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Exact Traveling Wave Solutions to Phi-4 Equation and Joseph-Egri (TRLW) Equation and Calogro-Degasperis (CD) Equation by Modified (G'/G^2)-Expansion Method

Maha Al-Harbi, Waleed Al-Hamdan, Luwai Wazzan

Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia
Email: mmutlaigalharbi@stu.kau.edu.sa, wmalhamdan@kau.edu.sa, lwazzan@kau.edu.sa

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Abstract

In this study, we will introduce the modified (G'/G^2)-expansion method to explore some of the exact traveling wave solutions of some nonlinear partial differential equations namely, Phi-4 equation, Joseph-Egri (TRLW) equation, and Calogro-Degasperis (CD) equation. As a result, we have obtained solutions for the equations expressed in terms of trigonometric, hyperbolic and rational functions. Moreover, some selected solutions are plotted using some specific values for the parameters.

Keywords

Exact Solutions, Modified (G'/G^2)-Expansion Method, Phi-4 Equation, Joseph-Egri (TRLW) Equation, Calogro-Degasperis (CD) Equation

1. Introduction

The world around us is inherently nonlinear. A large part of equations is nonlinear partial differential equations (NPDEs), which can be utilized to describe many important and sophisticated phenomena in physics, mathematical physics, engineering, and other various scientific fields. It is not easy to solve these equations, but exploration of their exact solutions becomes quite prominent due to the efficiency and reliability of symbolic software packages such as Maple or Mathematica.

In the past decades, authors constructed and developed various powerful and efficient methods for finding exact solutions of (NPDEs) with aid of symbolic software such as, the tanh-function method [1] [2], sine-cosine function method

[3] [4], Hirota bilinear method [5] [6], homogeneous balance method [7] [8], the exp-function method [9] [10], the sub-equation method, F-expansion method [11] [12], $(1/G')$-expansion method [13], $(G'/G, 1/G)$-expansion method [14] [15], (G'/G)-expansion method [16] [17], and so on.

Recently, the (G'/G^2)-expansion method [18] [19] have received a lot of attention for number of authors who work on the method to construct exact solutions of (NPDEs) in applied sciences and engineering. The modified (G'/G^2)-expansion method [20] [21] [22] has been used to find solutions for certain mathematical problems.

This work is summarized as follows. In Section 2, the description of the modified (G'/G^2)-expansion method is given. In Section 3, we apply the method to Phi-4 equation [23], Joseph-Egri (TRLW) equation [24] [25], and Calogro-Degasperis (CD) equation [26] to obtain some of their exact traveling wave solutions. Finally, conclusions and some remarks have been presented.

2. Description of the Method

Consider a general nonlinear partial differential equation (PDE) in the form:

$$P\left(u,u_t,u_x,u_y,u_{tt},u_{xx},u_{yy},u_{txy},\cdots\right)=0, \tag{1}$$

where $u=u\left(x,y,t,\cdots\right)$ is an unknown function, P is a polynomial in $u\left(x,y,t,\cdots\right)$ and its partial derivatives.

- **First step**

Reduction of Equation (1) to a nonlinear ordinary differential equation by introducing the traveling wave variable $\xi=k\left(x+y+\cdots\right)-Vt$, where t is a real positive number, k is the wave number and V is the velocity of travelling wave, we have:

$$Q\left(U,-VU',kU',kU',V^2U'',k^2U'',k^2U'',-Vk^2U''',\cdots\right)=0, \tag{2}$$

where $U\left(\xi\right)=u\left(x,y,t,\cdots\right)$ and Q is a polynomial of U and it's derivatives with respect to ξ.

- **Second step**

Assume that the traveling wave solution of Equation (2) can be expressed by a polynomial in $\frac{G'}{G^2}$ as follows:

$$U\left(\xi\right)=\sum_{i=0}^{N}a_i\left(\frac{G'}{G^2}\right)^{i}+\sum_{i=1}^{N}b_i\left(\frac{G'}{G^2}\right)^{-i}, \tag{3}$$

where $a_i\left(i=0,1,2,\cdots,N\right)$ and $b_i\left(i=1,2,\cdots,N\right)$ are constants which determined later and $G=G\left(\xi\right)$ satisfies the Riccati equation:

$$\left(\frac{G'}{G^2}\right)'=\rho\left(\frac{G'}{G^2}\right)^2+\mu\left(\frac{G'}{G^2}\right)+\sigma, \tag{4}$$

ρ, μ and σ are arbitrary constants.

- **Third step**

Calculate the value of the positive integer N by balancing the highest order nonlinear terms with the highest order derivatives appearing in Equation (2).

- **Fourth step**

The general solution of Equation (4), has five possible solutions as follows:

$$\frac{G'}{G^2}=\begin{cases} -\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}}{2\rho}\left(\frac{A\sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)+B\cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)}{A\cosh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)+B\sinh\left(\frac{1}{2}\sqrt{\Delta}\xi\right)}\right), & \text{if } \mu\neq 0,\Delta\geq 0, \\ -\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}}{2\rho}\left(\frac{-A\sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)+B\cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)}{A\cos\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)+B\sin\left(\frac{1}{2}\sqrt{-\Delta}\xi\right)}\right), & \text{if } \mu\neq 0,\Delta< 0, \\ \sqrt{\frac{\sigma}{\rho}}\left(\frac{A\cos\left(\sqrt{\sigma\rho}\xi\right)+B\sin\left(\sqrt{\sigma\rho}\xi\right)}{-A\sin\left(\sqrt{\sigma\rho}\xi\right)+B\cos\left(\sqrt{\sigma\rho}\xi\right)}\right), & \text{if } \sigma\rho>0,\mu=0, \\ -\frac{\sqrt{|\sigma\rho|}}{\rho}\left(\frac{A\sinh\left(2\sqrt{|\sigma\rho|}\xi\right)+A\cosh\left(2\sqrt{|\sigma\rho|}\xi\right)+B}{A\sinh\left(2\sqrt{|\sigma\rho|}\xi\right)+A\cosh\left(2\sqrt{|\sigma\rho|}\xi\right)-B}\right), & \text{if } \sigma\rho<0,\mu=0, \\ -\frac{A}{\rho(A\xi+B)}, & \text{if } \sigma=0,\mu=0,\rho\neq 0. \end{cases} \tag{5}$$

where A, B are constants and $\Delta=\mu^2-4\rho\sigma$.

