Journal of Applied Mathematics and Physics, 2023, 11, 2221-2230
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Lower Bounds of Decay Rates for Solution to the Single-Layer Quasi-Geostrophic Model

Haoyu Zhao[1,2]

[1]School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, China
[2]College of Data Science, Jiaxing University, Zhejiang, China
Email: zhaohaoyu_lyc@163.com

How to cite this paper: Zhao, H.Y. (2023) Lower Bounds of Decay Rates for Solution to the Single-Layer Quasi-Geostrophic Model. *Journal of Applied Mathematics and Physics*, **11**, 2221-2230.
https://doi.org/10.4236/jamp.2023.118144

Received: July 12, 2023
Accepted: August 12, 2023
Published: August 15, 2023

Abstract

In this paper, we study the long-time behavior of solutions of the single-layer quasi-geostrophic model arising from geophysical fluid dynamics. We obtain the lower bound of the decay estimate of the solution. Utilizing the Fourier splitting method, under suitable assumptions on the initial data, for any multi-index α, we show that the solution ψ satisfies

$$\left\|\nabla^\alpha \psi\right\|_{L^2} \geq C(1+t)^{-\frac{1+|\alpha|}{4}}, |\alpha| = 0,1,\cdots,m,\ m \geq 4,\ t \geq 1.$$

Keywords

Single-Layer Quasi-Geostrophic Model, Lower Bounds, Fourier Splitting Method

1. Introduction

We study the initial value problem of a single-layer quasi-geostrophic model with viscosity (see e.g. [1] [2]):

$$\frac{\partial}{\partial t}[\Delta\psi - F\psi] + J(\psi,\Delta\psi) + \beta\frac{\partial\psi}{\partial x} = \frac{1}{R_e}\Delta^2\psi, \tag{1}$$

where $\psi = \psi(x,y,t)$ denotes the stream functions, F is the rotational Froude number, β is the Coriolis parameter, R_e is the Reynolds number, $\frac{1}{R_e}\Delta^2\psi$ is the viscosity terms, and the nonlinear term $J(f,g)$ is defined by

$$J(f,g) := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

The quasi-geostrophic β-plane model is considered as a simplification of the shallow-water equations when the Rossby number is small and the magnitude of bottom topography variations is comparable to the Rossby number.

A generalization of the single-layer quasi-geostrophic model (1) is the two-layer (multi-layer) quasi-geostrophic model. There are many research results on the existence, uniqueness and long-time behavior of the solutions. For results in this regard, please refer to [3] [4] [5] [6].

In 2010, X. Pu and B. Guo [7] proved global existence of weak solutions for the fractional quasi-geostrophic equation with $\Delta^2\psi$ replaced by $(-\Delta)^{1+\alpha}\psi(\alpha\in(0,1])$ in Equation (1) and they also obtained long-time behavior of the solution when $\alpha\in\left(0,\frac{1}{2}\right]$.

In 2023, H. Li, J. Li and J. Zhang [8] showed that the existence and uniqueness of the global smooth solution to the Cauchy problem of Equation (1). They also obtain the upper bound of the decay estimates of the solution in L^2. And show that if $\psi_0\in L^1(\mathbb{R}^2)\cap H^m(\mathbb{R}^2)$ and $m\geq 4$ be an integer, then for any multi-index α, the solution ψ satisfies

$$\left\|\nabla^\alpha\psi\right\|_{L^2}\leq C(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{4}}, |\alpha|=0,1,\cdots,m-1,$$

and

$$\left\|\nabla^\alpha\psi\right\|_{L^2}\leq C(1+t)^{-\frac{|\alpha|}{4}}, |\alpha|=m.$$

Following the work of [8], we study the lower bound of the decay estimate of the solution for the initial-value problem (1) equipped with the initial data

$$\psi(x,y,0)=\psi_0(x,y). \tag{2}$$

To obtain the lower bound of the decay estimate of the solution, we not only use the Fourier splitting method which is originated from Schonbek [9] [10] and is improved by Zhang [11], we also need to construct a new expression

$$\omega=\phi-\psi$$

where ψ and ϕ are the solutions of Equation (1) itself and the linear part of Equation (1) with the given initial data ψ_0, respectively. In addition, in order to ensure that this method works, we must show the nonlinear solution ω decays faster than the linear part. To obtain the lower decay bound, we define the set for $\sigma,\gamma>0$, $M_\gamma^\sigma:=\left\{f\,|\,\left\|\hat{f}(\xi)\right\|_{L^\infty}\geq\gamma \text{ for } |\xi|\leq\sigma\right\}$. We have the following theorem.

