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# Nonstandard Unitary Transformations of Quantum States

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## Abstract

In quantum optics, unitary transformations of arbitrary states are evaluated by using the Taylor series expansion. However, this traditional approach can become cumbersome for the transformations involving non-commuting operators. Addressing this issue, a nonstandard unitary transformation technique is highlighted here with new perspective. In a spirit of "quantum" series expansions, the transition probabilities between initial and final states, such as displaced, squeezed and other nonlinearly transformed coherent states are obtained both numerically and analytically. This paper concludes that, although this technique is novel, its implementations for more extended systems are needed.

## Keywords

Taylor Series, Unitary Transformation, Temporal Evolution, Displaced State, Coherent State, Squeezed State, Two-Mode Squeezed State, Holstein-Primakoff State

## 1. Introduction

In quantum optics, a large cluster of quantum states can be constructed via unitary transformations from arbitrary initial states including the ground state [1] [2] [3] [4]. Unitary transformations can be interpreted as temporal evolutions of an initial state governed by specific Hamiltonians. For example, coherent (squeezed) state is constructed by applying an unitary transformation, so-called a displacement (squeeze) operator on initial states. Unlike the displacement operator, the squeeze operator consists of two-photon creation and annihilation operators, in which the nonlinear optical processes are involved. However, numerical evaluations for unitary transformations of arbitrary quantum state based on the classical Taylor series expansion become cumbersome [5] [6] [7] because

the system's Hamiltonian is time-independent and nonlinearly constructed by non-commuting operators.

This issue is addressed in this paper. The unique technique for studying the interactions of injected atoms with a cavity field presented by the author in his series of works [7] [8] [9] more than two decades ago is deepened here with an entirely new perspective in the spirit of "quantum" series expansions. This non-standard unitary transformation technique is established as follows. First, the systems with only time-independent Hamiltonians are considered here. Realizations of quantum dynamics of the systems governed by time-dependent Hamiltonians have been recently reported by the author in [10]. Second, similarly as in Ref. [10], the classical force terms are replaced with spin operators. Utilizing these two-state raising and lowering operators, various unitary transformations are realized here. As an example, both analytical and numerical results for the transition probabilities from arbitrary quantum states to the displaced [1] [3] [4] [7], single- [1] [4] [7] [11] [12] and two-mode squeezed [4] [9] [13] and Holstein-Primakoff $SU(1,1)$ transformed states [8] [14] are obtained.

The paper is organized as follows. In the next section, the standard approach for unitary transformations to produce the-above mentioned states is presented. In Section 3, our nonstandard technique for time-independent Hamiltonian systems is introduced and implemented demonstrating the realizations of the transformed arbitrary states. The final section concludes with some remarks.

## 2. Standard Unitary Transformations

Unitary transformations can be expressed as temporal evolutions as [1] [2] [3] [4]

$$\left|\psi(t)\right\rangle = \hat{U}\left|\psi(0)\right\rangle = \mathrm{e}^{-i\hat{V}t}\left|\psi(0)\right\rangle \tag{1}$$

here $\hbar \equiv 1$ and $\left|\psi(t)\right\rangle$ and $\left|\psi(0)\right\rangle$ are the transformed final state at time $t$ and the initial state at time zero. The Hamiltonian $\hat{V}$ is the Hermitian and satisfies $\hat{V} = \hat{V}^{\dagger}$. In the interaction picture and under the rotating wave approximation, the time-independent Hamiltonian for the given quantum system is explicitly written as [1]

$$\hat{V} = \hat{R}\varepsilon^{*} + \hat{R}^{\dagger}\varepsilon \tag{2}$$

where $\hat{R}$ and $\hat{R}^{\dagger}$ are non-commuting boson annihilation and creation operators; $\varepsilon$ and $\varepsilon^{*}$ are the time-independent force terms. Unitary operator $\hat{U}$ satisfies $\hat{U}^{\dagger} = \hat{U}^{-1}$. Operators $\hat{R}, \hat{R}^{\dagger}$ constitute the generalized boson oscillator algebra [15] [16] basis $\left\{\hat{1}, \hat{R}, \hat{R}^{\dagger}, \hat{n}\right\}$ which satisfies the relations,

$$\hat{R}^{\dagger}\hat{R} = \hat{\Psi}(n),\ \hat{R}\hat{R}^{\dagger} = \hat{\Psi}(n+1),\ \left[\hat{R}, \hat{n}\right] = \hat{R},\ \left[\hat{R}^{\dagger}, \hat{n}\right] = -\hat{R}^{\dagger}, \tag{3}$$

where $\hat{n}$ is the excitation number operator and real nonnegative structure function $\Psi(n)$ ($\Psi(n) \geq 0$, $\forall n \geq 0$) characterizes the given system. A vacuum (ground) state $\left|0\right\rangle$ is defined as $\hat{R}\left|0\right\rangle = 0$, $\Psi(n) > 0$, $\forall n > 0$, see **Table 1**. Using Equation (1) for $\hat{U}$, it is aimed to obtain final state $\left|\psi(t)\right\rangle$ for the initial state including vacuum (ground) state $\left|\psi(0)\right\rangle$.

