Journal of Applied Mathematics and Physics, 2023, 11, 1996-2052
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Dimensional Analysis for Relativistic and Quantum Outcomes

Sebastiano Tosto

ENEA Casaccia, Roma, Italy
Email: stosto@inwind.it, stosto44@gmail.com

How to cite this paper: Tosto, S. (2023) Dimensional Analysis for Relativistic and Quantum Outcomes. *Journal of Applied Mathematics and Physics*, **11**, 1996-2052.
https://doi.org/10.4236/jamp.2023.117129

Received: June 26, 2023
Accepted: July 23, 2023
Published: July 26, 2023

Abstract

The paper highlights the concept of dimensional analysis of dynamical variables to infer quantum and relativistic information. The mathematical model implements not only single dynamical variables, but also their appropriate combinations; this chance is the added value to infer physical information. The postulates of relativity are found as corollaries in this conceptual frame. In particular even the statistical formulation of the quantum uncertainty, which has been proven valuable source of physical information itself, is obtained as a straightforward corollary along with the wave equation and the relativistic invariants. It is shown in the paper how to infer information on the nature of black holes and dark matter.

Keywords

Relativity, Quantum Physics

1. Preliminary Considerations on the Physical Model

This section aims to show that simple considerations on the dimensional analysis of acceleration a and diffusion coefficient D enable valuable relativistic and quantum information. The text is organized in order to be as self-contained as possible.

1.1. Dimensional Analysis of Acceleration *a* and Diffusion Coefficient *D*

According to mere dimensional considerations, it is possible to introduce via a the basic dynamical variables

$$\frac{a}{c^2}=\frac{1}{\ell} \quad \frac{a}{c}=\frac{1}{t} \quad a=\frac{c}{\hbar}\varepsilon \quad a=\frac{c^2}{\hbar}p \quad a=\frac{c^3}{\hbar}m \quad a=|\mathbf{a}| \quad \mathbf{p}=|\mathbf{p}|, \tag{1.1}$$

being ℓ, m, t arbitrary length, mass and time in a reference system where the moduli a and p of acceleration and momentum are as indicated. These preliminary definitions implement c to emphasize the chance of relating the basic dynamical variables uniquely to a via a constant proportionality coefficient. This initial approach is classical, e.g. the derivatives have their standard meaning. The same holds to introduce combinations of dynamical variables as well as a function of a, in particular also

$$\sqrt{\hbar cG}=\sqrt{\frac{e^2}{\alpha}G}=\frac{\ell^3}{t^2}\quad\frac{\hbar c}{G}=m_{Pl}^2\quad\frac{\ell}{m}=\frac{G}{c^2}\quad\frac{a^2}{G}=\text{energy density}. \tag{1.2}$$

Moreover, since

$$aG=\frac{c^4}{m}\Rightarrow ma=F_{Pl}\quad F_{Pl}=\frac{c^4}{G}, \tag{1.3}$$

one finds

$$\frac{c^6}{aG}=mc^2=G\frac{m^2}{\ell}\quad\Rightarrow\quad\frac{mG}{c^3}=\sqrt{\frac{\ell}{a}}=\tau\quad\ell=\frac{mG}{c^2}, \tag{1.4}$$

being τ an arbitrary time; hence

$$\ell_\tau^3=mG\tau^2\quad\ell_\tau=c\tau. \tag{1.5}$$

Analogous considerations hold for D. A further example of straightforward proportionality link is obtained merging a and D, whose physical dimensions are *length*^{2}/*time*; e.g.

$$aD=c^3\quad\frac{D^2}{a}=\ell^3\quad\frac{D}{a^2}=t^3. \tag{1.6}$$

These definitions are not mere exercise of dimensional analysis: the fact that arbitrary values of a can be uniquely related to the basic ingredients of all physical equations via combinations of fundamental constants of nature, emphasizes one of the aims of this paper: regarding the concept of acceleration only as mere change of velocity of a matter body in a force field, is reductive. Rather it is useful to regard a in a more general and heuristic way, *i.e.* still implementing its physical dimensions but replacing more realistically c with a variable velocity modulus v depending on the specific physical problem. To account for the vector character of acceleration, for example, the dimensional Equations (1.1) are more conveniently rewritten as

$$a\ell=\begin{matrix}\nearrow v^2\\ \searrow c^2\end{matrix}\qquad v=\frac{\mathrm{d}\ell}{\mathrm{d}t}: \tag{1.7}$$

introducing the scalar function $\varphi=\varphi\left(velocity^2\right)$, the upper line yields

$$\mathbf{a}\propto\pm\nabla\varphi\quad\varphi\propto v^2, \tag{1.8}$$

whereas ℓ is such that the component a_ℓ of the vector $\mathbf{a}$ along the arbitrary direction $\boldsymbol{\ell}$ reads in general

$$a_\ell \propto \pm \frac{\mathrm{d}\varphi}{\mathrm{d}\ell}. \tag{1.9}$$

Note now that φ is potential energy per unit mass, so it takes in general the form appropriate to the specific purposes; here however it is defined in order to be suitable for gravitational problems. In fact, classically

$$\frac{\mathrm{d}v_\ell}{\mathrm{d}t} = \pm \frac{\mathrm{d}\varphi}{\mathrm{d}\ell} \Rightarrow \mathrm{d}v_\ell = \pm \frac{\mathrm{d}\varphi}{\mathrm{d}\ell/\mathrm{d}t} = \pm \frac{\mathrm{d}\varphi}{v_\ell} \Rightarrow \mathrm{d}\varphi = \pm v_\ell \mathrm{d}v_\ell \Rightarrow \varphi = \pm \frac{v_\ell^2}{2} + const. \tag{1.10}$$

So in general φ is a function defined by its own value of velocity; with the minus sign, in particular, it is the gravitational potential $-\varphi$. In the following φ is regarded as a positive quantity, whereas $-\varphi/2$ is the classical gravitational potential.

Follow some straightforward physical implications of this preliminary analysis.

1.2. Introductory Corollaries

What are the definitions (1.1) to (1.7) for? First of all it is necessary to show their self-consistency, in particular as concerns (1.7), and then their physical implications.

Consider first the lower line (1.7) that yields by differentiating $a\delta\ell + \ell\delta a = 0$ and thus

$$a = -\ell \frac{\delta a}{\delta \ell} = -\ell \frac{\delta}{\delta \ell}\frac{\delta v}{\delta t} = -\ell \frac{\delta}{\delta t}\frac{\delta v}{\delta \ell} \tag{1.11}$$

as $\delta\ell$ and δt are arbitrary and independent differential ranges in defining *v*, arbitrary itself as well. Suppose that exists a function *f* fulfilling the following requirements

$$\frac{a}{\ell} = -\frac{\delta}{\delta t}\frac{\delta v}{\delta \ell} \Rightarrow \frac{\delta v}{\delta \ell} = \frac{\delta L}{\delta \dot{f}} \quad \frac{a}{\ell} = -\frac{\delta L}{\delta f}, \tag{1.12}$$

where *L* is a new function to be found. Then it is necessary that *L* fulfills in turn

$$\delta L = \frac{a}{\ell}\delta f = \frac{\delta v}{\delta \ell}\delta \dot{f}.$$

This function does in fact exist because, according to (1.1)

$$\frac{1}{\ell}\frac{\delta v}{\delta t}\delta f = \frac{\delta v}{\delta \ell}\delta \dot{f} \Rightarrow \frac{1}{\ell}\frac{\delta f}{\delta t} = \frac{\delta \dot{f}}{\delta \ell} \Rightarrow \frac{v\delta t}{\ell}\frac{\delta f}{\delta t} = \delta \dot{f} \Rightarrow \frac{\delta \ell}{\ell}\delta \dot{f} = \delta \dot{f}:$$

clearly $\delta\ell/\ell = 1$ is fulfilled for example by $\delta\ell = \ell' - \ell_0 = \ell$ or by $\delta\ell = \ell - 0$ with $\ell_0 = 0$. Anyway (1.11) and (1.12) yield the Lagrange equation of the generalized coordinate *f*, *i.e.*

$$-\frac{a}{\ell} = -\frac{\delta}{\delta t}\frac{\delta v}{\delta \ell} \Rightarrow \frac{\delta L}{\delta f} = \frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{f}}. \tag{1.13}$$

Consider now the upper line of (1.7). First of all regard again (1.1) as follows

$$a = \frac{c^3}{\hbar}m = \frac{c^2}{\hbar}p = \frac{c}{\hbar}\varepsilon \Rightarrow a = \frac{v^3}{\hbar}m = \frac{v^2}{\hbar}p = \frac{v}{\hbar}\varepsilon$$

and calculate

$$\hbar\frac{\partial a}{\partial v}=3mv^2=2pv=1\varepsilon:$$

of course the three energies can be made equal selecting appropriately the arbitrary values of the respective dynamical variables ε, p, m. Dividing all terms of this chain by an arbitrary volume 3 *V*, this result reads

$$\frac{E}{V}=1\frac{\varepsilon''}{V}=\frac{2}{3}\frac{\varepsilon'}{V}=\frac{1}{3}\frac{\varepsilon}{V}\quad E=\frac{\hbar}{3}\frac{\partial a}{\partial v}\quad \varepsilon'=pv\quad \varepsilon''=mv^2,$$

which in turn defines

$$P=1\eta''=\frac{2}{3}\eta'=\frac{1}{3}\eta\quad P=\frac{\hbar}{3V}\frac{\partial a}{\partial v}\quad \eta''=\frac{\varepsilon''}{V}\quad \eta'=\frac{\varepsilon'}{V}\quad \eta=\frac{\varepsilon}{V}.$$

Clearly the primed symbols η are energy densities. The left hand side is different, but of course must have the same physical dimensions of energy density; then P can be nothing else but pressure. These results are eventually summarized as

$$P=\frac{j}{3}\eta\quad j=1,2,3,\tag{1.14}$$

which relates energy density inside an arbitrary volume to the surface pressure, depending on whether the particles inside V are reflected and bounce or not when impinging on the internal surface. The chance $P=\eta$ is the state equation of an ideal gas.