- **Fifth step**

Using Equation (4) to calculate the derivatives then substituting them and Equation (3) into Equation (2) then collecting all terms with same power of $\frac{G'}{G^2}$ and solving the system of algebraic equations by computational software such as Maple. Finally use Equation (5) with the results obtained to derive the exact solutions of Equation (1).

3. Applications

In this section, we will find exact traveling wave solutions to Phi-4 equation, Joseph-Egri (TRLW) equation and Calogro-Degasperis (CD) equation by applying the modified (G'/G^2)-expansion method.

3.1. First Application

Consider the Phi-4 equation in the form:

$$u_{tt}-u_{xx}+m^2u+cu^3=0, \tag{6}$$

which has a significant role in the study on the nuclear and particle physics, where m and c are real valued constants. Using the transformation $u(x,t)=U(\xi)$ where $\xi=kx-Vt$, Equation (6) changed into an ODE equation of the form:

$$\left(V^2-k^2\right)U''+m^2U+cU^3=0. \tag{7}$$

Balancing the highest order derivative term U'' and nonlinear term U^3 in

Equation (7), we obtain $N=1$. The solution of Equation (7) will be in the following form:

$$U(\xi)=a_0+a_1\left(\frac{G'}{G^2}\right)+b_1\left(\frac{G'}{G^2}\right)^{-1} \tag{8}$$

a_0, a_1 and b_1 are constants to be determined later. The derivatives U' and U'' compute as follows:

$$U'=a_1\rho\left(\frac{G'}{G^2}\right)^2+a_1\mu\left(\frac{G'}{G^2}\right)-b_1\rho+a_1\sigma-b_1\mu\left(\frac{G'}{G^2}\right)^{-1}-b_1\sigma\left(\frac{G'}{G^2}\right)^{-2}, \tag{9}$$

$$\begin{aligned}U''&=2a_1\rho^2\left(\frac{G'}{G^2}\right)^3+3a_1\mu\rho\left(\frac{G'}{G^2}\right)^2+\left(a_1\mu^2+2a_1\rho\sigma\right)\left(\frac{G'}{G^2}\right)+b_1\mu\rho+a_1\mu\sigma\\&+\left(b_1\mu^2+2b_1\sigma\rho\right)\left(\frac{G'}{G^2}\right)^{-1}+3b_1\sigma\mu\left(\frac{G'}{G^2}\right)^{-2}+2b_1\sigma^2\left(\frac{G'}{G^2}\right)^{-3}.\end{aligned} \tag{10}$$

Now, substituting Equation (8) and Equation (10) in Equation (7) we will get a polynomial of degree three in $\left(\frac{G'}{G^2}\right)$, since the polynomial is equal to zero then each of its coefficients are equal to zero, as follows:

$$\begin{aligned}&\left(\frac{G'}{G^2}\right)^3:\left(2\left(V^2-k^2\right)a_1\rho^2+ca_1^3\right)=0,\\&\left(\frac{G'}{G^2}\right)^2:\left(3\left(V^2-k^2\right)a_1\mu\rho+3ca_0a_1^2\right)=0,\\&\left(\frac{G'}{G^2}\right)^1:\left(V^2-k^2\right)\left(a_1\mu^2+2a_1\rho\sigma\right)+m^2a_1+c\left(b_1a_1^2+2a_0^2a_1+a_1\left(a_0^2+2b_1a_1\right)\right)=0,\\&\left(\frac{G'}{G^2}\right)^0:\left(V^2-k^2\right)\left(b_1\mu\rho+a_1\mu\sigma\right)+m^2a_0+c\left(4b_1a_0a_1+a_0\left(a_0^2+2b_1a_1\right)\right)=0,\\&\left(\frac{G'}{G^2}\right)^{-1}:\left(V^2-k^2\right)\left(b_1\mu^2+2b_1\sigma\rho\right)+m^2b_1+c\left(b_1\left(a_0^2+2b_1a_1\right)+2a_0^2b_1+a_1b_1^2\right)=0,\\&\left(\frac{G'}{G^2}\right)^{-2}:3\left(V^2-k^2\right)b_1\sigma\mu+3cb_1^2a_0=0,\\&\left(\frac{G'}{G^2}\right)^{-3}:2\left(V^2-k^2\right)b_1\sigma^2+cb_1^3=0.\end{aligned} \tag{11}$$

Solving the above algebraic equations with the assistance of Maple, we obtain the following results:

Result 1.

$$V=\pm\sqrt{-\frac{-k^2\mu^2+4k^2\rho\sigma-2m^2}{\Delta}},\quad a_0=-\frac{m\mu}{(\Delta)cs},\quad a_1=0,\quad b_1=2s\sigma m. \tag{12}$$

$$V=\pm\sqrt{-\frac{-k^2\mu^2+4k^2\rho\sigma-2m^2}{\Delta}},\quad a_0=\frac{m\mu}{(\Delta)cs},\quad a_1=0,\quad b_1=-2s\sigma m. \tag{13}$$

Result 2.

