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Relativistic Corrections to the Maxwellian Distribution for Astrophysical and Fusion Plasmas

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Abstract

We present calculations and improvement inspired by the work of Lorenzo Zaninetti, published in 2020, it concerns a problem whose origin dates back 1911 with so called Maxwell-Jüttner distribution these lies on the Lorentz factor $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, with $\beta = \frac{v}{c}$. This work uses powerful modern software for a reconstruction of Zaninetti work, which computes with special functions, these are included in the Mathematica software, as by instance Bessel and Meijer G-functions ready to manipulate. A progress is made, it is possible to perform an integral that is not computed in Zaninetti paper. This author connects the correct relativistic probability law: the Maxwell-Jüttner to the synchrotron emissivity with a magnetic $\boldsymbol{B}$ field, this work generalize these results, using the linear Stark effect and deals with an electric field $\boldsymbol{E}$.

Keywords

Maxwell-Jüttner Distribution, Relativity Modification to Maxwell Law, Lorentz Factor, Linear Stark Effect, Plasma Physics

1. Introduction

The Maxwell distribution is fundamental for statistical behavior of identical particles of mass *m*, in a environment of temperature $T(K)$, it starts with the Boltzmann theoretical thermodynamic law, which is $f(E) = \mathrm{e}^{-\frac{E}{k_B T}}$ [1]. Setting $E = \frac{m \cdot v^2}{2}$, one obtains the statistical distribution of identical particles of mass *m*. In order to obtain a true probability law, one integrates the Boltzmann law on a velocity range v from $0 \to \infty$, this leads to the correct classical Maxwellian distribution [2].

The theory of relativity developed by Albert Einstein (1905) imposes that the maximal physical velocity is $v = c$, with c is the velocity of light in vacuo.

Taking into account the modification of the velocity limit, for high energy particles, the Maxwell distribution is changed into the Maxwell-Jüttner distribution, (1911) [3]. This paper shows how is the change from standard Maxwellian, with numerical evidences.

Reminding the Maxwell distribution: that is:

$$F_{Maxwell}\left(v, m, k_B, T_{MB}\right) = m^{\frac{3}{2}} \times \frac{\sqrt{2} v^2 \times \mathrm{e}^{-\frac{m \times v^2}{k_B T}}}{\sqrt{\pi} \times \left(k_B T\right)^{\frac{3}{2}}} \tag{1}$$

[4] This function $F_{Maxwell}\left(v, m, k_B, T_{MB}\right)$ is a normalized probability law:

$$F_{Maxwell}\left(v, m, k_B, T\right) = \int_0^{\infty} m^{\frac{3}{2}} \times \frac{\sqrt{2} v^2 \times \mathrm{e}^{-\frac{m \times v^2}{k_B T}}}{\sqrt{\pi} \times \left(k_B T\right)^{\frac{3}{2}}} \mathrm{d}v = 1 \tag{2}$$

[4] At the origin of this article is the interest of the author (since a long time) on how to modify the Maxwellian distribution for particles of mass m, in surrounding environment of temperature T, considering the finite value of the velocity of light c in vacuo, then replacing $v = \infty$ into $v = c$ in the Maxwellian distribution is an interesting physical problem, it has important consequences for high temperatures up from $T = 10^5\ \mathrm{K}$ existing in laboratory fusion plasmas and astrophysical plasmas observed in X rays, notably in Supernovae explosions. The author came across the work of L. Zaninetti (2020) [4], in his paper changes of variables and the introduction of the Lorentz factor: γ are perfectly shown.

Dealing with relativity, the kinetic energy has to be changed according to the famous Albert Einstein formula: $E = mc^2 = \frac{m_0 c^2}{\sqrt{1-\beta^2}}$, where m_0 is the rest mass of the particle and $\beta = \frac{v}{c}$, and the Lorentz factor follows:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{3}$$

The relativistic energy follows from:

$$E_{Kin} = mc^2 - m_0 c^2 \tag{4}$$

$$E_{Kin} = \frac{m_0 c^2}{\sqrt{1-\beta^2}} - m_0 c^2 \tag{5}$$

$$E_{Kin} = m_0 c^2 \times \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) \tag{6}$$

$$E_{Kin} = m_0 c^2 \left(\gamma - 1\right) \tag{7}$$

This is the formula (3) [4] and in [5] formula (10). To install relativity in the

Maxwell distribution, I shall follow the Zaninetti way in [4], his function is:

$$f_v(v,T)=v^2\frac{\mathrm{e}^{\frac{1}{T}\left(1-\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)}}{\int_0^c w^2\mathrm{e}^{\frac{1}{T}\left(1-\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)}\mathrm{d}w} \tag{8}$$

It is a not invariant in relativistic transformations. Following: [4] it is necessary to change the variable v in this way:

$$v=\frac{\sqrt{\gamma^2-1}}{\gamma} \text{ and } \mathrm{d}v=\frac{1}{\sqrt{\gamma^2-1}\gamma^2}\mathrm{d}\gamma .$$

Thus the distribution becomes: the relativistic MB in the variable γ

$$f_\gamma(\gamma,T)=\frac{\sqrt{\gamma^2-1}}{\gamma^4}\times\mathrm{e}^{\frac{1}{T}(1-\gamma)} \tag{9}$$

The normalization of the distribution is performed:

$$N(T)=\int_1^\infty\frac{\sqrt{\gamma^2-1}}{\gamma^4}\times\mathrm{e}^{\frac{1}{T}(1-\gamma)}\mathrm{d}\gamma \tag{10}$$