A similar attempt can be made with (1.3)

$$aG=\frac{c^4}{m}\ \Rightarrow\ aG=\frac{v^4}{m},\tag{1.15}$$

which can be integrated and yields

$$\pm aG=\pm\frac{\partial v}{\partial t}G=\frac{v^4}{m}\ \Rightarrow\ v=\mp\left(\frac{mG/3}{t+mC}\right)^{1/3}\quad C=\frac{t_0}{m_0}\tag{1.16}$$

being C the integration constant. It appears that v is given by

$$0\le|v|\le v_0\quad 0\le t\le\infty\quad v_0=\left(\frac{G/3}{C}\right)^{1/3}:\tag{1.17}$$

whatever m_0 and t_0 might be, $\pm v$ is bound in a well defined range
$-v_0\le v\le v_0$ not dependent upon m itself above and below the time axis. Moreover, it also results

$$at=v\frac{t/3}{mC+t}\ \Rightarrow\ \lim_{t\to\infty}at\propto v\quad \lim_{t\to 0}at=0\quad \lim_{t\to 0}v=\left(\frac{G/3}{C}\right)^{1/3}:\tag{1.18}$$

i.e. the classical relationship is verified at times $t\gg mC$ only, *i.e.* at $v\ll v_0$.

Eventually, since the integration constant is arbitrary, it is possible to examine the asymptotic case $t\gg mC$, at which (1.16) reads

$$v^3t\approx\mp\frac{mG}{3}\ \Rightarrow\ \frac{(vt)^3}{t^2}=\ell^3\omega^2\approx\mp\frac{mG}{3}\quad \omega=\frac{1}{t}.\tag{1.19}$$

At this point, here is a summary to check the self-consistency of these results: the previous (1.9), (1.7), (1.22) and (1.15) are

$$a_\ell = \pm\frac{\partial\varphi}{\partial\ell} \quad a_\ell = \frac{v_\ell^4}{mG} \quad \varphi = \frac{v_\ell^2}{2} \Rightarrow \pm\frac{\partial\varphi}{\partial\ell} = \pm v_\ell \frac{\partial v_\ell}{\partial\ell} = \frac{v_\ell^4}{mG}.$$

Then the condition to be checked requires solving

$$\pm\frac{\partial v_\ell}{\partial\ell} = \frac{v_\ell^3}{mG} \Rightarrow \mp\frac{1}{2v_\ell^2} = \frac{(1+mGconst)\ell}{mG},$$

i.e.

$$v_\ell = \pm\sqrt{\frac{mG}{const'\ell}} \quad const' = 1 + mGconst;$$

in effect

$$\varphi = \pm\frac{v_\ell^2}{2} = \pm\frac{mG}{2const'\ell} \Rightarrow m'\varphi = \pm G\frac{m'm}{\ell} \quad 2const' = 1 \tag{1.20}$$

with the minus sign is the Newtonian potential energy of m' in the field of m.

Despite the classical approach, e.g. no care has been devoted to the reference system where are defined time and space coordinates and velocities, the outcomes are not at all classical, see in particular (1.18) and (1.4); also, note that merging (1.1) one finds various admissible forms of energy correlatable to a, *i.e.*

$$a = \frac{c^2}{\ell} \leftrightarrow \frac{c}{\hbar}\varepsilon \leftrightarrow \frac{c}{\hbar}pc \leftrightarrow \frac{c}{\hbar}mc^2 \Rightarrow \frac{\hbar c}{\ell} = E \leftrightarrow \varepsilon \leftrightarrow pc \leftrightarrow mc^2. \tag{1.21}$$

These energies deserve special attention for their importance in relativity and are concerned later. At this point it is convenient to deal with the special relativity directly.

1.3. Basics of Special Relativity

- Consider an arbitrary constant diffusion coefficient D_0 and define

$$D_0\delta t = D_0(t - t_0), \tag{1.22}$$

which has physical dimensions of a time dependent square length; t_0 is an arbitrary constant time, t an arbitrary time. Rewrite identically this expression as follows

$$D_0 t - D_0 t_0 = \ell^2 = \frac{D_0}{t_0}t_0 t - D_0 t_0 = v_0^2 t'^2 - \ell_0^2 \quad v_0^2 = \frac{D_0}{t_0} \quad t'^2 = t_0 t \quad \ell_0^2 = D_0 t_0 \tag{1.23}$$

whence $v_0^2 t' - \ell^2 = \ell_0^2$. Being $\ell_0 = const$ and t arbitrary, repeating the reasoning with $D_0 t'' - D_0 t_0$ one would find $v_0^2 t''' - \ell''^2 = \ell_0^2$, whence

$$v_0^2 t''' - \ell''^2 = \ell_0^2 = v_0^2 t' - \ell^2 \tag{1.24}$$

and so on; *i.e.*, whatever the primed quantities might be, one infers with simplified notation

$$v_0^2 t^2 - \ell^2 = invariant. \tag{1.25}$$

An elementary dimensional analysis has shown that (1.25) defines an invariant because anyway both sides share a common value fixed by the arbitrary constant ℓ_0. This result is particularly significant because it is shown in [1] [2] that the invariant interval is the foundation of all special relativity. Clearly v_0 is a finite velocity, it would be a nonsense to divide D_0 by $t_0 = 0$; usually this v_0 is indicated in the literature as *c*, which cannot be infinite because D_0/t_0 must be finite by definition.

Moreover put $v_0 t_0 = \ell_0$; multiplying both sides of (1.25) by the constant t_0^2 yields

$$\left(\ell_0 t\right)^2 - \left(\ell t_0\right)^2 = invariant' = \left(\ell_0 t'\right)^2 - \left(\ell' t_0\right)^2 \quad \Rightarrow \quad \ell t = invariant, \qquad (1.26)$$

which suggests that $\ell_0 t$ and ℓt_0 should be invariant themselves. This information is better highlighted writing, still with notation (1.23),

$$\delta s_{inv}^2 = v_0^2 t^2 - \ell^2 \quad \Rightarrow \quad \delta s_{inv}^2 = v_0^2 t^2 \left(1 - \frac{\ell^2/t^2}{v_0^2}\right)$$

whence

$$\frac{\delta s_{inv}^2}{v_0^2} = \tau^2 = t^2 \beta^2 \quad \delta s_{inv}^2 = \delta s^2 \left(1 - \frac{\ell^2/t^2}{v_0^2}\right) \Rightarrow t = \frac{\tau}{\beta} \quad \delta s' = \delta s \beta \quad \beta^2 = 1 - \frac{v^2}{v_0^2}: \qquad (1.27)$$

i.e. one finds length contraction and time dilation inherent the Lorentz transformation of the proper quantities when, in particular, $v = const$ concerns inertial reference frames reciprocally displacing. These considerations confirm that $v_0 \equiv c$. From these invariant forms follows the corollary of covariance of physical laws, well known and not concerned for brevity.

It is easy to show that the special relativity is elementary and straightforward generalization of the classical physics. In classical physics the energy ϵ of a particle is defined an arbitrary constant apart, due to the initial boundary conditions of the problem; write therefore $\epsilon' = \epsilon + const$. Yet the integration constant can also be in principle positive or negative, either because of different initial conditions of the problem or because one regards ϵ in a different reference frame or because of other specific reasons: e.g. it could be potential energy in attractive or repulsive force field. Thus is important to continue the reasoning simply considering also $\epsilon'' = \epsilon - const$, being ϵ' and ϵ'' arbitrary. At this point multiply side by side to merge these chances: one finds trivially
$\epsilon' \epsilon'' = \epsilon^2 - const^2$, *i.e.*

$$\epsilon^2 = \epsilon'''^2 + const^2 \Rightarrow \frac{\epsilon^2}{const^2} = \frac{\epsilon'''^2}{const^2} + 1 \quad \epsilon'''^2 = \epsilon' \epsilon''. \qquad (1.28)$$

To show as shortly as possible the implications of this result, note its consistency with

$$\epsilon^2 = \left(pc\right)^2 + \left(mc^2\right)^2 \quad p = \frac{\epsilon v}{c^2}$$
$$\Rightarrow \frac{\epsilon^2}{\left(mc^2\right)^2} = \frac{\left(pc\right)^2}{\left(mc^2\right)^2} + 1 \quad \epsilon_0 = const = mc^2 \quad \epsilon''' = pc, \qquad (1.29)$$

which also implies β found in (1.27). Elementary considerations, omitted for brevity, justify the rest energy and the momentum definition: e.g. ϵ_0^2 resulting for $p=0$ can be nothing but rest energy.