**Table 1.** Examples for unitary transformations with time-independent Hamiltonian.

| | Hamiltonian $\hat{H}$ | $\hat{R}$ | $\hat{n}$ | $\Psi(x)$ | States |
|---|---|---|---|---|---|
| 1. | $\hat{b}\varepsilon^* + \hat{b}^\dagger\varepsilon$ | $\hat{b}$ | $\hat{b}^\dagger\hat{b}$ | $x$ | Displaced |
| 2. | $\hat{b}^2\varepsilon^* + \hat{b}^{\dagger 2}\varepsilon$ | $\hat{b}^2$ | $\frac{\hat{b}^\dagger\hat{b}}{2}$ | $2x(2x-1)$ | Squeezed |
| 3. | $\hat{b}\sqrt{\hat{b}^\dagger\hat{b}}\varepsilon^* + \sqrt{\hat{b}^\dagger\hat{b}}\hat{b}^\dagger\varepsilon$ | $\hat{b}\left(\hat{b}^\dagger\hat{b}\right)^{1/2}$ | $\hat{b}^\dagger\hat{b}$ | $x^2$ | Holstein-Primakoff |
| 4. | $\hat{b}_1\hat{b}_2\varepsilon^* + \hat{b}_1^\dagger\hat{b}_2^\dagger\varepsilon$ | $\hat{b}_1\hat{b}_2$ | $\frac{\hat{b}_1^\dagger\hat{b}_1 + \hat{b}_2\hat{b}_2^\dagger}{2}$ | $x_1x_2$ | Squeezed |

### 2.1. Displaced Vacuum States: $\hat{R} \equiv \hat{b}$

Coherent state $|\gamma\rangle$ is defined as displaced vacuum state $|0\rangle$, which can also be expressed in terms of the Fock states $|n\rangle$ as [1] [3] [4]

$$\langle n|\gamma\rangle = \langle n|e^{\gamma\hat{b}^\dagger - \gamma^*\hat{b}}|0\rangle = e^{-\bar{n}/2}\frac{\gamma^n}{\sqrt{n!}}. \tag{4}$$

here the mean photon number is $\bar{n} = |\gamma|^2$. The probability of detecting $n$ photons in coherent state follows the Poissonian distribution $P(n) = |\langle n|\gamma\rangle|^2 = e^{-\bar{n}}\bar{n}^n/n!$. Equivalently, coherent state is also defined as an eigenstate of annihilation operator as $\hat{b}|\gamma\rangle = \gamma|\gamma\rangle$. In the Fock state representation, the ground state or vacuum state $|\psi(0)\rangle \equiv |0\rangle$ satisfies $\hat{b}|0\rangle = 0$ and $\langle\psi(0)|\hat{b}^\dagger = 0$, which are reduced from definitions such as $\hat{b}|n\rangle = \sqrt{n}|n-1\rangle$ and $\hat{b}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. From Equations (2) and (4), the unitary transformation and the displacement operator are equivalent when $\gamma = it\varepsilon$ and the final state reads as $|\psi(t)\rangle \equiv |\gamma\rangle$.

### 2.2. Squeezed Vacuum States: $\hat{R} \equiv \hat{b}^2$

Squeezed vacuum state $|\gamma\rangle$ is defined as single mode of the electromagnetic field where the fluctuations of one of the two quadrature become less than the short noise level [1] [3] [4]. Squeezing the ground (vacuum) state $|0\rangle$ is expressed in terms of the Fock states $|2n\rangle$ as

$$\langle 2n|\gamma\rangle = \langle 2n|e^{\frac{1}{2}\left(\gamma^*\hat{b}^2 - \gamma\hat{b}^{\dagger 2}\right)}|0\rangle = \frac{1}{\sqrt{\cosh r}}\left(-e^{i\phi}\tanh r\right)^n\frac{\sqrt{(2n)!}}{2^n n!} \tag{5}$$

here $\gamma = re^{i\phi}$ and the mean photon number is $\bar{n} = \sinh^2 r$. From Equations (2) and (5), it is worth to note again that the unitary transformation and the displacement operator are identical when $\gamma = -2it\varepsilon$.