This integral is given in [4] formula (7), giving the distribution $f_\gamma(\gamma,t)$ this contains a special function: the Meijer G-function. Using Mathematica 12.2 the library has this function ready to use the MeijerG function [6]:

It happens that the distribution PDF of [4] is given by:

$$f_\gamma(\gamma,T)=\frac{32\sqrt{\gamma^2-1}T^3\mathrm{e}^{\frac{1-\gamma}{T}-\frac{1}{T}}}{\gamma^4G_{1,3}^{3,0}\left(\frac{0.25}{T^2}\,\middle|\,\begin{matrix}1\\-\frac{3}{2},-1,-\frac{1}{2}\end{matrix}\right)} \tag{11}$$

The integral of the PDF follows: $F_\gamma(\gamma,T)=\int_1^\gamma f_\gamma(\gamma,T)\mathrm{d}\gamma$, but written as the PDF [4], is not analytical, thus can only be numerically integrated. The novelty of this work is to perform all relevant integrals and formal derivatives of these quantities bypassing the use MeijerG functions of the Zaninetti article [4].

At this stage, it is possible to get new results using Mathematica but also, without using MeijerG functions:

$F_\gamma(\gamma_{\max},T)=Integrate\left[\frac{\sqrt{\gamma^2-1}}{\gamma^4}\times\mathrm{e}^{\frac{1}{T}(1-\gamma)},\{\gamma,1,\gamma_{\max}\}\right]$ is not integrable analytically, but can be numerically integrated with the NIntegrate function producing a table with a fixed step, this is done in [4], thus constructing the DF (distribution function) with accuracy.

If the approximation $\sqrt{\gamma^2-1}\approx\gamma$, is made it simplifies.

The integral is analytical:

$$F_\gamma(T)=Integrate\left[\frac{1}{\gamma^3}\times\mathrm{e}^{\frac{1-\gamma}{T}},\gamma\right]=\frac{\gamma^2\mathrm{Ei}\left(-\frac{\gamma}{T}\right)+T\mathrm{e}^{-\frac{\gamma}{T}}(\gamma-T)}{2\gamma^2T^2} \tag{12}$$

The asymptotic probability law of the relativistic MB is given by:

$$f_\gamma(\gamma,T)=\frac{\gamma^2\mathrm{Ei}\left(-\frac{\gamma}{T}\right)+T\mathrm{e}^{-\frac{\gamma}{T}}(\gamma-T)}{2N(T)\gamma^2T^2} \quad (13)$$

$$N(T)=-\frac{\mathrm{e}^{1/T}\mathrm{Ei}\left(-\frac{1}{T}\right)-T^2+T}{2T^2} \quad (14)$$

It can be compared with the data of the numerical integral, for low γ values $\gamma\geq 1$ and $\gamma\leq 4$.

It gives the same DF than the function numerically integrated from the PDF [4], it is a new result.

Figure 1 shows the distribution of the relativistic Maxwell-Boltzmann formula given in Equation (9) for 3 temperatures T.

There are no differences in the Figures for the numerically integrated DF and these obtained with our approximation.

The mode of the relativistic MB is given by deriving and solving:

$$f_\gamma(T)'=0 \quad (15)$$

$$f_\gamma(T)'=\frac{\mathrm{e}^{\frac{1-\gamma}{T}}}{\gamma^3\sqrt{\gamma^2-1}}-\frac{4\sqrt{\gamma^2-1}\mathrm{e}^{\frac{1-\gamma}{T}}}{\gamma^5}-\frac{\sqrt{\gamma^2-1}\mathrm{e}^{\frac{1-\gamma}{T}}}{\gamma^4T} \quad (16)$$

Solving $f_\gamma(T)'=0$ gives the mode:

The real solution of the cubic equation (the same than obtained, formula (11) in [4]) without using the Meijer G-function.

$$\gamma^3-\gamma+3\gamma^2T-4T=0 \quad (17)$$

The real root is:

$$mode(T)=\frac{\sqrt[3]{-54T^3+\sqrt{-11664T^4+5589T^2-T-108}+81T}}{3\sqrt[3]{2}} \quad (18)$$

$$-\frac{\sqrt[3]{2}\left(-9T^2-3\right)}{3\sqrt[3]{-54T^3+\sqrt{-11664T^4+5589T^2-T-108}+81T}}-T \quad (19)$$

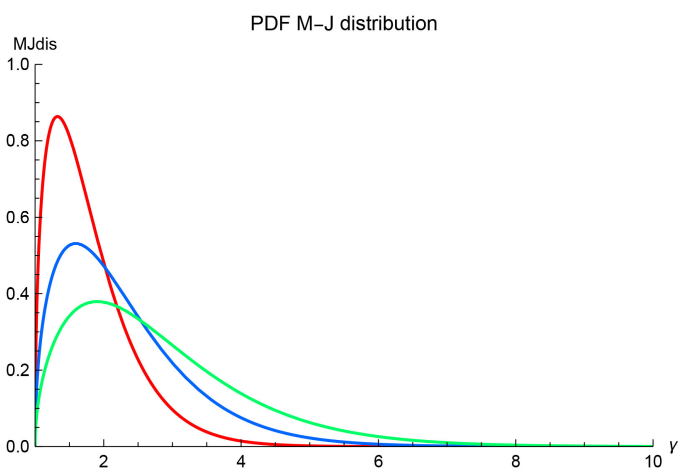

Figure 1. PDF of the relativistic MB as a function of γ for different values of T.

Another approach is possible, to find the distribution function DF, using the v velocity variable with $c = 1$.