Consider eventually one particle moving at rate v along an arbitrary axis with constant acceleration a_0 and let τ be its proper time; thus a_0 is 3D acceleration in a proper reference system, with components $a = a\left(0,0,0,a_0/c^2\right)$. Then $v = a_0\tau$ yields, owing to (1.27),

$$v^2 = \left(a_0 t\beta\right)^2 \quad t = \frac{\tau}{\beta} \quad v\left(t=0\right) = 0, \tag{1.30}$$

which in turn reads

$$v^2 = \left(a_0 t\right)^2 \left(1 - \frac{v^2}{c^2}\right) \Rightarrow v^2 = \frac{\left(a_0 t\right)^2}{1 + \frac{\left(a_0 t\right)^2}{c^2}}.$$

Hence the well known result is

$$v = \pm \frac{a_0 t}{\sqrt{1 + \frac{\left(a_0 t\right)^2}{c^2}}}. \tag{1.31}$$

Note that this result agrees with in principle with (1.18) and that (1.30) yields $v = \pm a_0 t\beta$ *i.e.*, as in (1.29),

$$v = \pm a_0 t \frac{mc^2}{\epsilon} \Rightarrow \epsilon = \frac{mc^2}{\beta} \quad p = ma_0 t = \pm \frac{\epsilon v}{c^2}. \tag{1.32}$$

These preliminary results highlight that the fundamental postulates of modern physics: e.g. space-time properties of dynamical variables, four-vectors, necessity of covariant physical laws and so on, are by passable being actually corollaries of elementary dimensional considerations having classical character.

Moreover, by differentiating (1.1) consider

$$\delta a = -\frac{c^2}{\ell^2}\delta\ell \quad \delta a = -\frac{c}{t^2}\delta t \Rightarrow \frac{\delta a}{\delta\ell} = -\frac{a}{\ell} \quad \frac{\delta a}{\delta t} = -\frac{a}{t} \tag{1.33}$$

to highlight how the signs of $\delta\ell$ and δt are related to that of δa.

A relevant feature of the chance to regard the physical equations via their dimensional analysis, is the result of their merging to introduce further dynamical variables into their initial definition. An example of this chance is the second (1.2) that reads with the help of the first (1.1)

$$\frac{a}{G} = \frac{m}{\ell^2}. \tag{1.34}$$

Then, with the positive sign of a it is possible to write

$$\frac{\partial a/G}{\partial\ell} = -\frac{2m}{\ell^3} \Rightarrow \frac{\partial a}{\partial\left(1/\ell\right)} = \frac{2mG}{\ell};$$

next, since owing to the first (1.1) $\partial a/\partial\left(1/\ell\right) = c^2$, the result is

$$\ell_{bh} = \frac{2mG}{c^2}. \tag{1.35}$$

The notation means that if the initial ℓ of (1.1), in principle arbitrary by definition, fulfills the requirement resulting from its merging with (1.2), which involves *G*; then ℓ_{bh} must surely have a new physical meaning. This relevant result will be concerned later.

As this kind of analysis appears also extensible to the general relativity, consider again G to introduce a further example. Since $G=\left(\rho_0 t_0^2\right)^{-1}$, being ρ_0 a constant density, define

$$G\delta\rho=\frac{1}{t^2}-\frac{1}{t_0^2}\quad \delta\rho=\rho-\rho_0\quad \rho=\rho\left(x,t\right) \tag{1.36}$$

Multiplying both sides by ℓ_0^2/c^2 one finds

$$G\frac{\ell_0^2}{c^2}\delta\rho=\left(\frac{\ell_0^2}{t^2}-\frac{\ell_0^2}{t_0^2}\right)\frac{1}{c^2}=\frac{v^2-v_0^2}{c^2}=\frac{v\delta v}{c^2}=\frac{\delta\varphi}{c^2}\quad v=\frac{\ell_0}{t}\quad \delta v=v-\frac{v_0^2}{v}\quad \delta\varphi<0; \tag{1.37}$$

the meaning of v_0 given by (1.23) requires $\delta v<0$ and thus the quoted sign of $\delta\varphi$. Write now

$$G\frac{\ell_0^2}{c^2}\delta\rho=\frac{\left(v+v_0\right)\ell'}{c^2}\frac{v-v_0}{\ell'}=\frac{1}{\omega''}\left(\omega'-\omega_0'\right)\quad \omega'=\frac{v}{\ell'}\quad \omega_0'=\frac{v_0}{\ell'}\quad \frac{1}{\omega''}=\frac{\left(v+v_0\right)\ell'}{c^2}, \tag{1.38}$$

being ℓ' an arbitrary length. This result therefore yields the well known redshift of the photon

$$\frac{\delta\varphi}{c^2}=\frac{\delta\omega}{\omega} \tag{1.39}$$

subjected to the attractive gravitational potential change $\delta\varphi$ along an arbitrary radial direction with respect to the source of φ.

Consider now that, however, nothing requires in principle v_0 defined by ℓ_0/t_0 to be exactly equal that of (1.23); actually ℓ_0 and t_0 concerned in (1.38) are arbitrary constants. Examine thus the case where

$$\frac{\ell_0}{t_0}=v_0'\quad \delta v=v-\frac{v_{0'}^2}{v}\gtreqless 0:$$

the notation does not exclude any chance in lack of specific information about v_0'. So (1.38) becomes now

$$G\frac{\ell_0^2}{c^2}\delta\rho=\frac{\left(v+v_0'\right)\ell'}{c^2}\frac{v-v_0'}{\ell'}=\frac{\delta\varphi}{c^2}\quad v=\frac{\ell_0}{t}\quad \delta v=v-\frac{v_{0'}^2}{v}\quad \delta\varphi\gtreqless 0. \tag{1.40}$$

Having changed the sign of $\delta\varphi$, which now is no longer gravitational potential, the physical meaning of (1.40) must be redefined. Nevertheless, as the form of φ is unchanged, it is still possible to write as in the previous (1.38)

$$G\frac{\ell_0^2}{c^2}\delta\rho=\frac{v-v_0'}{\ell'}\frac{\left(v+v_0'\right)\ell'}{c^2}=\omega'''\tau\quad \omega'''=\frac{v-v_0'}{\ell'}\quad \tau=\frac{\ell'\left(v+v_0'\right)}{c^2}\quad v\gtreqless v_0'. \tag{1.41}$$

The physical meaning of ω''' is by definition velocity per unit length, whereas the time factor τ defines the time function at the left hand side

$$\left(G\frac{\ell_0^2}{c^2}\right)\frac{\rho-\rho_0'}{\tau}=\omega''': \tag{1.42}$$

now the sign of ω''' depends on that of $\rho-\rho_0'$. The cosmological implication of (1.42) will be concerned later.

The aim of the paper is to better explain in detail these points and, more in general, to highlight how to infer further physical information including contextually quantum outcomes.

1.4. Classical and Relativistic Corollaries

Implement (1.7), (1.6) and (1.1) to assess the chance of replacing c with a characteristic average velocity $\overline{v}$: while the dimensional analysis still holds in principle, such a specific $\overline{v}$ does in fact exist and fits more realistically the problem. Three significant cases clarify the physical meaning of the respective positions

$$aD=c^3 \Rightarrow aD=\overline{v}^3 \quad \frac{\ell^3}{t^2}=\frac{c^4}{a} \Rightarrow \frac{\ell^3}{t^2}=\frac{\overline{v}^4}{\overline{a}} \quad a=m\frac{c^3}{\hbar} \Rightarrow a=m\frac{\overline{v}^3}{\hbar}. \tag{1.43}$$

The coordinate system is the one where are defined the moduli of dynamical variables of these equations.

1) As concerns the first (1.43), assuming $v=\partial\ell/\partial t$ and replacing v with its average $\overline{v}=\delta\overline{\ell}/\delta\overline{t}$, one finds

$$a\ell=v^2=\frac{Da}{v} \Rightarrow \overline{\ell}=\frac{D}{\overline{v}}=D\frac{\partial\overline{t}}{\partial\overline{\ell}} \Rightarrow \overline{\ell}\delta\overline{\ell}=\frac{1}{2}\delta\overline{\ell}^2=D\delta\overline{t} \Rightarrow \delta\overline{\ell}^2=2D\delta\overline{t}: \tag{1.44}$$

the result is the Einstein-Smoluchowski equation of the one-dimensional Brownian motion, having taken in the last step the average values of displacement and time at both sides. Hence $\overline{v}$ does effectively exist.

2) The second (1.43) is the third Kepler law with orbital parameters realistically hidden in $\overline{v}^4/\overline{a}$. Since

$$\frac{\ell^3}{t^2}=\frac{v^4}{\partial v/\partial t} \quad \ell=\ell(t) \quad v=v(t) \quad \Rightarrow \quad \frac{\ell^3 \mathrm{d}v}{v^4}=t^2\mathrm{d}t, \tag{1.45}$$

then owing to (1.5)

$$v=\left(-3\int\frac{t^2}{\ell(t)^3}\mathrm{d}t+const\right)^{-\frac{1}{3}}=\left(-3\int\frac{\mathrm{d}t}{mG}+const\right)^{-\frac{1}{3}} \Rightarrow \frac{\overline{v}^4}{\partial\overline{v}/\partial t}=mG=\frac{\overline{\ell}^3}{t^2}; \tag{1.46}$$

the last equation explains the chance of replacing c with the given $\overline{v}(t)$.