### 2.3. The Holstein-Primakoff $SU(1,1)$ Transformations: $\hat{R} \equiv \hat{b}\sqrt{\hat{b}\hat{b}^\dagger}$

The Holstein-Primakoff $SU(1,1)$ coherent state $|\gamma\rangle$ is defined as [8] [14] [17]

$$\langle n|\gamma\rangle = \langle n|e^{\gamma\sqrt{\hat{b}\hat{b}^\dagger}\hat{b}^\dagger - \gamma^*\hat{b}\sqrt{\hat{b}\hat{b}^\dagger}}|0\rangle = \frac{1}{\cosh r}\left(-e^{i\phi}\tanh r\right)^n \tag{6}$$

here $\gamma = -re^{i\phi}$ and the mean photon number is $\bar{n} = \sinh^2 r$. Comparing Equations (2) and (6), it is important to note that $\gamma = -it\varepsilon$.

### 2.4. Two-Mode Squeezed Vacuum States: $\hat{R} \equiv \hat{b}_1\hat{b}_2$

Two-mode squeezing of vacuum $|0,0\rangle$ is expressed in terms of the Fock states $|n,n\rangle$ as [4] [9] [13]

$$\langle n,n|\gamma\rangle = \langle n,n|e^{\gamma^*\hat{b}_1\hat{b}_2 - \gamma\hat{b}_1^\dagger\hat{b}_2^\dagger}|0,0\rangle = \frac{1}{\cosh r}\left(-e^{i\phi}\tanh r\right)^n \tag{7}$$

here $\gamma = re^{i\phi}$ and the mean photon number is $\bar{n} = \sinh^2 r$ and comparing Equations (2) and (7), it is identified as $\gamma = it\varepsilon$.

## 3. Nonstandard Unitary Transformations

The author aims to solve for final state $|\psi(t)\rangle$ in Equation (1) using our nonstandard technique previously introduced in Ref. ([10]) but, now, for the systems associated with time-independent Hamiltonians. It is formulated as: 1) the classical force terms are replaced with operators in the time-independent Hamiltonian Equation (2); 2) the density operator for the system under consideration is evolved after tracing over this force operator space. A time interval $t$ is divided into $N$ subintervals with a width of $\Delta t$ when $\Delta t \ll 1$ and $N \gg 1$ but $t = N\Delta t$ being finite. The Hamiltonian is expanded to an additional operator space $\hat{A}$ as

$$\hat{V} \to \hat{V} \otimes \hat{A} \tag{8}$$

where $\hat{V}$ and $\hat{A}$ commute and belong to different operator spaces. For example, new operator $\hat{A}$ can be chosen as

$$\hat{A} = |\alpha|^2|\uparrow\rangle\langle\uparrow| + \alpha^*\beta|\downarrow\rangle\langle\uparrow| + \alpha\beta^*|\uparrow\rangle\langle\downarrow| + |\beta|^2|\downarrow\rangle\langle\downarrow| \tag{9}$$

with $|\alpha|^2 + |\beta|^2 = 1$. For this choice, the force terms are replaced with the raising $|\uparrow\rangle\langle\downarrow|$ and lowering $|\downarrow\rangle\langle\uparrow|$ operators as

$$\begin{aligned} \varepsilon &\to \hat{\varepsilon} \equiv \eta|\downarrow\rangle\langle\uparrow| \\ \varepsilon^* &\to \hat{\varepsilon}^\dagger \equiv \eta^*|\uparrow\rangle\langle\downarrow| \end{aligned} \tag{10}$$

here $\hat{\varepsilon}$ and $\hat{R}$ commute. The original Hamiltonian $\hat{V}$ in Equation (2) is replaced with a modified Hamiltonian $\hat{V}_A$ as

$$\hat{V}_A = \hat{R}\hat{\varepsilon}^* + \hat{R}^\dagger\hat{\varepsilon} = \eta^*\hat{R}|\uparrow\rangle\langle\downarrow| + \eta\hat{R}^\dagger|\downarrow\rangle\langle\downarrow| \tag{11}$$

This is known as the unified Jaynes-Cummings Hamiltonian [15] [16]. A new iterative relation for the modified density operator $\rho_j$ using the modified Hamiltonian given in Equation (11) is introduced here as the recurrent solution of the von Neumann equation for time evolution in $\Delta t$ as

$$\hat{\rho}_j = \mathrm{Tr}_A\left[e^{-i\hat{V}_A\Delta t}\hat{A}\otimes\hat{\rho}_{j-1}e^{i\hat{V}_A\Delta t}\right] \tag{12}$$

here $\hat{A} = |\alpha|^2|\uparrow\rangle\langle\uparrow| + \alpha\beta^*|\uparrow\rangle\langle\downarrow| + \alpha^*\beta|\downarrow\rangle\langle\uparrow| + |\beta|^2|\downarrow\rangle\langle\downarrow|$ with $|\alpha|^2 + |\beta|^2 = 1$.