That is $\gamma\gamma(v) = \frac{1}{\sqrt{1-v^2}}$, the PDF function becomes:

$$\mathrm{FDibv}(v,T) = \frac{\sqrt{\gamma\gamma(v)^2 - 1}\exp\left(\frac{1-\gamma\gamma(v)}{T}\right)}{\gamma\gamma(v)^4} \tag{20}$$

and its derivative $\mathrm{FDibv}'(v,T)$ is:

$$\mathrm{FDibv}'(v,T) = -4\left(1-v^2\right)\sqrt{\frac{1}{1-v^2}-1}\,v\mathrm{e}^{\frac{1-\frac{1}{\sqrt{1-v^2}}}{T}}$$
$$-\frac{\sqrt{1-v^2}\sqrt{\frac{1}{1-v^2}-1}\,v\mathrm{e}^{\frac{1-\frac{1}{\sqrt{1-v^2}}}{T}}}{T} + \frac{v\mathrm{e}^{\frac{1-\frac{1}{\sqrt{1-v^2}}}{T}}}{\sqrt{\frac{1}{1-v^2}-1}}$$

Solving $\mathrm{FDibv}'(v,T) = 0$ gives the mode.

$$mode(T) = \left(-\frac{1}{12T^2} + \frac{\sqrt[3]{-216T^4 + 36T^2 + 24\sqrt{3}\sqrt{27T^8 - T^6} - 1}}{12T^2}\right.$$
$$-\frac{2}{\sqrt[3]{-216T^4 + 36T^2 + 24\sqrt{3}\sqrt{27T^8 - T^6} - 1}}$$
$$\left.+\frac{1}{12T^2\sqrt[3]{-216T^4 + 36T^2 + 24\sqrt{3}\sqrt{27T^8 - T^6} - 1}} + 1\right)^{\frac{1}{2}}$$

2. Main Results

It is accepted that relativistic effects are taken into account with the Maxwell-Jüttner distribution:

$$f_{MJ}(\gamma,\Theta) = \frac{\gamma\sqrt{\gamma^2 - 1}\times \mathrm{e}^{\frac{-\gamma}{\Theta}}}{\Theta \times K_2\left(\frac{1}{\Theta}\right)} \tag{21}$$

where $\Theta = \sqrt{\frac{k_B T_{MB}}{mc^2}}$.

Following: [4], m is the mass of the atom, and k_B is the Boltzmann constant, and the T_{MB} the temperature of the medium.

Using the Zaninetti formula for the Maxwell-Jüttner distribution and its average value which implies the function [6]:

$$G_{1,3}^{2,1}\left(\frac{0.25}{T^2}\,\middle|\,\begin{matrix}1\\-0.5,-1,-1.5\end{matrix}\right) \tag{22}$$

Instead it is possible to calculate all relevant quantities as average value or variance of the Maxwell-Jüttner distribution with the useful.

Figure 2 Results obtained by calculating DF with numerical integration [4].

Figure 3 Results obtained calculating DF by analytical integration, new way to get all quantities appearing in [4].

Figure 4 Exact formal Maxwell-Jüttner distribution, with the $\Theta = \sqrt{\frac{k_B T_{MB}}{mc^2}}$ variable [3]

$$f_{MJ}(\gamma, \Theta) = \frac{\gamma^2 \times \mathrm{e}^{\frac{-\gamma}{\Theta}}}{\Theta \times K_2\left(\frac{1}{\Theta}\right)} \tag{23}$$

It gives:

$$f_{MJ}(\gamma_{\max}, \Theta) = \frac{\mathrm{e}^{-1/\Theta}\Theta\left(2\Theta(\Theta+1)+1\right) - \Theta\mathrm{e}^{-\frac{\gamma_{\max}}{\Theta}}\left(\gamma_{\max}^2 + 2\gamma_{\max}\Theta + 2\Theta^2\right)}{\Theta K_2\left(\frac{1}{\Theta}\right)} \tag{24}$$

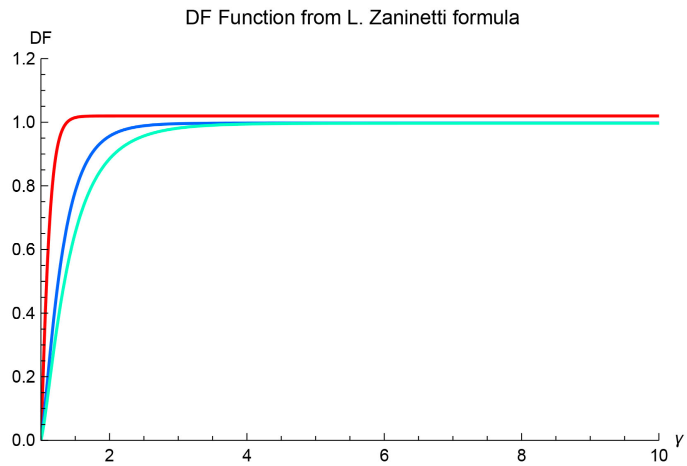

Figure 2. L. Zaninetti DF Distribution Function of the relativistic MB obtained by numerical integration: $T = 0.1$ red curve, $T = 0.5$ blue curve $T = 1$, green curve.