3) The third (1.43) reads in general

$$a_\pm=\pm\frac{\mathrm{d}}{\mathrm{d}t}\overline{v}=m\frac{\overline{v}^3}{\hbar} \quad \overline{v}=\overline{v}(t) \quad \overline{v}(t=0)=0, \tag{1.47}$$

which accounts for the two chances $a_\pm$. Integrate (1.47) keeping the notations of (1.23), the solutions read

$$\overline{v}_\pm=\pm\frac{v_0}{\sqrt{\dfrac{2mv_0^2t}{\hbar}+1}}: \tag{1.48}$$

both signs of (1.48) describe $-v_0 < -\overline{v}(t) < 0$ or $0 < \overline{v}(t) < v_0$ for $0 < t < \infty$; it is essential that in neither case $\overline{v}$ diverges, as it was already found in (1.31). Also, if $m = 0$ then $\overline{v} \equiv \pm v_0$. Reminding that actually $v_0 \equiv c$ according to (1.23) and (1.25), then (1.48) is sensible. To clarify further this point, keep the same notation and implement

$$\frac{\overline{v}_-\overline{v}_+}{v_0^2} = -\frac{V^2}{v_0^2} \quad \frac{V^2}{v_0^2} = \frac{1}{1 + mv_0^2/(\hbar/2t)}; \tag{1.49}$$

sum 1 at both sides of the first equation

$$1 + \frac{\overline{v}_-\overline{v}_+}{v_0^2} = \beta^2 \quad \beta^2 = 1 - \frac{V^2}{v_0^2}, \tag{1.50}$$

with $v_- < 0$ and $v_+ > 0$ by definition.

This result is implementable in two ways.

- Take the reciprocals of both sides of (1.50) and multiply the resulting equation by the further arbitrary velocity $\overline{v}'_+ + \overline{v}'_-$; this yields

$$\overline{v}'_+ + \overline{v}'_- = \frac{\overline{v}_+ + \overline{v}_-}{1 + \dfrac{\overline{v}_-\overline{v}_+}{c^2}} \quad \overline{v}_+ + \overline{v}_- = \frac{\overline{v}'_+ + \overline{v}'_-}{\beta^2}. \tag{1.51}$$

This result is known; is relevant the fact that (1.48) and (1.51) confirm the previous identification (1.23)

$$v_0 \equiv c. \tag{1.52}$$

Anyway, deserves attention (1.51) written with the general form $v' = \beta^2 v''$, which has physical meaning itself. Indeed, according to the upper line of (1.7),

$$v' = \frac{\ell'}{t'} \quad v'' = \frac{\ell''}{t''} \Rightarrow \frac{\ell'}{t'} = \beta^2 \frac{\ell''}{t''} \Rightarrow \ell' = \beta \ell'' \text{ and } t' = \frac{t''}{\beta}: \tag{1.53}$$

with V of (1.50) constant, in particular, appear the Lorentz transformations defining length contraction and time dilation with respect to the proper double primed variables, as already found in (1.27). Thus (1.51), (1.48) and (1.53) confirm the validity of (1.47).

- Rewrite now (1.50) as follows

$$1 - \frac{v_1 v_2}{c^2} = \frac{V'^2}{c^2} \quad \frac{V'^2}{c^2} = \frac{c^2 - V^2}{c^2} \quad -v_1 v_2 = v_+ v_-, \tag{1.54}$$

which does not requires $v_1 = v_2$, being enough that are both positive; this result reads

$$\frac{v_1}{c} = 1 - \frac{V'}{c} \quad \frac{v_2}{c} = 1 + \frac{V'}{c}, \tag{1.55}$$

which in turn yields

$$\frac{v_2}{v_1} = \frac{1 + V'/c}{1 - V'/c} \Rightarrow \frac{cv_2}{cv_1} = \frac{c + V'}{c - V'}.$$

Since vc has physical dimensions of square velocity, this result is formally compatible with

$$\frac{\ell_0^{-2} v_2 c}{\ell_0^{-2} v_1 c} = \frac{v_2^2}{v_1^2} = \frac{c+V'}{c-V'}. \tag{1.56}$$

Actually the square ratio at the left hand side is essential if V' is regarded as the velocity at which a luminous massive source emitting a given frequency moves with respect to an observer, in which case the observed frequency ν_1 is different from the emitted frequency ν_2.

In fact this result yields the relativistic Doppler effect where emitter and observer move with relative constant velocity *v*. Note that because of the time dilation with respect to the proper time, holds

$$\frac{\nu_1/\ell_0}{\nu_2/\ell_0} = \frac{\nu_1}{\nu_2} = \frac{\nu_{obs}}{\nu_{em}} = \frac{t_{em}}{t_{obs}} = \frac{1}{\sqrt{1-V^2/c^2}} = \frac{c}{\sqrt{(c-V)(c+V)}}, \tag{1.57}$$

having identified t_{em} as the proper time of the emitter and t_{obs} as the time of a mobile observer at the origin of the respective reference systems. With $t_{em} < t_{obs}$ the observed frequency is red shifted for the observer. In conclusion

$$t_{obs} = \frac{t_{em}}{\beta} \Leftrightarrow \nu_{em} = \frac{\nu_{obs}}{\beta} \Leftrightarrow \frac{\nu_2}{\nu_1} \equiv \frac{\nu_{em}}{\nu_{obs}} = \sqrt{\frac{c+V'}{c-V'}}: \tag{1.58}$$

the square root is not simply suggested by (1.56), it is essential to fit (1.58) and (1.57).

At this point however, since in agreement with (1.54)

$$t_{obs} = \frac{t_{em}}{\beta} \Rightarrow \frac{v^2}{c^2} + \frac{v'^2}{c^2} = 1, \tag{1.59}$$

the question rises: do v/c and v'/c have probabilistic meaning? In principle this question is sensible because the velocity range allowed to any particle is finite, so that in principle v/c can be regarded as a probabilistic property of a free particle.

Let $\delta v = v_2 - v_1$ be a possible velocity range enclosing an arbitrary local velocity v such that by definition $v_1 \le v \le v_2$. Consider then the actual velocity v of one massive particle to calculate its probability Π_v of being effectively included in δv. The reasoning is Bayes-like, *i.e.* the probability for an event to occur considers explicitly also the probability that the given event does not occur.

Let $\Pi_1 = v_1/c$ be the probability of velocity v_1, and $\Pi_2 = v_2/c$ that of v_2: then, for the particle to be described by the allowed range δv is necessary that $\Pi_v = \Pi_1\Pi_2 \ne 0$, as by definition both boundary velocities are attainable by the particle. Moreover is also necessary to define the residual probability $\Pi_{nv} = 1 - \Pi_v$ that the particle can take any velocity *v*, *i.e.* $v < v_1$ and $v > v_2$; as in fact v_2 can only tend asymptotically to c according to (1.49), then, whatever v_2 might be, it is possible to imagine another v' even closer to c than v_2. So Π_v is reasonably proportional to $\delta v/c$ and inversely proportional to Π_{nv}; in other words it should be true that

$$\Pi_v = \frac{\delta v/c}{\Pi_{nv}} = \frac{\delta v/c}{1-\Pi_v} = \frac{(v_2 - v_1)/c}{1 - v_1 v_2/c^2} \quad v_1 \le v \le v_2 \quad v_1 \ne 0, v_2 \ne 0. \tag{1.60}$$

This result is analogous to (1.51). Note that writing (1.60) as $\left(v_2''-v_1''\right)/\left(1-v_1'v_2''/c^2\right)$, if either $v_1''/c\to 1$ or $v_2''/c\to 1$ then $\Pi_v\to 1$, *i.e.* one finds again a probabilistic meaning of the asymptotic value of sum of velocities found in (VT1) and (B4B); moreover it also appears that $\Pi_v\to 1$ even for $v_2''\to v_1''$.

This section has shown that even relativistic results can take probabilistic meaning, which is a crucial step to merge relativity and quantum physics.

Once more, the classical definitions (1.43) have implied the relativistic (1.51), (1.48), (1.60) and (1.58).

1.5. Generalization of the Dimensional Analysis

In this section the dimensional analysis of the Section 1 is generalized to concern quantum and relativistic equations, not only physical dynamical variables.

In fact (1.4) is a Newtonian potential, which implies $a<0$ and thus $-\overline{v}$ at the right hand side of (1.43) to describe an attractive effect. The negative sign of a yields

$$\frac{c^6}{(-a)G}=-G\frac{m^2}{\ell}:$$

then the right hand side is a Newtonian potential energy in agreement with a negative acceleration.

Moreover, consider that (1.2) admits

$$\ell=\frac{mG}{c^2}\quad\Rightarrow\quad\ell_v=\frac{mG}{\overline{v}^2},\tag{1.61}$$

which yields

$$\overline{v}^2=\frac{mG}{\ell_v}\ \Rightarrow\ \delta\ell_v=-\frac{2mG}{\overline{v}^3}\delta\overline{v}=-\tau_v\delta v\quad\tau_v=\frac{2mG}{\overline{v}^3}.\tag{1.62}$$

Also here the symbol $\overline{v}$ takes the meaning of (1.43), *i.e.* a definite value of v suitable to replace c in a specific problem. As in (1.43), consider separately the two chances (1.61).