$$V=\pm\sqrt{-\frac{-k^2\mu^2+4k^2\rho\sigma-2m^2}{\Delta}},\quad a_0=-\frac{m\mu}{(\Delta)cs},\quad a_1=2s\rho m,\quad b_1=0. \tag{14}$$

$$V=\pm\sqrt{-\frac{-k^2\mu^2+4k^2\rho\sigma-2m^2}{\Delta}},\quad a_0=\frac{m\mu}{(\Delta)cs},\quad a_1=-2s\rho m,\quad b_1=0. \tag{15}$$

where $s=\sqrt{-\frac{1}{c\Delta}}$. Substituting (12) and (13) into the solution form (8) and using (5), we get the following solutions:

$$U_{1,1}(\xi)=\frac{2s\sigma m}{-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}}-\frac{m\mu}{(\Delta)cs}$$

$$U_{1,2}(\xi)=\frac{2s\sigma m}{-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}}-\frac{m\mu}{(\Delta)cs}$$

$$U_{1,3}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}\sigma m\left(-A\sin\left(\sqrt{\rho\sigma}\xi\right)+B\cos\left(\sqrt{\rho\sigma}\xi\right)\right)}{2\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}$$

$$U_{1,4}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}\sigma m\rho\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\xi\right)-B\right)}{2\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\xi\right)+B\right)}$$

$$U_{2,1}(\xi)=\frac{2s\sigma m}{-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}}+\frac{m\mu}{(\Delta)cs}$$

$$U_{2,2}(\xi)=-\frac{2s\sigma m}{-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}}+\frac{m\mu}{(\Delta)cs}$$

$$U_{2,3}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}\sigma m\left(-A\sin\left(\sqrt{\rho\sigma}\xi\right)+B\cos\left(\sqrt{\rho\sigma}\xi\right)\right)}{2\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}$$

$$U_{2,4}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}\sigma m\rho\left(A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)-B\right)}{2\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)+B\right)}$$

where $\xi=kx-\sqrt{-\frac{-k^2\mu^2+4k^2\rho\sigma-2m^2}{\Delta}}t$,

$$U_{3,1}(\xi)=\frac{2s\sigma m}{-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}}-\frac{m\mu}{(\Delta)cs}$$

$$U_{3,2}(\xi)=\frac{2s\sigma m}{-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}}-\frac{m\mu}{(\Delta)cs}$$

$$U_{3,3}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}\sigma m\left(-A\sin\left(\sqrt{\rho\sigma}\,\xi\right)+B\cos\left(\sqrt{\rho\sigma}\,\xi\right)\right)}{2\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\,\xi\right)+B\sin\left(\sqrt{\rho\sigma}\,\xi\right)\right)}$$

$$U_{3,4}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}\sigma m\rho\left(A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)-B\right)}{2\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)+B\right)}$$

$$U_{4,1}(\xi)=\frac{2s\sigma m}{-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}}+\frac{m\mu}{(\Delta)cs}$$

$$U_{4,2}(\xi)=\frac{2s\sigma m}{-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}}+\frac{m\mu}{(\Delta)cs}$$

$$U_{4,3}(\xi)=-\frac{\sqrt{\frac{4}{c\rho\sigma}}\sigma m\left(-A\sin\left(\sqrt{\rho\sigma}\,\xi\right)+B\cos\left(\sqrt{\rho\sigma}\,\xi\right)\right)}{2\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\,\xi\right)+B\sin\left(\sqrt{\rho\sigma}\,\xi\right)\right)}$$

$$U_{4,4}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}\sigma m\rho\left(A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)-B\right)}{2\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)+B\right)}$$

where $\xi = kx + \sqrt{-\frac{-k^2\mu^2 + 4k^2\rho\sigma - 2m^2}{\Delta}}t$. Now, substituting (14) and (15) into the solution form (8) and using (5), we get the following solutions:

$$U_{5,1}(\xi) = \frac{m\mu}{(\Delta)cs} + 2s\rho m\left(-\frac{\mu}{2\rho} - \frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}\right)$$

$$U_{5,2}(\xi) = \frac{m\mu}{(\Delta)cs} + 2s\rho m\left(-\frac{\mu}{2\rho} - \frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) - B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}\right)$$

$$U_{5,3}(\xi) = \frac{\sqrt{\frac{4}{c\rho\sigma}}\rho m\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right) + B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{2\left(-A\sin\left(\sqrt{\rho\sigma}\xi\right) + B\cos\left(\sqrt{\rho\sigma}\xi\right)\right)}$$

$$U_{5,4}(\xi) = \frac{\sqrt{\frac{4}{c\rho\sigma}}m\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right) + A\cosh\left(2\sqrt{-\rho\sigma}\xi\right) + B\right)}{2\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right) + A\cosh\left(2\sqrt{-\rho\sigma}\xi\right) - B\right)}$$

$$U_{6,1}(\xi) = \frac{m\mu}{(\Delta)cs} - 2s\rho m\left(-\frac{\mu}{2\rho} - \frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}\right)$$

$$U_{6,2}(\xi) = \frac{m\mu}{(\Delta)cs} - 2s\rho m\left(-\frac{\mu}{2\rho} - \frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right) - B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right) + B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}\right)$$

$$U_{6,3}(\xi) = \frac{\sqrt{\frac{4}{c\rho\sigma}}\rho m\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right) + B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{2\left(-A\sin\left(\sqrt{\rho\sigma}\xi\right) + B\cos\left(\sqrt{\rho\sigma}\xi\right)\right)}$$

$$U_{6,4}(\xi) = \frac{\sqrt{\frac{4}{c\rho\sigma}}m\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right) + A\cosh\left(2\sqrt{-\rho\sigma}\xi\right) + B\right)}{2\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right) + A\cosh\left(2\sqrt{-\rho\sigma}\xi\right) - B\right)}$$