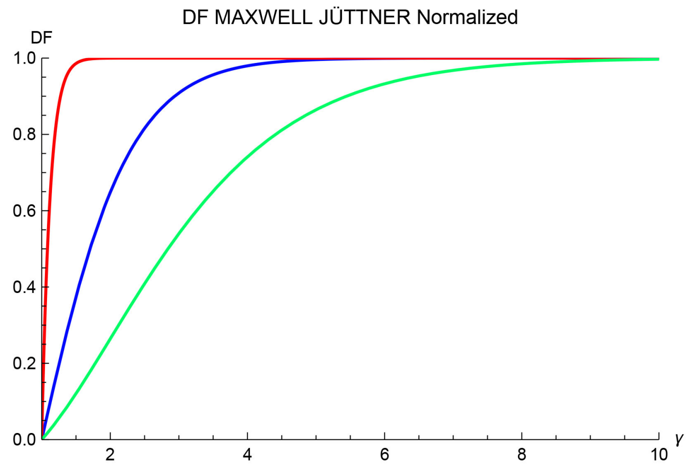

Figure 3. DF Distribution Function of the relativistic MB obtained by analytical integration: $T = 0.1$ red curve, $T = 0.5$, blue curve $T = 1$, green curve.

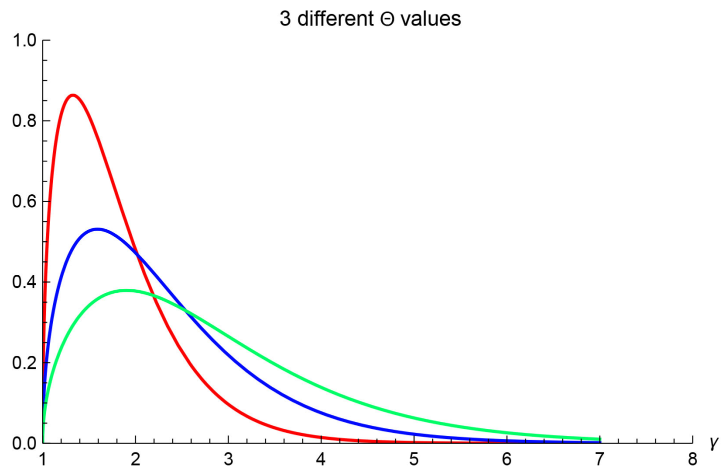

Figure 4. Maxwell-Jüttner relativistic PDF Distribution Function for different Θ red curve: $\Theta = 0.4$ blue curve $\Theta = 0.6$ green curve $\Theta = 0.8$.

This integral: the distribution function: the DF of [4], is normalized by:

$$Norm(\Theta) = \frac{e^{-1/\Theta}\left(2\Theta(\Theta+1)+1\right)}{K_2\left(\frac{1}{\Theta}\right)}$$

Thus the resulting $FDF_{MJ}(\gamma_{\max},\Theta)$ is:

$$fDF_{MJ}(\gamma_{\max},\Theta) = 1 - \frac{e^{\frac{1-\gamma_{\max}}{\Theta}}\left(\gamma_{\max}^2 + 2\gamma_{\max}\Theta + 2\Theta^2\right)}{2\Theta^2 + 2\Theta + 1} \tag{25}$$

This DF function fits perfectly the numerical DF of [4], obtained by numerical integration. It is possible to find the mode of the $f_{MJ}(\gamma,\Theta)$, that is solving: $f_{MJ}(\gamma,\Theta)' = 0$ this gives:

$$\begin{aligned}
&\mathrm{Mode}(\Theta) \\
&= \left(-\frac{1}{27\Theta^2} + \frac{1}{27}\sqrt[3]{-\frac{1}{\Theta^6} - \frac{9}{\Theta^4} - \frac{351}{\Theta^2} + \frac{54\sqrt{3}\sqrt{-375\Theta^4 - 13\Theta^2 - 1}}{\Theta^3} + 3375}\right. \\
&\quad + \frac{25}{3\sqrt[3]{-\frac{1}{\Theta^6} - \frac{9}{\Theta^4} - \frac{351}{\Theta^2} + \frac{54\sqrt{3}\sqrt{-375\Theta^4 - 13\Theta^2 - 1}}{\Theta^3} + 3375}} \\
&\quad + \frac{2}{9\Theta^2\sqrt[3]{-\frac{1}{\Theta^6} - \frac{9}{\Theta^4} - \frac{351}{\Theta^2} + \frac{54\sqrt{3}\sqrt{-375\Theta^4 - 13\Theta^2 - 1}}{\Theta^3} + 3375}} \\
&\quad \left. + \frac{1}{27\Theta^4\sqrt[3]{-\frac{1}{\Theta^6} - \frac{9}{\Theta^4} - \frac{351}{\Theta^2} + \frac{54\sqrt{3}\sqrt{-375\Theta^4 - 13\Theta^2 - 1}}{\Theta^3} + 3375}} - \frac{1}{9}\right)^{\frac{1}{2}}
\end{aligned}$$

[1] Figure 5 shows the mode, written Mode (Θ) obtained solving $f_{MJ}(\gamma,\Theta)' = 0$.

It has to be noticed that part of the Mode (Θ), the part containing $\sqrt{-375\Theta^4 - 13\Theta^2 - 1}$ is an imaginary number, because a negative square root, for all positive Θ, getting rid of the imaginary part enables a real curve (Figure 5), this occurs in [4] and remains a theoretical problem when solving: $f_{MJ}(\gamma,\Theta)' = 0$ with Mathematica or Maple software.