Theorem 1. If the initial data $\psi_0\in L^1(\mathbb{R}^2)\cap H^m(\mathbb{R}^2)\cap M_\gamma^\sigma$ and $m\geq 4$ be an integer, then for any multi-index α, the solution ψ of Equation (1) satisfies

$$\left\|\nabla^\alpha\psi\right\|_{L^2}\geq C(1+t)^{-\frac{1+|\alpha|}{4}}, |\alpha|=0,1,\cdots,m,\ t\geq 1.$$

This paper is organized as follows. In Section 2, we review the notations used throughout the paper. In Section 3, we consider the upper and lower bounds of the linear part of Equation (1). In Section 4, we estimate the upper bounds of the difference between the solution of linear part of Equation (1) and the solution of Equation (1). Furthermore, we show that the decay rate to the lower bounds of Equations (1)-(2).

2. Notations and Preliminaries

The Fourier transform $\hat{f} = \mathcal{F}(f)$ of a tempered distribution $f(x)$ on $\mathbb{R}^2$ is defined as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-ix\cdot\xi} f(x)\mathrm{d}x,$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$.

For $1 \le p \le +\infty$, we denote by $L^p(\mathbb{R}^2)$ the Lebesgue space equipped the norm

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}^2} |u(x)|^p\right)^{\frac{1}{p}},\ 1 \le p < +\infty,$$

and

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{\xi\in\mathbb{R}^2} |u(\xi)|.$$

For $s \in \mathbb{R}$, $H^s(\mathbb{R}^2)$ denotes the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}^2) = \left\{u \in S' \mid \|u\|_{H^s(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2}\left(1+|\xi|^2\right)^s |\hat{u}(\xi)|\mathrm{d}\xi < +\infty\right\},$$

where $\hat{u}(\xi)$ is the Fourier transform of *u*.

3. Decay Estimates for the Linear Part

In this section, we first study the lower and upper decay bounds on the decay rates for the following linear quasi-geostrophic model:

$$\frac{\partial}{\partial t}[\Delta\phi - F\phi] + \beta\frac{\partial\phi}{\partial x} = \frac{1}{R_e}\Delta^2\phi, \tag{3}$$

$$\phi(x,y,0) = \phi_0(x,y). \tag{4}$$

To solve Equations (3)-(4), we take the Fourier transform for the above equations to get

$$\frac{\partial}{\partial t}\left[-|\xi|^2\hat{\phi} - F\hat{\phi}\right] + \beta i\xi_1\hat{\phi} = \frac{1}{R_e}|\xi|^4\hat{\phi},$$

then we get

$$\hat{\phi} = \mathrm{e}^{\frac{\beta i\xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F}t}\hat{\phi}_0. \tag{5}$$

Since F is not zero, we note the following facts

$$c_1|\xi|^2 \le \frac{|\xi|^2}{|\xi|^2+F} \le c_2|\xi|^2,\ 0 \le |\xi| \le \sigma, \tag{6}$$

and

$$0 < \lambda \le \frac{|\xi|^2}{|\xi|^2 + F} < 1, \ |\xi| \ge \sigma, \tag{7}$$

where σ is an arbitrary positive number.

We now recall some estimates for the solutions of the liner equation which will be useful later.

Lemma 2. Let $\phi_0 \in L^1(\mathbb{R}^2) \cap H^m(\mathbb{R}^2)$ and $m \ge 4$ be an integer. Then the solution ϕ of Equations (3)-(4) satisfies

$$\left\| \nabla^\alpha \phi(\cdot,t) \right\|_{L^2} \le C(1+t)^{-\frac{1+|\alpha|}{4}}, \ |\alpha| = 0,1,2,\cdots,m, \tag{8}$$

where the constant C depends on R_e, F, σ and the initial data.