where $\xi = kx - \sqrt{-\frac{-k^2\mu^2 + 4k^2\rho\sigma - 2m^2}{\Delta}}t$,

$$U_{7,1}(\xi) = \frac{m\mu}{(\Delta)cs} + 2s\rho m\left(-\frac{\mu}{2\rho} - \frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right) + A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}\right)$$

$$U_{7,2}(\xi)=\frac{m\mu}{(\Delta)cs}+2s\rho m\left(-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}\right)$$

$$U_{7,3}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}\rho m\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{2\left(-A\sin\left(\sqrt{\rho\sigma}\xi\right)+B\cos\left(\sqrt{\rho\sigma}\xi\right)\right)}$$

$$U_{7,4}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}m\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\xi\right)+B\right)}{2\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\xi\right)-B\right)}$$

$$U_{8,1}(\xi)=\frac{m\mu}{(\Delta)cs}-2s\rho m\left(-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}\right)$$

$$U_{8,2}(\xi)=\frac{m\mu}{(\Delta)cs}-2s\rho m\left(-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}\right)$$

$$U_{8,3}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}\rho m\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\xi\right)+B\sin\left(\sqrt{\rho\sigma}\xi\right)\right)}{2\left(-A\sin\left(\sqrt{\rho\sigma}\xi\right)+B\cos\left(\sqrt{\rho\sigma}\xi\right)\right)}$$

$$U_{8,4}(\xi)=\frac{\sqrt{\frac{4}{c\rho\sigma}}m\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\xi\right)+B\right)}{2\left(A\sinh\left(2\sqrt{-\rho\sigma}\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\xi\right)-B\right)}$$

where $\xi=kx+\sqrt{-\frac{-k^2\mu^2+4k^2\rho\sigma-2m^2}{\Delta}}t$. Finally, using the fact that $u_{i,j}(x,t)=U_{i,j}(\xi)$, there are two indices *i* and *j*, where *i* represents the number of solutions of the ODE while *j* represents the number of solutions of the PDE (As an example of one of the above solutions is shown in **Figure 1**).

3.2. Second Application

Consider the Joseph-Egri (TRLW) equation in the form:

$$u_t+u_x+\alpha uu_x+u_{xtt}=0, \tag{16}$$

which an important model to describe the spreading wave, where α is a nonzero constant. Using the transformation $u(x,t)=U(\xi)$ where $\xi=kx-Vt$, Equation (16) changed into an ODE equation of the form:

$$(k-V)U'+\alpha kUU'+kV^2U'''=0. \tag{17}$$

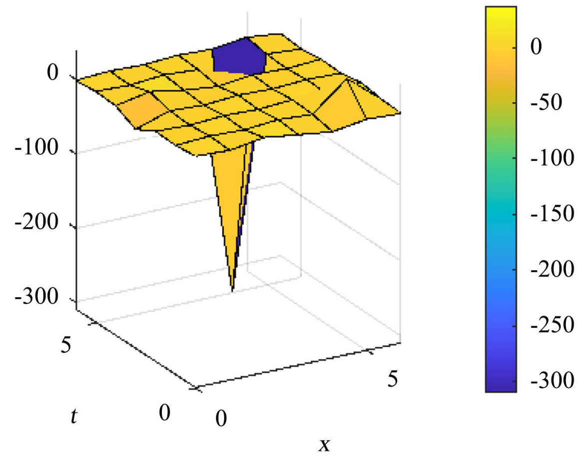

Figure 1. The solution $u_{1,3}(x,t)$ of Phi-4 equation with $A=1$, $B=2$, $k=1$, $\rho=10$, $\sigma=0.2$, $m=1$ and $c=1$.

Balancing the highest order derivative term U''' and nonlinear term UU' in Equation (17), we obtain $N=2$. The solution of Equation (17) will be in the following form:

$$U(\xi)=b_2\left(\frac{G'}{G^2}\right)^{-2}+b_1\left(\frac{G'}{G^2}\right)^{-1}+a_0+a_1\left(\frac{G'}{G^2}\right)+a_2\left(\frac{G'}{G^2}\right)^2 \tag{18}$$

a_0,a_1,a_2,b_1 and b_2 are constants to be determined later. Now, substituting Equation (18) and its derivatives in Equation (17) we will get a polynomial of degree five in $\left(\frac{G'}{G^2}\right)$, since the polynomial is equal to zero then each of its coefficients are equal to zero, as follows:

$$\left(\frac{G'}{G^2}\right)^5:\left(24kV^2a_2\rho^3+2\alpha ka_2^2\rho\right)=0,$$

$$\begin{aligned}\left(\frac{G'}{G^2}\right)^4&:2\alpha ka_1a_2\rho+\alpha ka_2\left(2a_2\mu+a_1\rho\right)\\&+kV^2\left(\left(3\left(4a_2\mu+2a_1\rho\right)\rho+18a_2\rho\mu\right)\rho+24a_2\rho^2\mu\right)=0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G^2}\right)^3&:2(k-V)a_2\rho+2\alpha ka_0a_2\rho+\alpha ka_1\left(2a_2\mu+a_1\rho\right)+\alpha ka_2\left(a_1\mu+2a_2\sigma\right)\\&+kV^2\left(\left(2\left(a_1\mu+2a_2\sigma\right)\rho+2\left(4a_2\mu+2a_1\rho\right)\mu+12a_2\rho\sigma\right)\rho\right)\\&+kV^2\left(\left(3\left(4a_2\mu+2a_1\rho\right)\rho+18a_2\rho\mu\right)\mu+24a_2\rho^2\sigma\right)=0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G^2}\right)^2&:(k-V)\left(2a_2\mu+a_1\rho\right)+2\alpha kb_1a_2\rho+\alpha ka_0\left(2a_2\mu+a_1\rho\right)\\&+\alpha ka_1\left(a_1\mu+2a_2\sigma\right)+\alpha ka_2\left(-b_1\rho+a_1\sigma\right)\\&+kV^2\left(\left(\left(a_1\mu+2a_2\sigma\right)\mu+\left(4a_2\mu+2a_1\rho\right)\sigma\right)\rho\right)\\&+kV^2\left(\left(2\left(a_1\mu+2a_2\sigma\right)\rho+2\left(4a_2\mu+2a_1\rho\right)\mu+12a_2\rho\sigma\right)\mu\right)\\&+kV^2\left(\left(3\left(4a_2\mu+2a_1\rho\right)\rho+18a_2\rho\mu\right)\sigma\right)=0,\end{aligned}$$