3. Application to Real Plasmas

I shall use the Zaninetti magnetic B field frequency that is: $\nu_g = \frac{eB}{2\pi m_e}$ inserted in the PDF, whose general formula is:

$$FDis(\nu,\nu_g,T) = \frac{16\nu_g \times T^3 \sqrt{\frac{\nu}{\nu_g} - 1} e^{\frac{1-\sqrt{\frac{\nu}{\nu_g}}}{T} - \frac{1}{T}}}{\nu^2 \sqrt{\frac{\nu}{\nu_g}} G_{1,3}^{3,0}\left(\frac{0.25}{T^2} \middle| \begin{matrix} 1 \\ -0.5,-1,-1.5 \end{matrix}\right)} \quad (26)$$

This leads to:

$$FDis(\nu,B,T) = \frac{7.49346 \times 10^{19} B \times T^3 \sqrt{\frac{3.572 \times 10^{-13}\nu}{B} - 1} e^{\frac{1-5.9769\times 10^{-7}\sqrt{\frac{\nu}{B}}}{T} - \frac{1}{T}}}{\nu^2 \sqrt{\frac{\nu}{B}} G_{1,3}^{3,0}\left(\frac{0.25}{T^2} \middle| \begin{matrix} 1 \\ -0.5,-1,-1.5 \end{matrix}\right)} \quad (27)$$

4. Maxwell-Jüttner Distribution with *Ee* Electric Field

This leads to: it is well known that applying an electric field *Ee* onto an atom, the interaction energy is: $E_{Stark} = eEe \times z$, this is the linear Stark effect, the factor $\frac{3}{2}$ appears when the Hamiltonian of the atom plus the energy associated with the electric field *Ee*:

$$-\frac{\hbar^2}{2m_e}\Delta_r^2 + \frac{e^2}{4\pi\varepsilon_0 r} + eEez = E \quad (28)$$

is solved.

That is: the Stark energy is given by the mean quantity $r_n = \langle n,l,m | r\cos(\theta) | n,l,m \rangle$, $|n,l,m\rangle$ being the hydrogen wave functions in spherical coordinates, yielding for the energy $E_{Stark} = eEe \times r_n$, with $z = r\cos(\theta)$

The frequencies associated with the linear Stark effect are defined as: $\nu_g = \frac{E_{Stark}}{h}$, h is the Planck constant and *Ee* the electric field are defined by:

$$E_{Stark} = \frac{3}{2} \times n^2 \times e \times Ee \times a_0 \quad (29)$$

[1] This equation has a meaning, that is gives a correct Mode(Θ) and the curve representing it, if one deletes its imaginary part $\sqrt{-375\Theta^4 - 13\Theta^2 - 1}$.

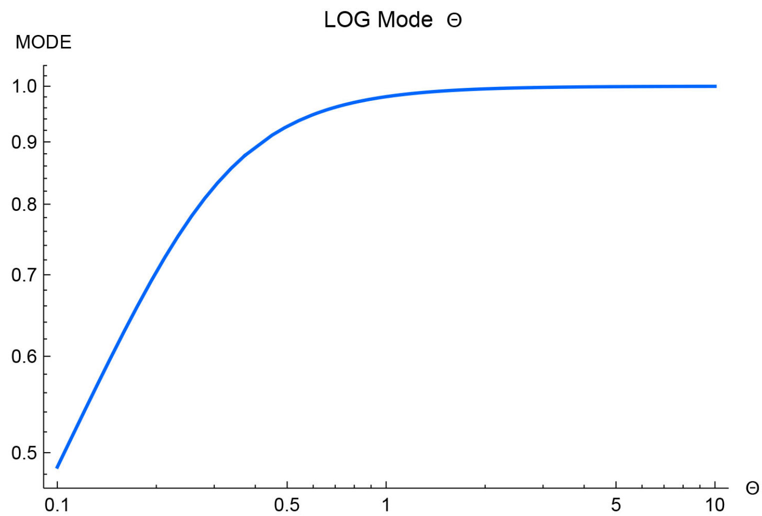

Figure 5. Maxwell-Jüttner relativistic mode as a function of Θ for different Θ, taking the real part of the theoretical mode(Θ).

$$\nu_g = \frac{E_{Stark}}{h} \tag{30}$$

n is the principal quantum number of the atoms in the plasmas, it can be integer when considering atoms in low states, or such as $n_* = n - \delta$ for atoms (Li, Na, K). For all calculations, the definition of temperature T of the plasmas is used as in [4]: $T_e = \frac{m_e c^2}{k_B}$, this temperature $T_e = 5.92991 \times 10^9\ \text{K}$, is quite high compared to the ionization energy of an hydrogen atom expressed in Kelvin $T_{Ionization} = 157821\ \text{K}$, thus even with a temperature $T = 0.1 \times T_e$, most of the atoms should be ionized, part of an hot plasma.

It is still possible define a Stark frequency using the Stark energy:

$E_{Stark} = eEez$, that is $\nu_g = \frac{E_{Stark}}{h}$, where the length z is a characteristic length, that could be the Debye length $\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T_e}{n_e e^2}}$.

I shall use the definition of the distribution $FDis(\nu, \nu_g, T)$ given by [4], that is Equation (26): changing the frequency $\nu_g = \frac{eB}{2\pi m_e}$ into $\nu_g = \frac{3en^2 Eea_0}{2h}$ for temperatures $T \le 0.1T_e$ and for higher temperatures $\nu_g = \frac{eEe \times z}{h}$ z being a length characteristic of the plasma, it could be the Debye length λ_D. The formula from [4] with the magnetic field B frequency distribution is changed with the electric field Ee giving the equation:

$$FDStark(\nu, Ee, T) = \frac{16 \times e \times Ee \times T^3 z \sqrt{\frac{h\nu}{eEez} - 1} \mathrm{e}^{\frac{1 - \sqrt{\frac{h\nu}{eEez}}}{T} - \frac{1}{T}}}{h\nu^2 G_{1,3}^{3,0}\left(\frac{0.25}{T^2} \middle| \begin{matrix} 1 \\ -0.5, -1, -1.5 \end{matrix}\right) \sqrt{\frac{h\nu}{eEez}}} \tag{31}$$

Figure 6 shows frequencies obtained with the Stark energy $E = \frac{3}{2} eEe \times z$ is inserted in $FDStark(\nu, Ee, T)$, with $z = a_0$.