Proof. When $t \ge 1$, from Equation (5), we have

$$\begin{aligned}
\iint_{\mathbb{R}^2} |\nabla^\alpha \phi|^2 \mathrm{d}x\mathrm{d}y &= \iint_{\mathbb{R}^2} |\xi|^{2|\alpha|} |\hat{\phi}_0|^2 \left| \mathrm{e}^{\frac{\beta i \xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F} t} \right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&= \iint_{|\xi|\le\sigma} |\xi|^{2|\alpha|} |\hat{\phi}_0|^2 \left| \mathrm{e}^{\frac{\beta i \xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F} t} \right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\quad + \iint_{|\xi|\ge\sigma} |\xi|^{2|\alpha|} |\hat{\phi}_0|^2 \left| \mathrm{e}^{\frac{\beta i \xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F} t} \right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\le C \iint_{|\xi|\le\sigma} |\xi|^{2|\alpha|} \mathrm{e}^{-\frac{2c_1}{R_e}|\xi|^4 t} \mathrm{d}\xi_1 \mathrm{d}\xi_2 + C \iint_{|\xi|\ge\sigma} |\xi|^{2|\alpha|} \mathrm{e}^{-\frac{2\lambda}{R_e}|\xi|^2 t} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\le Ct^{-\frac{1+|\alpha|}{2}} + Ct^{-(1+|\alpha|)} \le Ct^{-\frac{1+|\alpha|}{2}},
\end{aligned}$$

where the following observation is used in the last step

$$\lim_{t\to 0^+} \frac{\int_0^{\frac{2c_1}{R_e} t\sigma^4} u^{\frac{|\alpha|-1}{2}} \mathrm{e}^{-u} \mathrm{d}u}{t^{\frac{1+|\alpha|}{2}}} = \frac{2}{1+|\alpha|} \left(\frac{2c_1}{R_e} \right)^{\frac{1+|\alpha|}{2}} \sigma^{2+2|\alpha|},$$

$$\lim_{t\to 0^+} \frac{\int_0^{\frac{2\lambda}{R_e} t\sigma^2} u^{|\alpha|} \mathrm{e}^{-u} \mathrm{d}u}{t^{|\alpha|+1}} = \frac{1}{1+|\alpha|} \left(\frac{2\lambda}{R_e} \right)^{1+|\alpha|} \sigma^{2+2|\alpha|}.$$

Therefore Lemma 3.1 is proved. □

We recall the set for $\sigma, \gamma > 0$,

$$M_\gamma^\sigma := \left\{ f \mid \left\| \hat{f}(\xi) \right\|_{L^\infty} \ge \gamma \text{ for } |\xi| \le \sigma \right\}. \tag{9}$$

Lemma 3. Let $\phi_0 \in L^1(\mathbb{R}^2) \cap H^m(\mathbb{R}^2) \cap M_\gamma^\sigma$ and $m \ge 4$ be an integer. Then the solution ϕ of problem (3)-(4) satisfies

$$\left\|\nabla^{\alpha}\phi\right\|_{L^2} \geq C(1+t)^{-\frac{1+|\alpha|}{4}},\ |\alpha| = 0,1,2,\cdots,m, \tag{10}$$

where the constant C depends on F, R_e, γ, σ and initial data.

Proof. From Equations (5), (6) and (9), we have

$$\begin{aligned}
\iint_{\mathbb{R}^2} \left|\nabla^{\alpha}\phi\right|^2 \mathrm{d}x\mathrm{d}y &= \iint_{\mathbb{R}^2} |\xi|^{2|\alpha|} \left|\hat{\phi}_0\right|^2 \left| \mathrm{e}^{\frac{\beta i \xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F}t} \right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\geq \iint_{|\xi|\leq\sigma} |\xi|^{2|\alpha|} \left|\hat{\phi}_0\right|^2 \left| \mathrm{e}^{\frac{\beta i \xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F}t} \right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\geq \gamma^2 \iint_{|\xi|\leq\sigma} |\xi|^{2|\alpha|} \mathrm{e}^{-\frac{2c_2}{R_e}|\xi|^4 t} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\geq C(1+t)^{-\frac{1+|\alpha|}{2}},
\end{aligned}$$

where the following observation is used in the last step

$$\lim_{t\to 0^+} \frac{\int_0^{\frac{2c_2}{R_e}t\sigma^4} u^{\frac{|\alpha|-1}{2}} \mathrm{e}^{-u} \mathrm{d}u}{t^{\frac{1+|\alpha|}{2}}} = \frac{2}{1+|\alpha|}\left(\frac{2c_2}{R_e}\right)^{\frac{1+|\alpha|}{2}} \sigma^{2+2|\alpha|}.$$

Therefore Lemma 3.2 is proved. □

Lemma 4. Let $\phi_0 \in L^1(\mathbb{R}^2) \cap H^m(\mathbb{R}^2)$ and $m \geq 4$ be an integer. Then the solution ϕ of problem (3)-(4) satisfies

$$\left\|\nabla^{\alpha}\phi\right\|_{L^\infty} \leq C(1+t)^{-\frac{|\alpha|+2}{4}},\ |\alpha| = 0,1,2,\cdots,m, \tag{11}$$

where the constant C depends on F, R_e, σ and initial data.