$$\left(\frac{G'}{G^2}\right)^1 : (k-V)(a_1\mu+2a_2\sigma)+2\alpha kb_2a_2\rho+\alpha kb_1(2a_2\mu+a_1\rho)$$
$$+\alpha ka_0(a_1\mu+2a_2\sigma)+\alpha ka_1(-b_1\rho+a_1\sigma)+\alpha ka_2(-b_1\mu-2b_2\rho)$$
$$+kV^2\left(\left((a_1\mu+2a_2\sigma)\mu+(4a_2\mu+2a_1\rho)\sigma\right)\mu\right)$$
$$+kV^2\left(\left(2(a_1\mu+2a_2\sigma)\rho+2(4a_2\mu+2a_1\rho)\mu+12a_2\rho\sigma\right)\sigma\right)=0,$$

$$\left(\frac{G'}{G^2}\right)^0 : (k-V)(-b_1\rho+a_1\sigma)+\alpha kb_2(2a_2\mu+a_1\rho)+\alpha kb_1(a_1\mu+2a_2\sigma)$$
$$+\alpha ka_0(-b_1\rho+a_1\sigma)+\alpha ka_1(-b_1\mu-2b_2\rho)+\alpha ka_2(-2b_2\mu-b_1\sigma)$$
$$+kV^2\left(\left(-(4b_2\mu+2b_1\sigma)\rho-(b_1\mu+2b_2\rho)\mu\right)\rho\right)$$
$$+kV^2\left(\left((a_1\mu+2a_2\sigma)\mu+(4a_2\mu+2a_1\rho)\sigma\right)\sigma\right)=0,$$

$$\left(\frac{G'}{G^2}\right)^{-1} : (k-V)(-b_1\mu-2b_2\rho)+\alpha kb_2(a_1\mu+2a_2\sigma)+\alpha kb_1(-b_1\rho+a_1\sigma)$$
$$+\alpha ka_0(-b_1\mu-2b_2\rho)+\alpha ka_1(-2b_2\mu-b_1\sigma)-2\alpha ka_2b_2\sigma$$
$$+kV^2\left(\left(-12b_2\sigma\rho-2(4b_2\mu+2b_1\sigma)\mu-2(b_1\mu+2b_2\rho)\sigma\right)\rho\right)$$
$$+kV^2\left(\left(-(4b_2\mu+2b_1\sigma)\rho-(b_1\mu+2b_2\rho)\mu\right)\mu\right)=0,$$

$$\left(\frac{G'}{G^2}\right)^{-2} : (k-V)(-b_1\mu-2b_2\rho)+\alpha kb_2(a_1\mu+2a_2\sigma)+\alpha kb_1(-b_1\rho+a_1\sigma)$$
$$+\alpha ka_0(-b_1\mu-2b_2\rho)+\alpha ka_1(-2b_2\mu-b_1\sigma)-2\alpha ka_2b_2\sigma$$
$$+kV^2\left(\left(-12b_2\sigma\rho-2(4b_2\mu+2b_1\sigma)\mu-2(b_1\mu+2b_2\rho)\sigma\right)\rho\right)$$
$$+kV^2\left(\left(-(4b_2\mu+2b_1\sigma)\rho-(b_1\mu+2b_2\rho)\mu\right)\mu\right)=0,$$

$$\left(\frac{G'}{G^2}\right)^{-3} : -2(k-V)b_2\sigma+\alpha kb_2(-b_1\mu-2b_2\rho)+\alpha kb_1(-2b_2\mu-b_1\sigma)-2\alpha ka_0b_2\sigma$$
$$+kV^2\left(-24b_2\sigma^2\rho+\left(-18b_2\sigma\mu-3(4b_2\mu+2b_1\sigma)\sigma\right)\mu\right)$$
$$+kV^2\left(\left(-12b_2\sigma\rho-2(4b_2\mu+2b_1\sigma)\mu-2(b_1\mu+2b_2\rho)\sigma\right)\sigma\right)=0,$$

$$\left(\frac{G'}{G^2}\right)^{-4} : \alpha kb_2(-2b_2\mu-b_1\sigma)-2\alpha kb_1b_2\sigma$$
$$+kV^2\left(-24b_2\sigma^2\mu+\left(-18b_2\sigma\mu-3(4b_2\mu+2b_1\sigma)\sigma\right)\sigma\right)=0,$$

$$\left(\frac{G'}{G^2}\right)^{-5} : -24kV^2b_2\sigma^3-2\alpha kb_2^2\sigma=0.$$

Solving the above algebraic equations with the assistance of Maple, we obtain the following results:

Result 1.

$$a_0=-\frac{V^2k\mu^2+s}{\alpha k},\ a_1=-\frac{12V^2\mu\rho}{\alpha},\ a_2=-\frac{12V^2\rho^2}{\alpha},\ b_1=0,\ b_2=0. \tag{19}$$

Result 2.