Figure 7 gives frequencies[2,3] Numerically the distribution is:

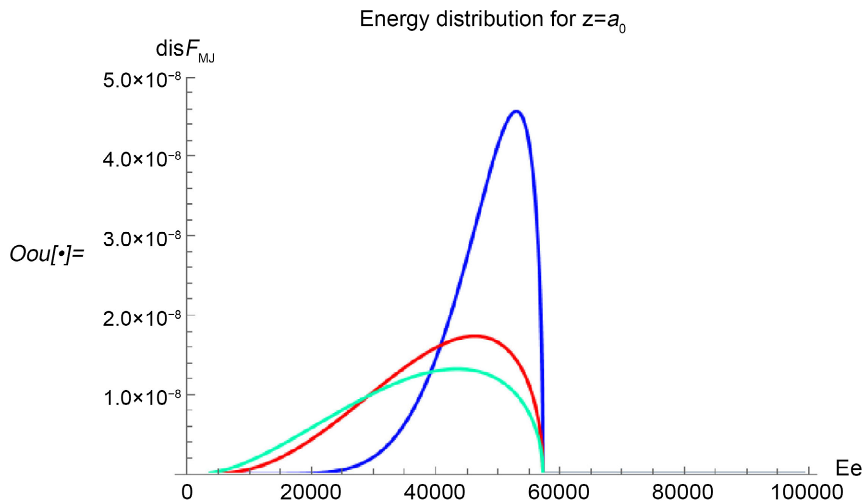

Figure 6. Stark frequencies versus variable *Ee* field using the distribution $FDStark(\nu, Ee, T)$ these: blue curve $T = 0.1$, red $T = 0.5$, green $T = 1$ in $\frac{m_e c^2}{k_B}$ units, the highest frequencies occur for small *Ee* field values and *z* length, here $z = a_0$, and low *T* temperatures.

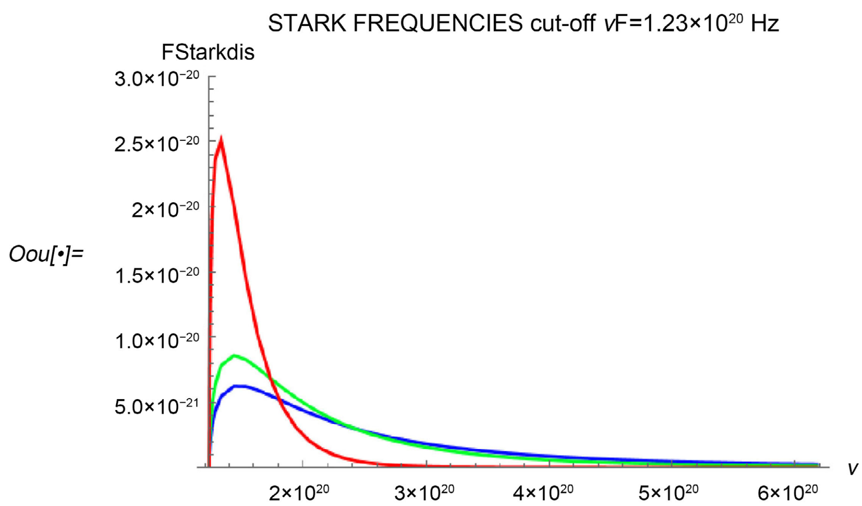

Figure 7. Stark Energy frequencies in the relativistic $FDStark(\nu, Ee, T)$ *T* varies from 0.1 to 1 in $\frac{m_e c^2}{k_B}$ units, blue curve $T = 1$, green curve $T = 0.5$, red curve $T = 0.1$.

[2] $a_0 = \frac{\hbar}{m_e c\alpha} = 5.29777210^{-11}\,\text{m}$ is the Bohr radius an $e = -1.602176 \times 10^{-19}\,\text{C}$ the electric charge of an electron, and $k_B = 6.022 \times 10^{23} \frac{\text{J}}{\text{K}}$ the Boltzmann constant, $m_e = 9.10938 \times 10^{-31}\,\text{kg}$.

[3] The electric field is written as *Ee* or *F* to distinguish these quantities from the energy written *En*.

$$FDStark(\nu, Ee, T) = \frac{Ee \times z \left(6.015 \times 10^{22} Ee \times T^3 z \left(\sqrt{\frac{4.135 \times 10^{-15} \nu}{Eez}} - 1\right) e^{\frac{1-6.430\times10^{-8}\sqrt{\frac{\nu}{Eez}}}{T} - \frac{1}{T}}\right)}{\nu^2 \sqrt{\frac{\nu}{Ez}} G_{1,3}^{3,0}\left(\frac{0.25}{T^2} \middle| \begin{matrix} 1 \\ -0.5, -1, -1.5 \end{matrix}\right)} \tag{32}$$

Comments of **Figure 7**: the distribution $FDStark(\nu, Ee, T)$ shows that the lesser are the values of the energy $e \times Ee \times z$ the highest are the frequencies because of the factor $\frac{h\nu}{e \times Ee \times z}$ is higher for small values *z*, if used on atom scale such as the Bohr radius a_0 gives an higher frequency distribution than a *z* factor of $z = 10^{-2}$ m, the frequencies from a *z* near a_0 compared to the $z = 10^{-2}$ m are enhanced by a factor of fac = $\sqrt{\frac{10^{-2}}{0.529 \times 10^{-10}}} = 13749$. Using the Maxwell-Jüttner PDF distribution Equation (22) and defining $\gamma = \frac{E}{m_e c^2}$ yields:

$$F_{MJ}Stark(\nu, Ee, \Theta) = \frac{\left(h\sqrt{\frac{h\nu}{eEez}} - 1 e^{-\frac{\sqrt{\frac{h\nu}{eEez}}}{\Theta}}\right)}{2eEe\Theta z K_2\left(\frac{1}{\Theta}\right)} \tag{33}$$