Proof. When $t \geq 1$, we use Hausdorff-Young inequality to obtain

$$\begin{aligned}
\left\|\nabla^{\alpha}\phi\right\|_{L^\infty} &\leq C\left\|\mathcal{F}\left(\nabla^{\alpha}\phi\right)\right\|_{L^1} \\
&\leq \iint_{\mathbb{R}^2} |\xi|^{|\alpha|} \left|\hat{\phi}_0\right| \left| \mathrm{e}^{\frac{\beta i \xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F}t} \right| \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\leq \left\|\hat{\phi}_0\right\|_{L^\infty} \left(\iint_{|\xi|\leq\sigma} + \iint_{|\xi|\geq\sigma}\right) |\xi|^{|\alpha|} \left| \mathrm{e}^{\frac{\beta i \xi_1 - \frac{1}{R_e}|\xi|^4}{|\xi|^2+F}t} \right| \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\leq C\iint_{|\xi|\leq\sigma} |\xi|^{|\alpha|} \mathrm{e}^{-\frac{c_1}{R_e}|\xi|^4 t} \mathrm{d}\xi_1 \mathrm{d}\xi_2 + C\iint_{|\xi|\geq\sigma} |\xi|^{|\alpha|} \mathrm{e}^{-\frac{\lambda}{R_e}|\xi|^2 t} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\leq Ct^{-\frac{2+|\alpha|}{4}} + Ct^{-\frac{2+|\alpha|}{2}} \leq Ct^{-\frac{2+|\alpha|}{4}},
\end{aligned}$$

where the following observation is used in the last step

$$\lim_{t\to 0^+}\frac{\int_0^{\frac{c_1}{R_e}t\sigma^4} u^{\frac{|\alpha|-2}{4}}\mathrm{e}^{-u}\mathrm{d}u}{t^{\frac{2+|\alpha|}{4}}}=\frac{4}{2+|\alpha|}\left(\frac{c_1}{R_e}\right)^{\frac{3}{4}}\sigma^{2+|\alpha|},$$

$$\lim_{t\to 0^+}\frac{\int_0^{\frac{\lambda}{R_e}t\sigma^2} u^{\frac{|\alpha|}{2}}\mathrm{e}^{-u}\mathrm{d}u}{t^{\frac{2+|\alpha|}{2}}}=\frac{2}{2+|\alpha|}\left(\frac{\lambda}{R_e}\right)^{\frac{3}{2}}\sigma^{2+|\alpha|}.$$

Therefore Lemma 3.3 is proved. □

4. Lower Bound on the Decay Rates

In this section, we study the lower bound of the decay rate of the solution of the nonlinear quasi-geostrophic equation. In [8], the existence of smooth solutions has been obtained, and the following upper bound is also proved.

Lemma 5. [8] Let $\psi_0\in H^m(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$, $m\ge 4$ is an integer, and ψ is the solution of Equation (1), then for any multi-index α, we have the decay estimates

$$|\left\|\nabla^\alpha\psi\right\|_{L^2}\le C(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{4}},|\alpha|=0,1,\cdots,m-1, \tag{12}$$

$$\left\|\nabla^\alpha\psi\right\|_{L^2}\le C(1+t)^{-\frac{m}{4}},|\alpha|=m. \tag{13}$$