Next it is demonstrated here that the results obtained by the two techniques merge as

$$\hat{\rho}_N \simeq \hat{\rho}_N \tag{13}$$

for the same pure initial state $|\psi(0)\rangle$ which stands for $\hat{\rho}_0 = \hat{\rho}_0 = |\psi(0)\rangle\langle\psi(0)|$ and $\hat{\rho}_N = |\psi(t)\rangle\langle\psi(t)|$. In the Fock state representation, the iterative relation for the $j$th density matrix element $\rho_j(0,n) = \langle 0|\hat{\rho}_j|n\rangle$ is given by

$$\begin{aligned}\rho_j(0,n) = &\left[|\alpha|^2 C_1 C_{n+1} + |\beta|^2 C_n\right]\rho_{j-1}(0,n) + |\beta|^2 S_1 S_{n+1}\rho_{j-1}(1,n+1)\\ &+ i\alpha_j\beta^* C_1 S_{n+1}\rho_{j-1}(0,n+1) + i\alpha^*\beta S_n\rho_{j-1}(0,n-1)\\ &- i\alpha^*\beta S_1 C_{n+1}\rho_{j-1}(1,n)\end{aligned} \tag{14}$$

where coefficients $C_n = \cos\left(\eta\Delta t\sqrt{\Psi(n)}\right)$ and $S_n = S_n(\eta\Delta t) = \sin\left(\eta\Delta t\sqrt{\Psi(n)}\right)$. For $\Delta t \to 0$, the author uses the approximations $C_n \simeq 1$ and $S_n \simeq \eta_j\Delta t\sqrt{\Psi(n)}$. A transformation is introduced as [7] [8] [9] [18]

$$\rho_j(0,n) = S_n(\eta\Delta t)!\tilde{\rho}_j(0,n) \simeq (\eta\Delta t)^n\sqrt{\Psi(n)!}\tilde{\rho}_j(0,n) \tag{15}$$

here the former is denoted $\prod_{k=1}^{n} S_k(y) \equiv S_n(y)!$ with $S_0(y)! \equiv 1$. Similarly, the latter is denoted $\prod_{k=1}^{n}\Psi(k) \equiv \Psi(n)!$ with $\Psi(0)! \equiv 1$ (see **Table 1**. for the explicit expressions of $\Psi(k)$). Therefore, the reduced equation for $\tilde{\rho}_j(0,n)$ becomes

$$\tilde{\rho}_j(0,n) = \tilde{\rho}_{j-1}(0,n) + i\zeta\tilde{\rho}_{j-1}(0,n-1) \tag{16}$$

here $\zeta \equiv \alpha^*\beta$ is a parameter representing coherence between $|\uparrow\rangle$ and $|\downarrow\rangle$ spin states. Vacuum state is expressed as $\rho_0(0,n) = \tilde{\rho}_0(0,n) = \delta_{0,n}$. Solving Equation (16) and using the transformation given in Equation (15), the density matrix elements for the output state $\rho_N(0,n)$ are expressed in terms of the density matrix elements for arbitrary, but not for necessarily ground state as input state $\rho_0(0,n)$ are found to be

$$\rho_N(0,n) \simeq \mathcal{A}_0^{(0)}(\gamma_A)\sum_{k=0}^{n}\mathcal{A}_k^{(n-k)}(\gamma_A^*)\rho_0(0,n-k) \tag{17}$$

with $\gamma_A = -i\alpha\beta^*\eta\Delta tN = -i\eta\zeta^* t$ and "$\mathcal{A}$-functions" $\mathcal{A}_k^{(n)}(x)$ are defined as

$$\mathcal{A}_k^{(n)}(x) = \frac{1}{\sqrt{L}}\frac{x^k}{k!}\frac{S_{n+k}(y)!}{y^k S_n(y)!} \simeq \frac{1}{\sqrt{N}}\frac{x^k}{k!}\left(\frac{\Psi(n+k)!}{\Psi(n)!}\right)^{1/2} \tag{18}$$

here $\mathcal{L}$ and $\mathcal{N}$ are normalization coefficients and $\mathcal{A}_0^{(0)}(x)$ is either $1/\sqrt{\mathcal{L}}$ or $1/\sqrt{\mathcal{N}}$. Thus, the relation for the probability amplitudes between (pure) input and (pure) output states in terms of wave functions are obtained

$$\langle n|\psi(t)\rangle \simeq \sum_{k=0}^{n}\mathcal{A}_k^{(n-k)}(\gamma_A)\langle n-k|\psi(0)\rangle \tag{19}$$