Numerical results are obtained inserting the Planck constant h and electric charge of the electron *e*, and giving to the electric field $E = 10^5 \frac{\text{V}}{\text{m}}$ this field E is 26 times higher than the ionizing field for hydrogen $E_{Ionization} = 3816 \frac{\text{V}}{\text{m}}$

$$F_{MJ}Stark(\nu, Ee, \Theta, z) = \frac{2.0678 \times 10^{-20} e^{-\frac{2.0336\times10^{-10}\sqrt{\frac{\nu}{z}}}{\Theta}} \sqrt{\frac{4.1356 \times 10^{-20} \nu}{z}} - 1}{z\Theta K_2\left(\frac{1}{\Theta}\right)} \tag{34}$$

For a hot plasma, $z = 10^{-2}$ m and a field $Ee = 10^5 \frac{\text{V}}{\text{m}}$ the numerical distribution is:

$$F_{MJ}Stark(\nu, 10^5, \Theta, 10^{-2}) = \frac{2.0678 \times 10^{-20} \sqrt{4.1356 \times 10^{-18} \nu - 1} e^{-\frac{2.0336\times10^{-9}\sqrt{\nu}}{\Theta}}}{\Theta K_2\left(\frac{1}{\Theta}\right)} \tag{35}$$

and the graphical representation is:

Figure 8 gives the energy distribution for a field value $Ee = 10^5 \frac{\text{V}}{\text{m}}$ and a *z* length $z = 10^{-2}\ \text{m}$.

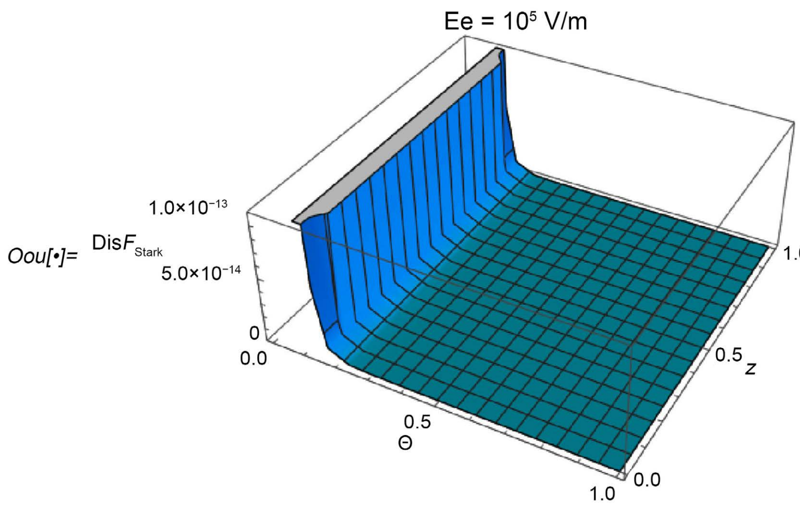

Figure 8. Stark frequencies in the Maxwell-Jüttner distribution function of Θ and z the characteristic plasma length.

5. Correct Relativistic M-B Distribution for Any Energy *En*

Providing an energy written En in the equation, gives the good relativistic behavior of the Maxwell-Boltzmann distribution:

$$F_{Rel}\left(\nu, En, T\right) = \frac{16 \times En \times T^{3} \times \mathrm{e}^{\frac{1-\sqrt{\frac{h\nu}{En}}}{T} - \frac{1}{T}} \Re\left(\sqrt{\frac{h\nu}{En} - 1}\right)}{h\nu^{2}\sqrt{\frac{h\nu}{En}}\, G_{1,3}^{3,0}\left(\frac{0.25}{T^{2}} \,\middle|\, \begin{matrix} 1 \\ -0.5, -1, -1.5 \end{matrix}\right)} \tag{36}$$

The same applies to the Maxwell-Jüttner distribution that is:

$$F_{MJ}\left(\nu, En, \Theta\right) = \frac{h\sqrt{1-\frac{h\nu}{En}}\,\mathrm{e}^{-\frac{\sqrt{\frac{h\nu}{En}}}{\Theta}}}{2En \times \Theta \times K_{2}\left(\frac{1}{\Theta}\right)} \tag{37}$$

Numerically:

$$F_{MJ}\left(\nu, En, \Theta\right) = \frac{3.3130 \times 10^{-34}\sqrt{1-\frac{6.6261 \times 10^{-34}\nu}{En}}\,\mathrm{e}^{-\frac{2.5746 \times 10^{-17}\sqrt{\frac{\nu}{En}}}{\Theta}}}{En \times \Theta \times K_{2}\left(\frac{1}{\Theta}\right)} \tag{38}$$

It is readily seen that to get real values it is necessary that any kind of energy En in Joule units has to be $En \geq h\nu$ numerically $En \geq 6.626 \times 10^{-34}\nu$.