Combining equation $\psi=\phi-\omega$. Since we have already obtained the lower decay bound for ϕ in Section 3, the key point is to estimate estimate the upper bound for ω. The equations for ω can be written in the form

$$\frac{\partial}{\partial t}[\Delta\omega-F\omega]-J(\psi,\Delta\psi)+\beta\frac{\partial\omega}{\partial x}=\frac{1}{R_e}\Delta^2\omega, \tag{14}$$

$$\omega_0(t)=0, \tag{15}$$

which implies that

$$\begin{aligned}|\hat{\omega}|&\le\int_0^t\left|\mathrm{e}^{(t-s)\frac{\beta i\xi_1-\frac{1}{R_e}|\xi|^4}{|\xi|^4+F}}\right|\frac{\left|\hat{J}(\psi,\Delta\psi)\right|}{|\xi|^2+F}\mathrm{d}s\\&\le C\int_0^t\left|\hat{J}(\psi,\Delta\psi)\right|\mathrm{d}s\le C\int_0^t|\xi|^2\left\|\psi\right\|_{L^2}\left\|\Delta\psi\right\|_{L^2}\mathrm{d}s\\&\le C\int_0^t|\xi|^2(1+s)^{-\frac{1}{4}}(1+s)^{-\frac{3}{4}}\mathrm{d}s\le C|\xi|^2\ln(1+t).\end{aligned} \tag{16}$$

This involves inequalities with respect to $\left|\hat{J}(\psi,\Delta\psi)\right|\le|\xi|^2\left\|\psi\right\|_{L^2}\left\|\Delta\psi\right\|_{L^2}$, please refer to [8] for detailed proof.

Proof of Theorem 1.1. We first show

$$\left\|\psi\right\|_{L^2}\ge C(1+t)^{-\frac{1}{4}}. \tag{17}$$

We multiply Equation (14) with 2ω, then integrate it in $\mathbb{R}^2$ with (x,y), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^2}\left[|\nabla\omega|^2+F|\omega|^2\right]\mathrm{d}x\mathrm{d}y+\frac{2}{R_e}\iint_{\mathbb{R}^2}|\Delta\omega|^2\,\mathrm{d}x\mathrm{d}y\\&=2\iint_{\mathbb{R}^2}J(\psi,\Delta\psi)(\phi-\psi)\mathrm{d}x\mathrm{d}y=\iint_{\mathbb{R}^2}J(\psi,\Delta\psi)\phi\mathrm{d}x\mathrm{d}y\\&\le\left|\iint_{\mathbb{R}^2}J(\psi,\Delta\psi)\phi\mathrm{d}x\mathrm{d}y\right|=\left|-\iint_{\mathbb{R}^2}\nabla\phi J(\psi,\nabla\psi)\mathrm{d}x\mathrm{d}y\right|\\&\le\|\nabla\phi\|_{L^\infty}\|\nabla\psi\|_{L^2}\|\Delta\psi\|_{L^2}\le C(1+t)^{-2}.\end{aligned}\tag{18}$$

Applying Plancherel's theorem to the Equation (18), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\iint_{\mathbb{R}^2}\left(|\xi|^2+F\right)|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\right]+\frac{2}{R_e}\iint_{\mathbb{R}^2}|\xi|^4|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\le C(1+t)^{-2}.\tag{19}$$

Let

$$\mathbb{S}_1(t)=\left\{(\xi_1,\xi_2)\in\mathbb{R}^2:\frac{|\xi|^4}{F+|\xi|^2}\le\frac{R_e}{1+t}\right\}.$$

Then the second term of Equation (19) becomes

$$\begin{aligned}&\frac{2}{R_e}\iint_{\mathbb{R}^2}|\xi|^4|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\\&=\iint_{\mathbb{R}^2\setminus\mathbb{S}_1(t)}\frac{2\left(F+|\xi|^2\right)}{1+t}|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2+\iint_{\mathbb{S}_1(t)}|\xi|^4|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\\&\ge2\iint_{\mathbb{R}^2}\frac{F+|\xi|^2}{1+t}|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2-2\iint_{\mathbb{S}_1(t)}\frac{F+|\xi|^2}{1+t}|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2.\end{aligned}\tag{20}$$

Inserting Equation (20) into Equation (19), using the estimate (16), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\iint_{\mathbb{R}^2}\left(|\xi|^2+F\right)|\hat{\omega}|^2\right]+2\iint_{\mathbb{R}^2}\frac{F+|\xi|^2}{1+t}|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\\&\le\frac{4}{1+\alpha}\iint_{\mathbb{S}_1(t)}\frac{F+|\xi|^2}{1+t}|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2+C(1+t)^{-2}\\&\le\frac{C}{1+t}\iint_{\mathbb{S}_1(t)}\left(F+|\xi|^2\right)|\xi|^4\ln^2(1+t)\mathrm{d}\xi_1\mathrm{d}\xi_2+C(1+t)^{-2}\\&\le\frac{C}{1+t}\int_0^{2\pi}\int_0^{(1+t)^{-\frac{1}{4}}}\left(F+r^2\right)r^4\ln^2(1+t)r\mathrm{d}r\mathrm{d}\theta+C(1+t)^{-2}\\&\le C(1+t)^{-\frac{9}{4}}+C(1+t)^{-2}\le C(1+t)^{-2}.\end{aligned}\tag{21}$$

