = \left[\frac{\hbar}{i}\frac{\partial}{\partial t} - \mathrm{e}^{-\lambda q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}\right]\psi \tag{35}$$

After some calculations, we can show that in the limit $\hbar \to 0$, $\hat{H}\psi = 0$. This satisfies the quantization condition.

4. Conclusion

One dimensional Lagrangian of a free particle of mass m in the presence of damping is discussed as an example of damping time independent systems using Lagrangian mechanics and Hamiltonian mechanics which help us to form the Hamilton Jacobi equation from this equation and by using separation of variables method we can find the Hamilton Jacobi function *S*, by using Hamilton Jacobi equation as an operator we find that these systems have an ability to quantize, in the limit $\hbar \to 0$, then, $\hat{H}\psi = 0$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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