It is possible to check the normalization of this distribution integrating on the variables ν with a initial value $\nu_{F} = 1.23554 \times 10^{20}$, $\nu_{F} = \frac{k_{B}Te}{h}$, and on the variable Θ, these calculations are quite heavy if one wants an analytical integration. The integrated distributions are obtained with parameters: $\nu_{\max}$ and $\Theta_{\max}$, these replace integration to infinity and give analytical results with: $\nu_{\max} \geq 10 \times \nu_{F}$

and $\Theta_{\max} \geq 10$. This means that instead of an integration of the variables ν and Θ towards infinity, the function Integrate works well, if the limits of integration to Infinity, are replaced by a integration with: $\int_{\nu_F}^{\nu_{\max}}$ rather than $\int_{\nu_F}^{\infty}$, the same apply for the variable Θ that is again, $\int_{\Theta}^{\Theta_{\max}}$ replaces $\int_{\Theta}^{\infty}$ in that way the normalization of $F_{MJ}(\nu, En, \Theta)$ is obtained.

Figure 9 shows the variation of the energy distribution $f_{M\text{-}J}(En)$ when any energy is quantized by the relation $En = h\nu$.

Defining a thermal energy by: $En = k_B T$ gives the following figure:

Figure 10 plots the Maxwell Jüttner distribution, for $En = k_B T$ and $\nu = \frac{k_B T}{h}$, the temperatures T vary from 0 to 100, these are given in $\frac{m_e c^2}{k_B}$ units, to get the temperatures in Kelvin one has to multiply T by the factor: 5.92965 × 10^9.

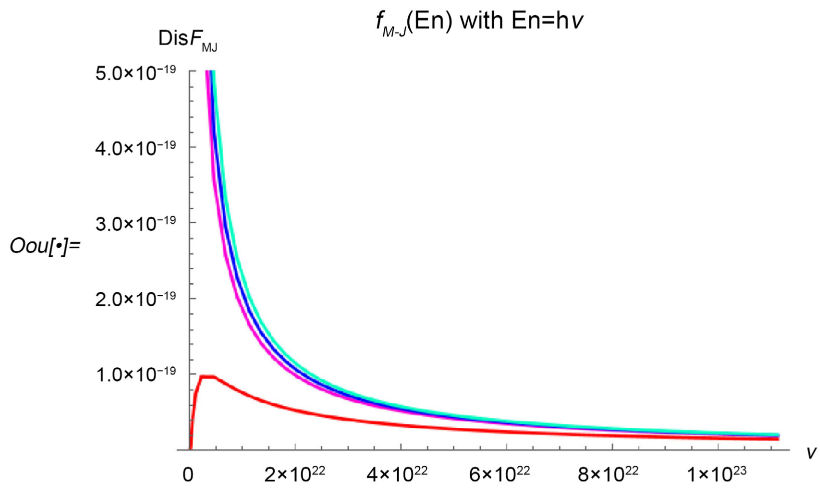

Figure 9. Maxwell-Jüttner relativistic distribution of energy $En = h\nu$. Different Θ values. Red curve is for $\Theta = 0.1$. Other curves are for $\Theta \geq 0.5$ and $\Theta \leq 10$. These curves merge for $\Theta \geq 0.5$.

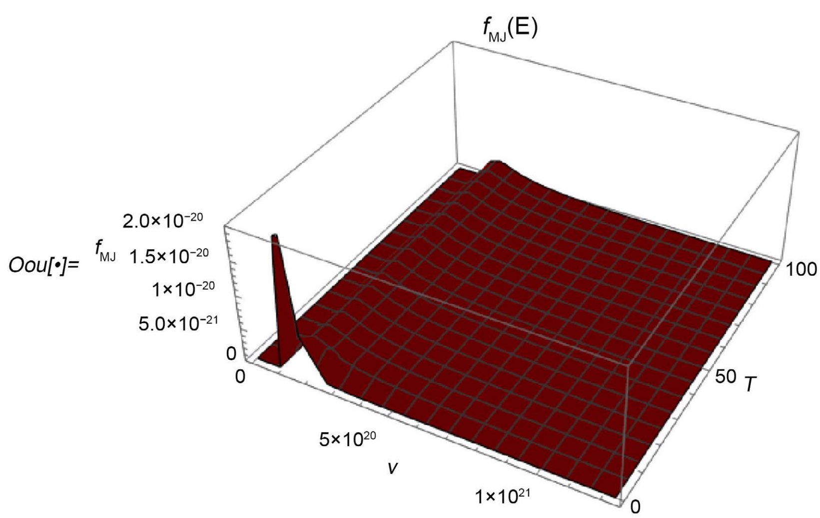

Figure 10. Maxwell-Jüttner relativistic distribution of energy $En = k_B T$.

6. Conclusions

This article reviews significant progress made for this interesting problem to adapt the Maxwell probability distribution to Einstein relativity theory, (no physical velocity should be greater than *c* velocity of light), this leads to a correction given partly by the Maxwell-Jüttner distribution. It is known that this correction is important for high temperatures obtained in astrophysical or fusion plasmas, such high temperatures are such that:

$$T \geq \frac{mc^2}{k_B} \text{ or } T \geq \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}k_B}.$$

It is desirable that the manipulation of these quantities: PDF and DF distributions relativity compatible with the correct Maxwell-Jüttner theoretical expression, should give a better handling of real plasmas.

The use of modern software like Mathematica gives an unique way to deal with special functions, all the results obtained by L. Zaninetti with Maple are found again, with the special function $G_{1,3}^{3,0}\left(\frac{0.25}{T^2}\middle|\begin{matrix}1\\-0.5,-1,-1.5\end{matrix}\right)$.

The basic approximation $\sqrt{\gamma^2-1} \approx \gamma$ brings analytical results that fit perfectly the numerical construction of the distribution function DF of [4]. Notebooks developed by the author can be sent to interested readers of this article.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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[4]Mathematica notebooks concerning calculations of this article can be furnished on request.

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