$$a_0=-\frac{V^2k\mu^2+s}{\alpha k},\ a_1=0,\ a_2=0,\ b_1=-\frac{12V^2\mu\sigma}{\alpha},\ b_2=-\frac{12V^2\sigma^2}{\alpha}. \tag{20}$$

where $s=8V^2k\rho\sigma-V+k$. Substituting (19) into the solution form (18) and using (5), we get the following solutions:

$$U_{1,1}(\xi)=-\frac{V^2k\mu^2+s}{\alpha k}-\frac{12V^2\mu\rho\left(-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}\right)}{\alpha}-\frac{12V^2\rho^2\left(-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}\right)^2}{\alpha}$$

$$U_{1,2}(\xi)=-\frac{V^2k\mu^2+s}{\alpha k}-\frac{12V^2\mu\rho\left(-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}\right)}{\alpha}-\frac{12V^2\rho^2\left(-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta\rho}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\mu^2+4\sigma\rho}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\mu^2+4\sigma\rho}\xi}{2}\right)\right)}\right)^2}{\alpha}$$

$$U_{1,3}(\xi)=\frac{s}{\alpha k}-\frac{12V^2\rho\sigma\left(A\cos\left(\sqrt{\sigma\rho}\xi\right)+B\sin\left(\sqrt{\sigma\rho}\xi\right)\right)^2}{\alpha\left(-A\sin\left(\sqrt{\sigma\rho}\xi\right)+B\cos\left(\sqrt{\sigma\rho}\xi\right)\right)^2}$$

$$U_{1,4}(\xi)=\frac{s}{\alpha k}+\frac{12V^2\rho\sigma\left(A\sinh\left(2\sqrt{-\sigma\rho}\xi\right)+A\cosh\left(2\sqrt{-\sigma\rho}\xi\right)+B\right)^2}{\alpha\left(A\sinh\left(2\sqrt{-\sigma\rho}\xi\right)+A\cosh\left(2\sqrt{-\sigma\rho}\xi\right)-B\right)^2}$$

$$U_{1,5}(\xi)=\frac{k-V}{\alpha k}-\frac{12V^2A^2}{\alpha(A\xi+B)^2}$$

Now, substituting (20) into the solution form (18) and using (5), we get the following solutions:

$$U_{2,1}(\xi)=\frac{12V^2\sigma^2}{\alpha\left(-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}\right)^2}$$

$$-\frac{12V^2\mu\sigma}{\alpha\left(-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}\right)}-\frac{V^2k\mu^2+s}{\alpha k}$$

$$U_{2,2}(\xi)=\frac{12V^2\sigma^2}{\alpha\left(-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}\right)^2}$$

$$-\frac{12V^2\mu\sigma}{\alpha\left(-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}\right)}-\frac{V^2k\mu^2+s}{\alpha k}$$

$$U_{2,3}(\xi)=\frac{12V^2\sigma\rho\left(-A\sin\left(\sqrt{\sigma\rho}\,\xi\right)+B\cos\left(\sqrt{\sigma\rho}\,\xi\right)\right)^2}{\alpha\left(A\cos\left(\sqrt{\sigma\rho}\,\xi\right)+B\sin\left(\sqrt{\sigma\rho}\,\xi\right)\right)^2}-\frac{s}{\alpha k}$$

$$U_{2,4}(\xi)=\frac{12V^2\sigma\rho\left(A\sinh\left(2\sqrt{-\sigma\rho}\,\xi\right)+A\cosh\left(2\sqrt{-\sigma\rho}\,\xi\right)-B\right)^2}{\alpha\left(A\sinh\left(2\sqrt{-\sigma\rho}\,\xi\right)+A\cosh\left(2\sqrt{-\sigma\rho}\,\xi\right)+B\right)^2}-\frac{s}{\alpha k}$$

Finally, using the fact that $u_{i,j}(x,t)=U_{i,j}(\xi)$ (As an example of one of the above solutions is shown in Figure 2).

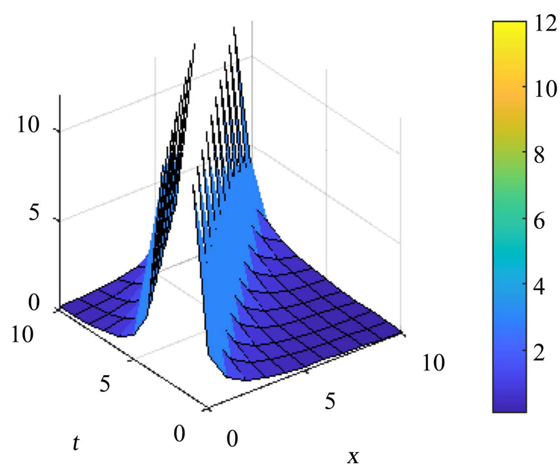

Figure 2. The solution $u_{1,5}(x,t)$ of TRLW equation with $A=1$, $B=2$, $k=1$, $V=1$ and $\alpha=-1$.