- Differentiating both sides of (1.4)

$$-\frac{c^6\delta a}{a^2G}=-\frac{Gm^2\delta\ell}{\ell^2},$$

which reads, owing to (1.33),

$$\frac{c^4}{G}\frac{\delta a}{\delta\ell}\ell^2=G(ma)^2\quad\Rightarrow\quad\frac{\delta a}{\delta\ell}(c\ell)^2=-\frac{a}{\ell}(c\ell)^2=\frac{G(ma)^2}{F_{Pl}}\tag{1.63}$$

- The second chance (1.61) yields

$$\delta\ell'=-\overline{v}\frac{\delta\ell_v}{\delta\overline{v}}=\tau_v\overline{v}=\frac{2mG}{\overline{v}^2}\tag{1.64}$$

and

$$\delta\ell'=\frac{mG}{\varphi}\tag{1.65}$$

i.e., multiplying both equations by an arbitrary m', one finds

$$m'\varphi = \frac{m'mG}{\delta\ell'} \quad m'\varphi = \frac{m'v^2}{2}. \tag{1.66}$$

The physical meaning of $\delta\ell'$ also appears noting that

$$v_{esc} = \sqrt{\frac{2mG}{\delta\ell'}} \tag{1.67}$$

is the classical escape velocity of any body of matter initially $\delta\ell'$ apart from the gravity source m with attractive gravitational potential.

Considering (1.64) and (1.67), for $v \to c$ one finds again (1.35)

$$\delta\ell_{bh} = \frac{2m_{bh}G}{c^2} \quad \delta\ell_{bh} = \delta\ell', \tag{1.68}$$

which is nothing else but the lower line (1.7). This last result will be concerned later; yet two properties of (1.68) deserve attention.

- On the one hand, (1.65) shows that $\delta\ell_{bh} \equiv \delta\ell'_{min}$ for a given m because, as it is obvious according to (1.52),

$$\delta\ell'_{\min} \equiv \varphi_{\max} \quad \varphi_{\max} = -\frac{c^2}{2}. \tag{1.69}$$

- On the other hand, consider that the ratio $m_{bh}/\delta\ell_{bh}$ reads $m_{Pl}/(\delta\ell_{bh}/\xi)$ once having introduced an arbitrary scale factor ξ such that $m_{bh}/\xi = m_{Pl}$. The corresponding order of magnitude of $\delta\ell_{bh}/\xi$ is then

$$\frac{\delta\ell_{bh}}{\xi} = 2\sqrt{\frac{\hbar c}{G}}\frac{G}{c^2} = 2\sqrt{\frac{\hbar G}{c^3}} = 2\ell_{Pl} = 2\frac{\hbar}{m_{Pl}c}. \tag{1.70}$$

It is however worth noticing that not even (1.65) is merely formal. Consider indeed, as in (1.10),

$$\frac{\partial\varphi}{\partial\ell} = \frac{\partial\varphi}{\partial v}\frac{\partial v}{\partial\ell} \quad \varphi = \frac{v^2}{2} \quad \frac{\partial\varphi}{\partial v} = v \quad v = \frac{\partial\ell}{\partial t}: \tag{1.71}$$

since

$$v = \frac{\partial\ell}{\partial t} = \frac{\dfrac{\partial v}{\partial t}}{\dfrac{\partial v}{\partial\ell}} = \frac{\dfrac{\partial}{\partial t}\dfrac{\partial\varphi}{\partial v}}{\dfrac{\partial v}{\partial\ell}} = \frac{\dfrac{\partial\varphi}{\partial\ell}}{\dfrac{\partial v}{\partial\ell}} = \frac{\partial\varphi}{\partial v} = v,$$

where the intermediate steps must be true because of the consistency of the left and right sides, (1.71) yield the Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial\varphi}{\partial v} = \frac{\partial\varphi}{\partial\ell}. \tag{1.72}$$

This confirms (1.10). The remainder of this section shows the relevant physical worth of analogous dimensional considerations on the diffusion coefficient D. Note the interesting relationship with $\hbar$ given by

$$D = \frac{\hbar}{m} \quad \frac{D}{c} = \frac{\hbar}{mc} = \lambda_C, \tag{1.73}$$

i.e. D is directly related to $\hbar$ and proportional to the reduced Compton length of *m*; also, is significant its link to *a* suggested by the last (1.73), *i.e.*

$$\frac{D}{wavelength} = velocity \quad \nabla \cdot (D\mathbf{p}) = energy. \tag{1.74}$$

The diffusion coefficient has further relevant properties directly related to these elementary definitions; e.g., merging the first (1.73) and (1.74) it follows

$$\frac{\hbar}{m \times wavelength} = velocity \Rightarrow \frac{\hbar}{wavelength} = momentum. \tag{1.75}$$

The De Broglie momentum is found here. Moreover from the last (1.74) it also follows

$$\hbar\nabla \cdot (D\nabla\chi) = energy \quad \nabla \cdot (D\nabla\chi) = \dot{\chi} \quad \dot{\chi} = \frac{energy}{\hbar}, \tag{1.76}$$

being χ any dimensionless scalar field. In practice any field is describable by χ: for example the thermal field characterized by a temperature distribution $T = T(\ell, t)$ and $D = D(\chi)$ requires the dimensionless form $\chi = T/T_0$, with $T_0 = const$. Note that (1.74) and (1.76) imply

$$\nabla \cdot (D\xi_1\hbar\nabla\chi) = energy \Rightarrow \mathbf{p} = \xi_1\hbar\nabla\chi + \xi_0 \quad \xi_0 = constant: \tag{1.77}$$

(1.74) is a differential equation for $\mathbf{p} = \mathbf{p}(x,t)$, the second (1.77) introduces the operator $\hbar\nabla$ that acts on $\mathbf{p}$ and relates it to an appropriate scalar field χ via an arbitrary proportionality factor ξ_1 admissible for sake of generality. Eventually, consider the chance of $\mathbf{p}$ of being real even though χ does not: reasonably then χ has a logarithmic form, *i.e.* $\chi = \log(\psi)$, in order that for example

$$\delta\dot{\chi} = \frac{\delta\dot{\psi}}{\psi} \equiv \frac{\delta(i\dot{\psi})}{i\psi}. \tag{1.78}$$

Thus the chance of real $\mathbf{p}$ for non real ψ allows writing (1.77) as

$$\mathbf{p} = \frac{\hbar}{i\psi}\nabla(i\psi) + \xi_0 \quad \xi_1 = \frac{1}{i\psi} \Rightarrow \mathbf{p}\psi = \frac{\hbar}{i}\nabla(\psi) + \xi_0 \quad \psi = i\chi \quad \xi_0 = const; \tag{1.79}$$

analogous reasoning holds for $\dot{\chi}$ complex function to calculate the energy. Also, a relevant generalization of (1.77) is

$$\nabla \cdot (D\xi_1\hbar\nabla\chi) + energy' = energy'': \tag{1.80}$$

this is not a trivial repetition of (1.77), as $energy'$ has the physical meaning of source term of the diffused term $energy''$. Eventually rewrite (1.80) identically as follows

$$\begin{gathered} \nabla \cdot \left(\frac{D\hbar}{\tau_0\ell_0^3}\nabla\chi\right) + \frac{energy'}{\tau_0\ell_0^3} = \frac{energy''}{\tau_0\ell_0^3} \\ \Rightarrow \nabla \cdot (\mathrm{K}\nabla T) + \frac{source}{\tau_0\ell_0^3} = \frac{energy''}{\tau_0\ell_0^3} \quad \mathrm{K} = \frac{D\hbar}{\tau_0\ell_0^3 T_0}: \end{gathered} \tag{1.81}$$

in this way K is the heat conductivity inferred as a function of D as watt per unit length and temperature, whereas (1.80) turns into the Fourier heat Equation (1.81) simply expressing also the right hand side as a function of T as $\rho C\dot{T}$.

The usefulness of the dimensional analysis is extensible also to the electromagnetic vector frame. Consider for example the possible chances

$$\mathbf{v}\cdot\mathbf{force} \quad \mathbf{v}\times\mathbf{force}, \tag{1.82}$$

both purposely conceived to infer the two corresponding kinds of field physically definable via an appropriate velocity and involving the charge e as well.

- The first (1.82) yields

$$\frac{\mathbf{v}\cdot\mathbf{force}}{ce}\mathbf{u}=\frac{1}{e}\frac{power}{c}\mathbf{u}=\frac{force'}{e}\mathbf{u}=\frac{\mathbf{force}''}{e}=\mathbf{field}$$

$$force'=\frac{power}{c} \quad \mathbf{force}''=force'\mathbf{u},$$

being u an arbitrary dimensionless vector; as the left hand side of (1.82) reads

$$\frac{\mathbf{v}\cdot\mathbf{force}}{ce}\mathbf{u}=ev'\mathbf{u}\frac{\mathbf{v}\cdot\mathbf{force}}{v'ce^2}=\frac{\mathbf{J}}{\sigma} \quad \mathbf{J}=\frac{e\mathbf{v}'}{V}=\rho\mathbf{v}' \quad \sigma=\frac{v'ce^2V}{\mathbf{v}\cdot\mathbf{force}} \quad \rho=\frac{e}{V} \quad \mathbf{v}'=v'\mathbf{u},$$

where V is volume, the result is

$$\frac{\mathbf{J}}{\sigma}=\mathbf{field}=\mathbf{E} \quad e\mathbf{E}=\mathbf{force}. \tag{1.83}$$

The first (1.83) is thus acknowledged as the Ohm law with conductivity σ.