And both sides of Equation (21) are multiplied by $(1+t)^2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)^2\iint_{\mathbb{R}^2}\left[|\nabla\omega|^2+F|\omega|^2\right]\mathrm{d}x\mathrm{d}y\right]\le C.$$

Integrating the above inequality over interval $[0,t]$, we get

$$\iint_{\mathbb{R}^2}\left[|\nabla\omega|^2+F|\omega|^2\right]\mathrm{d}x\mathrm{d}y\le C(1+t)^{-1}.$$

Therefore

$$\|\psi\|_{L^2}=\|\phi-\omega\|_{L^2}\ge\|\phi\|_{L^2}-\|\omega\|_{L^2}\ge C(1+t)^{-\frac{1}{4}},$$

where the constant C depends on R_e,F,σ,γ and initial data.

Next, we want to prove

$$\|\nabla\psi\|_{L^2}\ge C(1+t)^{-\frac{1}{2}}. \tag{22}$$

We multiply Equation (14) with $2\Delta\omega$, then integrate it in $\mathbb{R}^2$ with (x,y), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^2}\left[|\Delta\omega|^2+F|\nabla\omega|^2\right]\mathrm{d}x\mathrm{d}y+\frac{2}{R_e}\iint_{\mathbb{R}^2}|\nabla\Delta\omega|^2\,\mathrm{d}x\mathrm{d}y\\&=2\iint_{\mathbb{R}^2}J(\psi,\Delta\psi)(\Delta\phi-\Delta\psi)\mathrm{d}x\mathrm{d}y\\&=\iint_{\mathbb{R}^2}J(\psi,\Delta\psi)\Delta\phi\mathrm{d}x\mathrm{d}y\\&\le\|\nabla\Delta\phi\|_{L^\infty}\|\Delta\psi\|_{L^2}\|\nabla\psi\|_{L^2}\le C(1+t)^{-\frac{5}{2}}.\end{aligned} \tag{23}$$

Applying Plancherel's theorem to the equations Equation (23), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\iint_{\mathbb{R}^2}\left(|\xi|^4+F|\xi|^2\right)|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\right]+\frac{2}{R_e}\iint_{\mathbb{R}^2}|\xi|^6|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\le C(1+t)^{-\frac{5}{2}}. \tag{24}$$

Let

$$\mathbb{S}_2(t)=\left\{(\xi_1,\xi_2)\in\mathbb{R}^2:\frac{|\xi|^6}{F|\xi|^2+|\xi|^4}\le\frac{5R_e}{4(1+t)}\right\}.$$

Then the second term of Equation (24) becomes

$$\begin{aligned}&\frac{2}{R_e}\iint_{\mathbb{R}^2}|\xi|^6|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\\&\ge\frac{5}{2}\iint_{\mathbb{R}^2}\frac{F|\xi|^2+|\xi|^4}{1+t}|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2-\frac{5}{2}\iint_{\mathbb{S}_2(t)}\frac{F|\xi|^2+|\xi|^4}{1+t}|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2.\end{aligned} \tag{25}$$

Inserting Equation (25) into Equation (24), using the estimate (16), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\iint_{\mathbb{R}^2}\left(|\xi|^4+F|\xi|^2\right)|\hat{\omega}|^2\right]+\frac{5}{2}\iint_{\mathbb{R}^2}\frac{|\xi|^4+F|\xi|^2}{1+t}|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2\\&\le\frac{5}{1+\alpha}\iint_{\mathbb{S}_2}\frac{|\xi|^4+F|\xi|^2}{1+t}|\hat{\omega}|^2\,\mathrm{d}\xi_1\mathrm{d}\xi_2+C(1+t)^{-\frac{5}{2}}\\&\le\frac{C}{1+t}\iint_{\mathbb{S}_2}\left(|\xi|^4+F|\xi|^2\right)|\xi|^4\ln^2(1+t)\mathrm{d}\xi_1\mathrm{d}\xi_2+C(1+t)^{-\frac{5}{2}}\\&\le C(1+t)^{-\frac{11}{4}}+C(1+t)^{-\frac{5}{2}}\le C(1+t)^{-\frac{5}{2}}.\end{aligned} \tag{26}$$