3.3. Third Application

Consider the Calogro-Degasperis (CD) equation in the form:

$$u_{xt} - 4u_x u_{xx} - 2u_y u_{xx} + u_{xxxy} = 0, \tag{21}$$

which arises in fluid mechanics, solid state physics and plasma physics. Using the transformation $u(x,y,t) = U(\xi)$ where $\xi = k(x + y - Vt)$, Equation (21) changed into an ODE equation of the form:

$$-VU'' - 4kU'U'' - 2kU'U'' + k^2U'''' = 0. \tag{22}$$

Balancing the highest order derivative term U'''' and nonlinear term $U'U''$ in Equation (22), we obtain $N = 1$. The solution of Equation (22) will be in the following form:

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G^2}\right) + b_1\left(\frac{G'}{G^2}\right)^{-1} \tag{23}$$

a_0, a_1 and b_1 are constants to be determined later. Now, substituting Equation (23) and its derivatives in Equation (22) we will get a polynomial of degree five in $\left(\frac{G'}{G^2}\right)$, since the polynomial is equal to zero then each of its coefficients are equal to zero, as follows:

$$\left(\frac{G'}{G^2}\right)^5 : 24a_1\rho^4k^2 - 12a_1^2\rho^3k = 0,$$

$$\left(\frac{G'}{G^2}\right)^4 : 60a_1\mu\rho^3k^2 - 30a_1^2\mu\rho^2k = 0,$$

$$\begin{aligned}\left(\frac{G'}{G^2}\right)^3 :& -6\left(2(-b_1\rho + a_1\sigma)a_1\rho^2 + 3a_1^2\mu^2\rho + a_1\rho\left(a_1\mu^2 + 2a_1\rho\sigma\right)\right)k \\ & - 2a_1\rho^2V + \left(\left(2\left(a_1\mu^2 + 2a_1\rho\sigma\right)\rho + 12a_1\mu^2\rho + 12a_1\rho^2\sigma\right)\rho \right.\\ & \left. + 36a_1\mu^2\rho^2 + 24a_1\rho^3\sigma\right)k^2 = 0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G^2}\right)^2 :& -6\left(-2b_1\mu a_1\rho^2 + 3(-b_1\rho + a_1\sigma)a_1\mu\rho + a_1\mu\left(a_1\mu^2 + 2a_1\rho\sigma\right)\right. \\ & \left. + a_1\rho(b_1\mu\rho + a_1\mu\sigma)\right)k + \left(\left(\left(a_1\mu^2 + 2a_1\rho\sigma\right)\mu + 6a_1\mu\rho\sigma\right)\rho\right. \\ & \left. + \left(2\left(a_1\mu^2 + 2a_1\rho\sigma\right)\rho + 12a_1\mu^2\rho + 12a_1\rho^2\sigma\right)\mu + 36a_1\mu\rho^2\sigma\right)k^2 \\ & - 3a_1\mu\rho V = 0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G^2}\right)^1 :& -6\left(-2b_1\sigma a_1\rho^2 - 3b_1\mu^2a_1\rho + (-b_1\rho + a_1\sigma)\left(a_1\mu^2 + 2a_1\rho\sigma\right)\right. \\ & \left. + a_1\mu(b_1\mu\rho + a_1\mu\sigma) + a_1\rho\left(b_1\mu^2 + 2b_1\sigma\rho\right)\right)k + \left(\left(\left(a_1\mu^2 + 2a_1\rho\sigma\right)\mu\right.\right. \\ & \left.\left. + 6a_1\mu\rho\sigma\right)\mu + \left(2\left(a_1\mu^2 + 2a_1\rho\sigma\right)\rho + 12a_1\mu^2\rho + 12a_1\rho^2\sigma\right)\sigma\right)k^2 \\ & - \left(a_1\mu^2 + 2a_1\rho\sigma\right)V = 0,\end{aligned}$$

$$\begin{aligned}\left(\frac{G'}{G^2}\right)^0 :& \; 6\left(-b_1\mu\left(a_1\mu^2 + 2a_1\rho\sigma\right) + (-b_1\rho + a_1\sigma)(b_1\mu\rho + a_1\mu\sigma)\right. \\ & \left. + a_1\mu\left(b_1\mu^2 + 2b_1\sigma\rho\right)\right)k + \left(\left(6b_1\sigma\mu\rho - \left(-b_1\mu^2 - 2b_1\sigma\rho\right)\mu\right)\rho\right. \\ & \left. + \left(\left(a_1\mu^2 + 2a_1\rho\sigma\right)\mu + 6a_1\mu\rho\sigma\right)\sigma\right)k^2 - (b_1\mu\rho + a_1\mu\sigma)V = 0,\end{aligned}$$

$$\left(\frac{G'}{G^2}\right)^{-1}: -6\Big(-b_1\sigma\left(a_1\mu^2+2a_1\rho\sigma\right)-b_1\mu\left(b_1\mu\rho+a_1\mu\sigma\right)+\left(-b_1\rho+a_1\sigma\right)\left(b_1\mu^2+2b_1\sigma\rho\right)+3a_1\mu^2 b_1\sigma+2a_1\rho b_1\sigma^2\Big)k+\Big(\left(12b_1\sigma^2\rho+12b_1\sigma\mu^2-2\left(-b_1\mu^2-2b_1\sigma\rho\right)\sigma\right)\rho+\left(6b_1\sigma\mu\rho-\left(-b_1\mu^2-2b_1\sigma\rho\right)\mu\right)\mu\Big)k^2-\left(b_1\mu^2+2b_1\sigma\rho\right)V=0,$$

$$\left(\frac{G'}{G^2}\right)^{-2}: -6\Big(-b_1\sigma\left(b_1\mu\rho+a_1\mu\sigma\right)-b_1\mu\left(b_1\mu^2+2b_1\sigma\rho\right)+3\left(-b_1\rho+a_1\sigma\right)b_1\sigma\mu+2a_1\mu b_1\sigma^2\Big)k+\Big(36b_1\sigma^2\mu\rho+\left(12b_1\sigma^2\rho+12b_1\sigma\mu^2-2\left(-b_1\mu^2-2b_1\sigma\rho\right)\sigma\right)\mu+\left(6b_1\sigma\mu\rho-\left(-b_1\mu^2-2b_1\sigma\rho\right)\mu\right)\sigma\Big)k^2-3b_1\sigma\mu V=0,$$

$$\left(\frac{G'}{G^2}\right)^{-3}: -6\Big(-b_1\sigma\left(b_1\mu^2+2b_1\sigma\rho\right)-3b_1^2\mu^2\sigma+2\left(-b_1\rho+a_1\sigma\right)b_1\sigma^2\Big)k+\Big(24b_1\sigma^3\rho+36b_1\sigma^2\mu^2+\left(12b_1\sigma^2\rho+12b_1\sigma\mu^2-2\left(-b_1\mu^2-2b_1\sigma\rho\right)\sigma\right)\sigma\Big)k^2-2b_1\sigma^2 V=0,$$

$$\left(\frac{G'}{G^2}\right)^{-4}: 60b_1\sigma^3\mu k^2+30b_1^2\sigma^2\mu k=0,$$

$$\left(\frac{G'}{G^2}\right)^{-5}: 24b_1\sigma^4 k^2+12b_1^2\sigma^3 k=0. \tag{24}$$