- The second (1.82) yields

$$\frac{\mathbf{v}}{c}\times\mathbf{force}'=e\frac{\mathbf{v}}{c}\times\frac{\mathbf{force}'}{e}=e\frac{\mathbf{v}}{c}\times\mathbf{field}'=\mathbf{force}'';$$

i.e.

$$\mathbf{force}''=\frac{e}{c}\mathbf{v}\times\mathbf{H} \quad \mathbf{field}'=\mathbf{H}.$$

With the given premises the classical Lorentz force is obtained as an identity. Hence write the Lorentz magnetic force $\mathbf{force}'$ as to be summed to the electric field force $\mathbf{F}=e\mathbf{E}$ of (1.83) in order that

$$\mathbf{F}_L=e\mathbf{E}+\frac{e}{c}\mathbf{v}\times\mathbf{H}. \tag{1.84}$$

Once having verified that in fact the dimensional analysis gives the naive (1.1) added value, consider that owing to (1.6) and (1.1)

$$\frac{a}{c^2}=\frac{1}{\ell}=\frac{c}{D}.$$

Then introduce

$$\frac{c}{D}=\frac{v_1}{D_1}+\frac{v_2}{D_2}=\frac{1}{\ell_1}+\frac{1}{\ell_2}=\frac{v_1D_2+D_1v_2}{D_1D_2}=\frac{m_1G+m_2G}{D_1D_2}=\frac{m_1+m_2}{\ell m} \quad v=\frac{mG}{D}, \tag{1.85}$$

which expresses c/D as a sum of two corresponding addends and yields

$$\frac{mG}{D^2}=\frac{m_1+m_2}{\ell m} \quad \frac{(mG)^2}{D^2}=v^2=\frac{(m_1+m_2)G}{\ell} \quad \ell=\frac{(m_1+m_2)G}{v^2}; \tag{1.86}$$

then, owing to (1.73),

$$\frac{c}{D}=\frac{1}{\ell_1}+\frac{1}{\ell_2}=\frac{1}{\ell}\frac{m_1+m_2}{m} \quad F=\frac{\epsilon c}{D}=m\frac{\epsilon c}{\hbar} \tag{1.87}$$

and thus

$$F = \epsilon \frac{c}{D} = \epsilon \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right). \tag{1.88}$$

This classical result is sensible. Dividing both sides by an arbitrary surface A, one finds

$$P = \sigma \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} \right) \quad P = \frac{F}{A} \quad \sigma = \frac{\epsilon}{A}: \tag{1.89}$$

this is the Young-Laplace equation, being σ the surface tension of a liquid phase ascending through a capillary column in opposition to the gravity force under the net pressure P resulting on A. The term in parenthesis expresses thus the curvature of the liquid surface A.

Note that the second (1.86) can be rewritten in general according to (1.8) and (1.65) as follows

$$\frac{(m_1 + m_2) G}{2\ell} = \frac{\overline{m} G}{\ell} = \frac{v^2}{2} = \pm \varphi \quad \overline{m} = \frac{m_1 + m_2}{2} \tag{1.90}$$

and then, owing to (1.62),

$$\pm \varphi = \frac{\overline{m} G}{\ell} \quad \Rightarrow \quad \pm \delta \frac{\varphi}{c^2} = \frac{1}{2} \delta \frac{v^2}{c^2} = \delta \frac{\overline{m} G / c^2}{\ell}; \tag{1.91}$$

also, in general (1.87) reads

$$\frac{1}{\ell_1} + \frac{1}{\ell_2} = \frac{1}{\ell} \frac{2\overline{m}}{m} = \frac{2\varphi}{mG}, \tag{1.92}$$

where the resulting ℓ^{-1} depends upon the ratio $\overline{m}/m$. Note that the definition of φ is sensible, as (1.92) suggests $\varphi/mG \leftrightarrow \ell^{-1}$ in agreement with with (1.65) and yields

$$\delta\varphi = v\delta v = v\delta t \frac{\delta v}{\delta t} = a\delta\ell.$$

It is necessary to verify that both signs (1.90) imply in fact their own physical meaning; $\overline{m} / G / \ell$ is the key term in this respect, where actually $\overline{m}$ is an arbitrary mass, because m_1 and m_2 are arbitrary themselves.

- If in particular $-\varphi$ is the gravitational potential, then taking its minus sign in (1.91) one finds

$$a_\ell = -\frac{\partial \varphi}{\partial \ell} = \frac{\overline{m} G}{\ell^2} = -\frac{\varphi}{\ell} \quad \varphi = -\frac{\overline{m} G}{\ell} \tag{1.93}$$

and then

$$\frac{\delta\varphi}{c^2} = \frac{\overline{m} G}{c^2 \ell^2} \delta\ell = \frac{\overline{m} G}{c^2 \ell^2} (\ell_2 - \ell_1) = \frac{\overline{m} G}{c^3} \left(\frac{\ell_2 c}{\ell^2} - \frac{\ell_1 c}{\ell^2} \right) = \frac{\omega_2 - \omega_1}{\omega}$$
$$\omega_2 = \frac{\ell_2 c}{\ell^2} \quad \omega_1 = \frac{\ell_1 c}{\ell^2} \quad \omega = \frac{c^3}{\overline{m} G}. \tag{1.94}$$

Of course the actual sign of $\delta\omega$ is related to that of $\delta\varphi$ depending on whether one photon, implied by c, moves radially outwards or towards the gravitational source. This is the gravitational red shift of light.

The Equations (1.25), (1.26) and (1.85) have already shown relevant implications of (1.1) on the special relativity; now the gravitational red shift Equation (1.94) concerns the general relativity.

- Instead the positive sign in (1.91) yields

$$\varphi = \frac{1}{2}v_\ell^2 = \frac{mG}{\delta\ell'}; \tag{1.95}$$

one finds thus

$$v_\ell = \sqrt{\frac{2mG}{\delta\ell'}} \tag{1.96}$$

that for $v_\ell \to c$ requires $\delta\ell' \to \delta\ell_{bh}$, which reads instead

$$\delta\ell_{bh} = \frac{2mG}{c^2}, \tag{1.97}$$

in agreement with (1.64) and then with (1.68) as well. Furthermore it is possible to calculate

$$\frac{\delta\varphi}{c^2} = \frac{v\delta v}{c^2} = \frac{v\delta t}{c^2}\frac{\delta v}{\delta t} = \delta\ell\frac{a_\ell}{c^2} = \frac{\delta\ell}{\ell} = \delta\frac{\overline{m}G/c^2}{\ell} \quad \Rightarrow \quad \delta(\log\ell) = \delta\frac{\overline{m}G/c^2}{\ell},$$

which implies

$$\log\ell = \frac{\overline{m}G/c^2}{\ell} + const; \tag{1.98}$$

putting $const = \log\ell_0$, being ℓ_0 a constant length, then

$$\ell_0\frac{\ell}{\ell_0}\log\frac{\ell}{\ell_0} = -\ell_0 S = \frac{\overline{m}G}{c^2} \quad S = -\frac{\ell}{\ell_0}\log\frac{\ell}{\ell_0} = -w\log w \quad w = \frac{\ell}{\ell_0} \tag{1.99}$$

while being also

$$S = -\frac{\overline{m}G}{\ell_0 c^2} = -\frac{\ell_{bh}}{2\ell_0} = -\frac{\epsilon_0}{m_0 c^2} \quad \epsilon_0 = \frac{m_0\overline{m}G}{\ell_0}. \tag{1.100}$$

Of course the dimensionless S of (1.99) refers to the i_{th} quantum state of a given system, to which corresponds specifically ℓ/ℓ_0; so regarding ℓ as ℓ_i related to the i_{th} thermodynamic state w_i of the system, e.g. any bond length or energy of a solid body and everything else related to it, the entropy of the whole system is actually

$$S_{sys} = -\Sigma_i w_i \log w_i. \tag{1.101}$$

Reasonably the statistical form of S consits of a sum of terms, each one of which corresponds to the classical ratio (1.100) of two energies.

Consider now (1.27) and check the physical meaning of t and τ in the particular case where a body of matter is at an arbitrary distance ℓ from a gravitational source and $v = v_{esc}$. Then, owing to (1.4),

$$\tau = t\sqrt{1 - \frac{v_{esc}^2}{c^2}}; \tag{1.102}$$

in this case (1.67) suggests the chance of examining how t deviates from τ because of the presence of a mass m, since the escape velocity of the body allows in principle comparing the the effect of m on t even in the absence of gravity field when the body is infinitely apart from m. Then, owing to (1.96) and (1.90),

$$\tau = t\sqrt{1-\frac{2mG/c^2}{\ell}} = t\sqrt{1-\frac{2\varphi/c^2}{\ell}} \tag{1.103}$$

which also shows that the proper time of the body stops running at the peculiar distance $\ell_{bh} = 2mG/c^2$ from the gravity center of m whatever t far from ℓ_{bh} might be.

Check now the validity of (1.93) considering two possible ways to rewrite without additional hypotheses

$$\frac{\delta\ell}{\ell} = \frac{\delta\varphi}{\varphi}, \tag{1.104}$$

simply exploiting the fact that both $\delta\ell$ and ℓ itself are arbitrary.

Foremost his result is correct, as it implies a relevant consequence simply rewriting identically

$$\frac{\delta\varphi}{c^2} = \frac{\varphi}{c^2\ell}\delta\ell \Rightarrow \frac{\varphi_2-\varphi_1}{c^2} = \frac{\varphi}{c^2\ell}\left(\ell_2-\ell_1\right) \equiv \frac{\ell_0^2\varphi}{c^3\ell}\left(\frac{c\ell_2}{\ell_0^2}-\frac{c\ell_1}{\ell_0^2}\right);$$

next make explicit the meaning of $\delta\varphi$ and $\delta\ell$. The notation at the right hand side does not add anything new but allows introducing the reciprocal times $c\ell_2/\ell_0^2 = 1/t_2$ and $c\ell_1/\ell_0^2 = 1/t_1$ at which one photon moving radially at the constant rate c with respect to a source of gravitational potential φ reaches the distances ℓ_2 and ℓ_1 where the field is described by the values φ_2 and φ_1. Then, in agreement with (1.94),

$$\frac{\varphi_2-\varphi_1}{c^2} = \frac{\omega_2-\omega_1}{\omega} \quad \omega_2 = \frac{c\ell_2}{\ell_0^2} \quad \omega_1 = \frac{c\ell_1}{\ell_0^2} \quad \omega = \frac{c^2}{\varphi}\frac{c\ell}{\ell_0^2}: \tag{1.105}$$

of course the sign of $\omega_2-\omega_1$ depends on whether $\varphi_2 \lesseqgtr \varphi_1$ at $\ell_2 \lesseqgtr \ell_1$.