And both sides of Equation (26) are multiplied by $(1+t)^{\frac{5}{2}}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)^{\frac{5}{2}}\iint_{\mathbb{R}^2}\left[|\Delta\omega|^2+F|\nabla\omega|^2\right]\mathrm{d}x\mathrm{d}y\right]\le C.$$

Integrating the above inequality over interval $[0,t]$, we get

$$\iint_{\mathbb{R}^2}\left[|\Delta\omega|^2+F|\nabla\omega|^2\right]\mathrm{d}x\mathrm{d}y\le C(1+t)^{-\frac{3}{2}}.$$

Therefore

$$\|\nabla\psi\|_{L^2}=\|\nabla\phi-\nabla\omega\|_{L^2}\ge\|\nabla\phi\|_{L^2}-\|\nabla\omega\|_{L^2}\ge C(1+t)^{-\frac{1}{2}}.$$

Then we will prove

$$\left\|\Delta\psi\right\|_{L^2}\geq C\left(1+t\right)^{-\frac{3}{4}}. \tag{27}$$

We multiply Equation (14) with $2\Delta^2\omega$, then integrate it in $\mathbb{R}^2$ with (x,y), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^2}\left[\left|\nabla\Delta\omega\right|^2+F\left|\Delta\omega\right|^2\right]\mathrm{d}x\mathrm{d}y+\frac{2}{R_e}\iint_{\mathbb{R}^2}\left|\Delta^2\omega\right|^2\mathrm{d}x\mathrm{d}y\\
&=2\iint_{\mathbb{R}^2}J\left(\psi,\Delta\psi\right)\left(\Delta^2\phi-\Delta^2\psi\right)\mathrm{d}x\mathrm{d}y\\
&\leq\left\|\nabla\Delta^2\phi\right\|_{L^\infty}\left\|\Delta\psi\right\|_{L^2}\left\|\nabla\psi\right\|_{L^2}+C\left\|\Delta^2\psi\right\|_{L^2}^{\frac{3}{2}}\left\|\Delta\psi\right\|_{L^2}^{\frac{3}{2}}\\
&\leq C\left(1+t\right)^{-3}.
\end{aligned} \tag{28}$$

Applying Plancherel's theorem to the equations Equation (28), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^2}\left(\left|\xi\right|^6+F\left|\xi\right|^4\right)\left|\hat{\omega}\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2+\frac{2}{R_e}\iint_{\mathbb{R}^2}\left|\xi\right|^8\left|\hat{\omega}\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\leq C\left(1+t\right)^{-3}. \tag{29}$$

Let

$$\mathbb{S}_3\left(t\right)=\left\{\xi\in\mathbb{R}^2:\frac{\left|\xi\right|^8}{F\left|\xi\right|^4+\left|\xi\right|^6}\leq\frac{3R_e}{2\left(1+t\right)}\right\}.$$

Then the second term of Equation (29) becomes

$$\begin{aligned}
&\frac{2}{R_e}\iint_{\mathbb{R}^2}\left|\xi\right|^8\left|\hat{\omega}\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\\
&\geq3\iint_{\mathbb{R}^2}\frac{F\left|\xi\right|^4+\left|\xi\right|^6}{1+t}\left|\hat{\omega}\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2-3\iint_{\mathbb{S}_3(t)}\frac{F\left|\xi\right|^4+\left|\xi\right|^6}{1+t}\left|\hat{\omega}\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2.
\end{aligned} \tag{30}$$

Inserting Equation (30) into Equation (29), using the estimate (16), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\iint_{\mathbb{R}^2}\left(\left|\xi\right|^6+F\left|\xi\right|^4\right)\left|\hat{\omega}\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\right]+3\iint_{\mathbb{R}^2}\frac{\left|\xi\right|^6+F\left|\xi\right|^4}{1+t}\left|\hat{\omega}\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\\
&\leq3\iint_{\mathbb{S}_3}\frac{\left|\xi\right|^6+F\left|\xi\right|^4}{1+t}\left|\hat{\omega}\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2+C\left(1+t\right)^{-3}\\
&\leq\frac{C}{1+t}\iint_{\mathbb{S}_3}\left(\left|\xi\right|^6+F\left|\xi\right|^4\right)\left|\xi\right|^4\ln^2\left(1+t\right)\mathrm{d}\xi_1\mathrm{d}\xi_2+C\left(1+t\right)^{-3}\\
&\leq C\left(1+t\right)^{-\frac{7}{2}}+C\left(1+t\right)^{-3}\leq C\left(1+t\right)^{-3}.
\end{aligned} \tag{31}$$