Solving the above algebraic equations with the assistance of Maple, we obtain the following results:

Result 1.

$$V=k^2\mu^2-4k^2\rho\sigma,\ a_0=a_0,\ a_1=2k\rho,\ b_1=0, \tag{25}$$

Result 2.

$$V=k^2\mu^2-4k^2\rho\sigma,\ a_0=a_0,\ a_1=0,\ b_1=-2k\sigma. \tag{26}$$

Substituting (25) into the solution form (23) and using (5), we get the following solutions:

$$U_{1,1}(\xi)=2k\rho\left(-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}\right)+a_0$$

$$U_{1,2}(\xi)=2k\rho\left(-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}\right)+a_0$$

$$U_{1,3}(\xi)=\frac{2k\rho\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\,\xi\right)+B\sin\left(\sqrt{\rho\sigma}\,\xi\right)\right)}{-A\sin\left(\sqrt{\rho\sigma}\,\xi\right)+B\cos\left(\sqrt{\rho\sigma}\,\xi\right)}+a_0,\quad \sigma,\rho>0$$

$$U_{1,4}(\xi)=\frac{2k\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)+B\right)}{A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)-B}+a_0$$

$$U_{1,5}(\xi)=\frac{2kA}{A\xi+B}+a_0$$

Now, substituting (26) into the solution form (23) and using (5), we get the following solutions:

$$U_{2,1}(\xi)=\frac{2k\sigma}{-\frac{\mu}{2\rho}-\frac{\sqrt{\Delta}\left(A\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+B\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}{2\rho\left(B\cosh\left(\frac{\sqrt{\Delta}\xi}{2}\right)+A\sinh\left(\frac{\sqrt{\Delta}\xi}{2}\right)\right)}}+a_0$$

$$U_{2,2}(\xi)=-\frac{2k\sigma}{-\frac{\mu}{2\rho}-\frac{\sqrt{-\Delta}\left(A\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)-B\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}{2\rho\left(A\sin\left(\frac{\sqrt{-\Delta}\xi}{2}\right)+B\cos\left(\frac{\sqrt{-\Delta}\xi}{2}\right)\right)}}+a_0$$

$$U_{2,3}(\xi)=-\frac{2k\sigma\left(-A\sin\left(\sqrt{\rho\sigma}\,\xi\right)+B\cos\left(\sqrt{\rho\sigma}\,\xi\right)\right)}{\sqrt{\frac{\sigma}{\rho}}\left(A\cos\left(\sqrt{\rho\sigma}\,\xi\right)+B\sin\left(\sqrt{\rho\sigma}\,\xi\right)\right)}+a_0,\quad \sigma,\rho>0$$

$$U_{2,4}(\xi)=\frac{2k\sigma\rho\left(A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)-B\right)}{\sqrt{-\rho\sigma}\left(A\sinh\left(2\sqrt{-\rho\sigma}\,\xi\right)+A\cosh\left(2\sqrt{-\rho\sigma}\,\xi\right)+B\right)}+a_0$$

Finally, using the fact that $u_{i,j}(x,y,t)=U_{i,j}(\xi)$, where $\xi=k\left(x+y-\left(k^2\mu^2-4k^2\right)t\right)$ (As an example of one of the above solutions is shown in **Figure 3**).

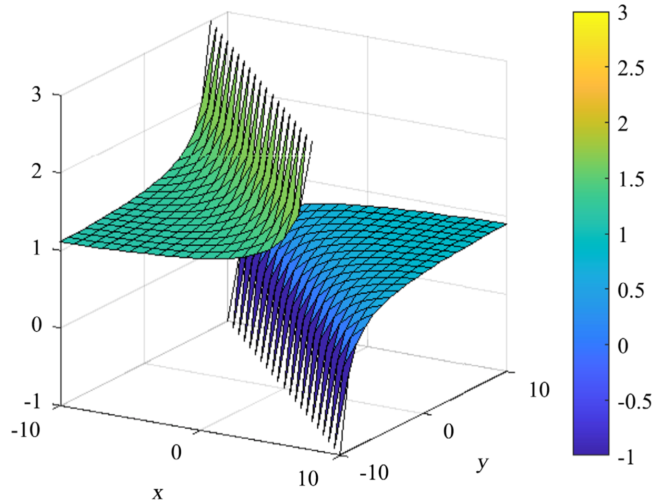

Figure 3. The solution $u_{1,5}(x,y)$ of CD equation with $A=1$, $B=1$, $k=1$, and $a_0=1$.

4. Conclusion and Future Work

A modified (G'/G^2)-expansion method has been proposed in this paper to obtain exact traveling wave solutions to the Phi-4 equation, the Joseph-Egri (TRLW) equation, and the Calogro-Degasperis (CD) equation. All solutions obtained to these equations have been verified by substitution into the original equation. With the help of the symbolic computation software Maple, this method is easy to implement and powerful. For future work, we will try to modify the auxiliary equation in order to get different new solutions.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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