- On the one hand (1.104) yields also

$$\delta\log\ell = \log\ell_2 - \log\ell_1 = \log\frac{\ell_2}{\ell_1} = \frac{\delta\varphi}{\varphi} \quad \Rightarrow \quad \frac{\ell_2}{\ell_1} = \exp\left(\frac{\delta\varphi}{\varphi}\right);$$

hence defining reasonably the arbitrary length ℓ_2 as $\ell_3 \pm \ell_1$, *i.e.* as the combination of two lengths ℓ_3 and ℓ_1 itself, it follows

$$\frac{\ell_3 \pm \ell_1}{\ell_1} = \exp\left(\frac{\delta\varphi}{\varphi}\right) \quad \frac{\ell_3}{\ell_1} = \exp\left(\frac{\delta\varphi}{\varphi}\right) \mp 1$$

and thus

$$\ell_1 = \frac{\ell_3}{\exp\left(\frac{\delta(m\varphi)}{m\varphi}\right) \mp 1} \quad \frac{\delta(m\varphi)}{m\varphi} = \frac{\epsilon_2-\epsilon_1}{\epsilon}. \tag{1.106}$$

This result reads also

$$N_1 = \frac{N_3}{\exp\left(\frac{\delta\varepsilon}{\varepsilon}\right)\mp 1} \Leftrightarrow \overline{N_j} = \frac{1}{\exp(-b)\exp\left(\frac{\delta\varepsilon_j}{\varepsilon}\right)\mp 1}$$
$$N_1 = \frac{\ell_1}{\ell_0} \quad N_3 = \frac{\ell_3}{\ell_0} = \exp(-b), \tag{1.107}$$

being ℓ_0 an arbitrary length. This serendipitous result and its physical meaning are reasonably recognizable. The notation emphasizes that $\delta\varepsilon$ defining the generic N_1 has nothing special or any particular properties, rather it is actually one among all arbitrary ranges $\delta\varepsilon_j$ definable by $m\delta\varphi_j$ of the j_{th} quantum state; this is because the same holds for (1.106), where ℓ_3 and its corresponding ℓ_1 are lengths in turn relatable for example to bond lengths $\delta\ell_j$ in a molecule or in a body of solid matter. For this reason (1.107) is not based on a specific assumption, rather this way to regard (1.106) gives itself physical meaning to the successive (1.107): N_j are the averages numbers of particles in a given quantum states, whereas clearly $\exp(-b)$ fits, as usual, the condition $\sum_j \overline{N_j} = N$ for the total number N of particles of the physical system.

- On the other hand, it is also possible to write (1.104) as

$$\frac{\delta\ell}{\ell} = \delta\log\varphi = \log\frac{\varphi_2}{\varphi_1} \Rightarrow \frac{\varphi_2}{\varphi_1}\frac{\delta\ell}{\ell} = \frac{\varphi_2}{\varphi_1}\log\frac{\varphi_2}{\varphi_1} \Rightarrow \frac{\varphi_2\delta\ell}{\varphi_1\ell} = -S \quad S = -\frac{\varphi_2}{\varphi_1}\log\frac{\varphi_2}{\varphi_1}. \tag{1.108}$$

Hold again the last considerations. If (1.93) is sensible, then the physical meaning of both results must be recognizable as well. The left hand side of (1.108) is in fact the ratio of two energies $(m_0\varphi_2)/(m_0\varphi_1) = \delta\varepsilon/\varepsilon$, likewise in (1.100); hence nothing hinders to identify $\delta\varepsilon = (\ell_2 - \ell_1)m_0\varphi_2$ and $\varepsilon = m_0\varphi_1\ell$ as proportional to the usual δQ and $k_B T$, so that (1.108) reads more expressively, with the notation previously introduced for the i_{th} allowed state,

$$S_i = -w_i\log w_i \quad w_i = \frac{\varphi_{i2}}{\varphi_{i1}}. \tag{1.109}$$

The crucial fact is that even the simple dimensional analysis here carried out has produced results with an identifiable physical meaning, *i.e.*: the added value due to (1.1) does exist and provides useful information.

To implement further this strategy, consider (1.85) and define now

$$\frac{v_1}{D_1} + \frac{v_2}{D_2} \ge \frac{v_1 + v_2}{D_1 + D_2},$$

where the equality sign holds for $D_1 = D_2$ and $v_1 = v_2$; the inequality is evident noting that the right hand side is $v_1/(D_1 + D_2)$ plus $v_2/(D_1 + D_2)$. Then it is possible to rewrite

$$\frac{v_1}{D_1} + \frac{v_2}{D_2} = \frac{v_1 + v_2}{D_1 + D_2}\frac{c^2}{c^2 - v^2}, \quad v < c$$

where now the equality holds for $v = 0$. Then solving

$$\frac{\zeta + \xi}{\zeta} = \frac{\xi + 1}{\zeta + 1}\beta^{-2} \quad \beta = \sqrt{1 - \frac{v^2}{c^2}} \quad v_1 = \xi v_2 \quad D_1 = \zeta D_2 \tag{1.110}$$

with respect to ξ, one finds

$$\xi = -\frac{(\beta^2\zeta + \beta^2 - 1)\zeta}{\beta^2\zeta + \beta^2 - \zeta}$$

$$\xi = -\zeta\frac{(v/c)^2\zeta + (v/c)^2 - \zeta}{(v/c)^2\zeta + (v/c)^2 - 1} = -\zeta\frac{(v/c)^2(\zeta + 1) - \zeta}{(v/c)^2(\zeta + 1) - 1} = -\zeta\frac{\zeta - (v/c)^2(\zeta + 1)}{1 - (v/c)^2(\zeta + 1)}.$$

As it is admissible that in particular $\xi = 0$, in which case the first (1.110) reads $1 = (\zeta + 1)\beta^2$, one finds

$$\zeta + 1 = \frac{D_2 + D_1}{D_2} = \beta^{-2};$$

in turn, since $D \times time = length^2$ by dimensional definition, this result reads $length_2 = length_{12}\beta$ with $length_{12} > length_2$. So, if $v = const$, then $length_2$ is knowledgeable as the Lorentz contraction of the proper length $length_{12}$ in two different inertial reference systems with reciprocal motion at $v = const$. Also, owing to (1.26), one infers the time dilation via $length_{12} time_{12} = length_2 time_2$ *i.e.*

$$length_2 = length_{12}\beta \quad time_2 = \frac{time_{12}}{\beta}. \tag{1.111}$$

in agreement with (1.53) obtained through a completely different reasoning. Eventually note that the last (1.86) $G(m_1 + m_2)/\ell = v^2$ yields

$$G\frac{m_{12} m'}{\ell} = m'v^2 = 2\mathrm{T} = -\mathrm{U} \quad m_{12} = m_1 + m_2, \tag{1.112}$$

which merges the classical potential-U and kinetic energy T. This result is the classical virial theorem and has an interesting corollary

$$a = \frac{\partial v}{\partial t} = \frac{1}{m}\frac{\partial}{\partial t}\frac{\partial \varepsilon}{\partial v} = \frac{1}{m}\frac{\partial \varepsilon}{\partial \ell}$$

$$\text{because } v = \frac{\partial \varepsilon}{\partial v} \quad v\delta v = \frac{1}{2}\delta v^2 \quad \delta v = \delta\dot{\ell} = \delta\frac{\partial \ell}{\partial t} = \left.\frac{\partial \ell}{\partial t}\right|_{\ell_2, t_2} - \left.\frac{\partial \ell}{\partial t}\right|_{\ell_1, t_1};$$

since $\mathrm{T} = -\mathrm{U}/2$, then $\mathrm{T} - \mathrm{U} = \mathrm{U}/2 = -\mathrm{T}$; *i.e.* $m\varphi = \mathrm{T} - \mathrm{U}$. So, regarding $\partial\varepsilon = \varepsilon_2 - \varepsilon_1 = \mathrm{T} - \mathrm{U}$, one finds

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\ell}} = \frac{\partial \mathcal{L}}{\partial \ell} \tag{1.113}$$

i.e. the Lagrangian already found in (1.72).

1.6. Classical Force Laws and Relativistic Corollaries

Equations (1.88) and (1.61) fulfill the physical dimensions of force, yet they waive the concepts of propagation time and carrier of the interaction. It is easy however to demonstrate how to modify these equations to introduce even in a classical frame these concepts, in fact required once having found the upper limit v_0 of (1.23) admissible for any propagation rate. Consider first (1.87): if in particular $m = m_1 + m_2$, then the equation consists of a sterile relationship between reciprocal lengths only; if instead $m_1 + m_2 \neq m$, then the mass appears explicitly in the equation and determines in general the resulting ℓ^{-1}. This is more interesting, as in effect (1.89) suggests that the mass is somehow linked to the concept of curvature of the capillary liquid surface and to the pressure gradient across the curved surface. To generalize this idea via the gravitational potential $-\varphi$, consider that (1.91) yields

$$\frac{\partial(m'\varphi)}{\partial \ell} = \frac{m'\overline{m}G}{\ell^2} = -F_N; \tag{1.114}$$

even in this result, having the form of the Newton equation in agreement with (1.93), is missing the propagation time of the interaction despite ℓ concerns the distance between m' and $\bar{m}$. This idea suggests the necessity of adding further information to (1.114) by modifying its form.