And both sides of Equation (31) are multiplied by $\left(1+t\right)^3$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(1+t\right)^3\iint_{\mathbb{R}^2}\left[\left|\nabla\Delta\omega\right|^2+F\left|\Delta\omega\right|^2\right]\mathrm{d}x\mathrm{d}y\right]\leq C.$$

Integrating the above inequality over interval $[0,t]$, we get

$$\iint_{\mathbb{R}^2}\left[\left|\nabla\Delta\omega\right|^2+F\left|\Delta\omega\right|^2\right]\mathrm{d}x\mathrm{d}y\leq C\left(1+t\right)^{-2}.$$

Therefore

$$\left\|\Delta\psi\right\|_{L^2}=\left\|\Delta\phi-\Delta\omega\right\|_{L^2}\geq\left\|\Delta\phi\right\|_{L^2}-\left\|\Delta\omega\right\|_{L^2}\geq C\left(1+t\right)^{-\frac{3}{4}}.$$

Finally, applying the same processing method as above, we can get

$$\left\|\nabla^\alpha\psi\right\|_{L^2}\geq C\left(1+t\right)^{-\frac{1+|\alpha|}{4}},\left|\alpha\right|=0,1,2,\cdots,m.$$

Because the way of proof is similar to Equations (17), (22) and (27), it is omitted here. □

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Dijkstra, H.A. (2005) Nonlinear Physical Oceanography: A Dynamical Systems Approach to the Large Scale Ocean Circulationand el nino. Springer, Berlin.

[2] Pedlosky, J. (1987) Geophysical Fluid Dynamics. Springer-Verlag, New York. https://doi.org/10.1007/978-1-4612-4650-3

[3] Guo, B., Han, Y., Huang, D., Bian, D. and Zhang, L. (2017) Global Smooth Solution of a Two-Dimensional Nonlinear Singular System of Differential Equations Arising from Geostrophics. *Journal of Difference Equations*, **262**, 3980-4020. https://doi.org/10.1016/j.jde.2016.12.005

[4] Guo, B. and Zhang, L. (1995) Decay of Solutions to Magnetohydrodynamics Equations in Two Space Dimensions. *Proceedings of the Royal Society of London, Series A*, **449**, 79-91. https://doi.org/10.1098/rspa.1995.0033

[5] Guo, B., Huang, D. and Zhang, J. (2017) Decay of Solutions to a Two-Layer Quasi-Geostrophic Model. *Analysis and Applications*, **15**, 595-606. https://doi.org/10.1142/S0219530517500117

[6] Medjo, T.T. (2008) On Strong Solutions of the Multi-Layer Quasi-Geostrophic Equations of the Ocean. *Nonlinear Analysis*, **68**, 3550-3564. https://doi.org/10.1016/j.na.2007.03.046

[7] Pu, X. and Guo, B. (2010) Existence and Decay of Solutions to the Two-Dimensional Fractional Quasigeostrophic Equation. *Journal of Mathematical Physics*, **51**, Article 083101. https://doi.org/10.1063/1.3460319

[8] Li, H., Li, J. and Zhang, J. (2023) Existence and Decay Estimates of Solution for a Fourth Order Quasi-Geostrophic Equation, Preprint.

[9] Schonbek, M.E. (1985) L^2 Decay for Weak Solutions of the Nonlinear Navier-Stokes Equations. *Archive for Rational Mechanics and Analysis*, **88**, 209-222. https://doi.org/10.1007/BF00752111

[10] Schonbek, M.E. (1986) Large Time Behaviour to the Navier-Stokes Equations. *Communications in Partial Differential Equations*, **11**, 733-763. https://doi.org/10.1080/03605308608820443

[11] Zhang, L. (1995) Sharp Rate of Decay of Solutions to 2-Dimensional Navier-Stokes Equations. *Communications in Partial Differential Equations*, **20**, 119-127. https://doi.org/10.1080/03605309508821089