Moreover, the solution in Equations (17) and (18) are obtained under the conditions $\alpha,\beta \neq 0$, $N \ll 1$, $\eta\Delta t\sqrt{\Psi(n)} \ll 1$ and $n \ll N$. In particular, for nonlinear Hamiltonians, it must be $|\gamma_A| < 1$, which excludes the case $\hat{R} = \hat{b}$ for

displaced states. In the case of the input vacuum state $\langle n-k|\psi(0)\rangle=\delta_{n-k,0}$, the solution in Equation (19) simplified to

$$\langle n|\psi(t)\rangle\simeq\mathcal{A}_n^{(0)}(\gamma_A)=\frac{1}{\sqrt{\mathcal{L}}}\frac{\gamma_A^n}{n!}\frac{S_n(\eta\Delta t)!}{(\eta\Delta t)^n}\simeq\frac{1}{\sqrt{\mathcal{N}}}\frac{\gamma_A^n\sqrt{\Psi(n)!}}{n!} \tag{20}$$

Thus, it reads explicitly as

$$|\psi(t)\rangle\simeq\sum_{n=0}^{\infty}\mathcal{A}_n^{(0)}(\gamma_A)|n\rangle=\frac{1}{\sqrt{\mathcal{L}}}\sum_{n=0}^{\infty}\frac{\gamma_A^n}{n!}\frac{S_n(\eta\Delta t)!}{(\eta\Delta t)^n}|n\rangle\simeq\frac{1}{\sqrt{\mathcal{N}}}\sum_{n=0}^{\infty}\frac{\gamma_A^n\sqrt{\Psi(n)!}}{n!}|n\rangle \tag{21}$$

with $\gamma_A=-i\eta\zeta^*t$. Depending on the explicit form of the structure function $\Psi(n)$, the displaced, squeezed, $SU(1,1)$ transformed and two-mode squeezed vacuum states are realized by Equation (21) by using our nonstandard technique. Next, it is assumed that an initial state is not vacuum but given as the same type of state $|\gamma_0\rangle$ such as

$$\langle n|\psi(0)\rangle=\mathcal{A}_n^{(0)}(\gamma_0)=\frac{1}{\sqrt{\mathcal{N}_0}}\frac{\gamma_0^n\sqrt{\Psi(n)!}}{n!}=\langle n|\gamma_0\rangle \tag{22}$$

From Equation (19), the final state is found to be $|\gamma_A+\gamma_0\rangle$ as

$$\langle n|\psi(t)\rangle\simeq\frac{1}{\sqrt{\mathcal{N}\mathcal{N}_0}}\frac{(\gamma_A+\gamma_0)^n\sqrt{\Psi(n)!}}{n!}\simeq\langle n|\gamma_A+\gamma_0\rangle \tag{23}$$

where the initial displaced, squeezed and $SU(1,1)$ transformed $|\gamma_0\rangle$ states are further displaced (or squeezed) to be $|\gamma_A+\gamma_0\rangle$. It is worth to note that further combinations of types of these states and transformations are clearly realized. For example, squeezing (displacing) of initially displaced (squeezed) vacuum state leads to the realization of squeezed (coherent) coherent (squeezed) states and so on [7].

## 4. Conclusion

One of the challenges in quantum optics is that when the system's Hamiltonian is time-independent and composed of non-commuting operators, then the numerical evaluations for the unitary transformations of arbitrary quantum state become cumbersome. Addressing this issue, the previous nonstandard unitary transformation technique associated with the unified Jaynes-Cummings model is extended with a new perspective as "quantum" series expansions against the classical Taylor series expansions. In particular, 1) The systems with time-independent Hamiltonians are considered; 2) The classical force terms are replaced with quantum operators; 3) The density matrix for the system under consideration is evolved after tracing over the force operator space. As a proof of principle of this nonstandard technique, utilizing the spin operators as force operators, both analytical and numerical results for the transition probabilities from arbitrary initial states to the final states such as displaced, single- and two-mode squeezed and Holstein-Primakoff $SU(1,1)$ transformed states are obtained. Finally, it is concluded that, although, this technique is novel, its implementations

for more general and extended systems are needed.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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