On the one hand (1.87) can be formally rewritten as a function of a time τ as

$$\begin{gathered}\frac{c^2}{D}=\frac{1}{\ell_1/c}+\frac{1}{\ell_2/c}=\frac{1}{\ell/c}\mathcal{M}\quad \mathcal{M}=\frac{m_1+m_2}{m}\\ \Rightarrow \frac{1}{\ell_1}+\frac{1}{\ell_2}=\frac{1}{\ell}\mathcal{M}\quad \frac{1}{t_1'}+\frac{1}{t_2'}=\frac{1}{\tau}\mathcal{M}\end{gathered} \tag{1.115}$$

i.e.

$$\tau=\tau(\mathcal{M})\quad \ell=\ell(\mathcal{M}). \tag{1.116}$$

On the other hand rewrite (1.87) also introducing the reduced mass of a two body gravitational system

$$\begin{gathered}-\frac{F_N}{G}=\frac{m_{tot}}{\ell_{tot}}\mu\mathcal{R}\quad \mathcal{R}=\frac{1}{\ell_1}+\frac{1}{\ell_2}\equiv\frac{\ell_{tot}}{\ell^2}\quad \ell_{tot}=\ell_1+\ell_2\\ \mu=\frac{m_1m_2}{m_1+m_2}\quad m_{tot}=m_1+m_2\end{gathered} \tag{1.117}$$

and next rewrite identically this result via (1.88) and (1.89) putting

$$\begin{gathered}\ell_1+\ell_2=v(t_2'+t_1')\\ \Rightarrow -\frac{F_N}{G}=\frac{m_{tot}}{v(t_1'+t_2')}\mu\mathcal{R}=\frac{m_{tot}}{v_{gr}(t_2'-t_1')}\mu\mathcal{R}\quad v(t_2'+t_1')=v_{gr}(t_2'-t_1').\end{gathered} \tag{1.118}$$

Equation (1.118) is more interesting than (1.114). The mere fact of having expressed the Newton law via $\mathcal{R}$ implies having introduced two lengths $\ell_1^{-1}+\ell_2^{-1}$ whence, without resorting to additional assumptions, follows in principle the idea of time lapse corresponding to the space range size $\ell_2-\ell_1$: the velocity *v*, introduced in (1.118) without specific reference to a previous definition, is purposely related to the actual propagation rate v_{gr} of gravitons, the hypothetical carriers that mediate the gravity interaction. Whatever the physical properties of these vector bosons might be, e.g. their spin, if $v_{gr}\equiv c$ then the proposed definition of *v* via v_{gr} is sensible: if the graviton actually existed, then *v* fulfilling the condition (1.118) links the time coordintes t_1' and t_2' to the space distance ℓ_{tot} between m_1 and m_2. However $\mathcal{R}$ regarded also here as a curvature likewise in (1.89) rises the problem of clarifying what is actually curved; reasonably $\mathcal{R}$ should concern the space gap $\delta\ell_{tot}$ or the time lapse between t_1' and t_2' itself, simply because these are the only parameters available and suitable to define it in (1.118). A possible implication of (1.115) is that the presence of $\mathcal{M}\neq 0$ curves the space range defined by ℓ_1 and ℓ_2 and modifies the time lapse between t_1' at ℓ_1 and t_2' at ℓ_2 as well, with respect to an empty universe; in this respect (1.116) remark that if it is true, then $\mathcal{M}$ modifies both space and time ranges.

This is the motivation of this reasoning: to show that the necessity of a finite propagation rate implied by (1.23) requires a curved space time even in a mere Newtonian classical frame.

Nevertheless if the graviton actually exists and propagates at rate $v_{gr} \equiv c$, (1.116) must be regarded as the ratio of two energies

$$\mathcal{M} \Rightarrow \mathcal{M}_{gr} = \frac{(m_1 + m_2)c^2}{mc^2}$$

rather than a massive carrier itself; this fulfills the aforesaid corollary of (1.48). Also, if really it consists of two massless vector bosons defining $\mathcal{M}$, its spin should be at least 2. Eventually, if $\boldsymbol{F}$ propagates along a curved path from m_1 to m_2, then the Euclidean $\boldsymbol{F}$ and $\boldsymbol{a}$ are no longer self-aligned, which explains why the naive relationship of $\mathbf{F}$ and $m\mathbf{a}$ is inadequate in this modified conceptual frame where

$$-\frac{F_N}{G} = \frac{m_{tot}\mu}{\delta\ell'}\mathcal{MR} \quad \delta\ell' = c(t_2' - t_1'): \tag{1.119}$$

the Newtonian masses m_1 and m_2 are replaced by $m_{tot}\mu$, with $\mathcal{M}$ playing the role of massless carrier. Eventually (1.62) and (1.2) yield

$$-\frac{F_N}{G} = \frac{\mu c^2 \mathcal{R}}{G} \quad \frac{m_{tot}}{\delta\ell'} = \frac{\overline{v}^2}{G} \equiv \frac{c^2}{G}. \tag{1.120}$$

It is significant that F_N is proportional to a curvature $\mathcal{R}$ and that the propagation time lapse and the interaction carriers, possibly hidden in $\mathcal{M}$, are in fact inherent the naive Newton law simply given by $\mu c^2 \mathcal{R}$ even skipping G, whereas the mass μ only defines F_N in the field of which all masses behave comprehensively in the same way.

At least in principle, this reasoning should hold even for the Coulomb law between static charges e_1 and e_2: indeed $F_C = (e_1/\ell)(e_2/\ell)$ yields now

$$\frac{F_C \delta\ell_{tot}}{e^2} = \pm\mathcal{R}' \Leftrightarrow \frac{F_C \delta\ell_{tot}}{\hbar c} = \pm\alpha\mathcal{R}'. \tag{1.121}$$

Equations (6JK) and (1.121) compare directly the relative strengths $F_N/F_C = m_{tot}^2 G/e^2$ of the respective forces with the same $\delta\ell_{tot}$.

Before going on to emphasize the consequences of these considerations it is necessary to examine the quantum corollaries of the Section 2.

2. Physical Implications

This section infers relevant corollaries of the previous considerations. It consists of four subsections purposely aimed to exemplify the cases of classical and quantum physics, special and general relativity. It is necessary to specify, likewise as in (1.7), whether $v = c$ or $v \neq c$.

2.1. The Quantum Corollary: The Quantization

In addition to (1.75) and (1.79), consider again (1.1). Multiply side by side $p = a\hbar/c^2$ and $\ell = c^2/a$, which yields $p\ell = \hbar$; also, multiply $\varepsilon = a\hbar/c$ by

$t = c/a$, which yields $\varepsilon t = \hbar$. Hence $p\ell = \hbar = \epsilon t$: rewriting n times all possible $p_n \ell_n$ and summing, this result yields $np\ell = n\hbar = n\epsilon t$. It is crucial the fact that the integer n multiplies $\hbar$ and both products $p\ell$ and εt; however $n\hbar$ is uniquely defined with integer *n*, whereas $np\ell = (n'p)(n''\ell)$ requires $n'n'' = n$ with arbitrary n' and n'' not necessarily integers. So it is possible to define $p = p_\ell$ and ℓ as conjugate variables and write

$$\begin{aligned} &\delta p_\ell \delta \ell = n\hbar = \delta\varepsilon\delta t \quad \delta p_\ell = n_p^* p_\ell \quad \delta\ell = n_\ell^* \ell \\ &\delta\varepsilon = n_\varepsilon^* \varepsilon \quad \delta t = n_t^* t \quad n_p^* n_\ell^* = n = n_\varepsilon^* n_t^* : \end{aligned} \tag{2.1}$$

i.e. n_p^* multiplying p_ℓ defines a range of values $p_1 \le p_\ell \le p_2$ since $n_{p1} \le n_p^* \le n_{p2}$, and so on. Thus, by definition, *n* is integer, n_i^* are real numbers. Owing to the way of defining the ranges of dynamical variables, nothing is known about their sizes and position in a given reference system *R*: for example the lower boundary coordinate of a given uncertainty range $\delta x = x_2 - x_1$ could determine the position of the range with respect to the origin of a corresponding *R*, whereas the upper boundary coordinate could be related to its size; however, neither of them is known and conceptually knowable by corollary of (2.1). This corollary is the fundamental assumption of the present physical model. Moreover, without knowing at least x_1 it is impossible to establish whether δx includes the origin of any assumed *R*, or it lies on the negative side of this *R* or on its positive side; *i.e.* in lack of any information about both range boundaries there is no way to identify how δx is related to the origin of this *R* rather than to that of any other *R'*. In this sense any consideration based on uncertainty ranges only is not related in particular to some specific *R*. Everything that can be deduced with this information (2.1) is valid in any coordinate system, because the form of this principle remains unchanged.

Actually, no *R* is definable in such agnostic model based on (2.1), as it is evident observing that actually the former formulation of uncertainty is identical to the last one. In other words, considering $\delta p_\ell \delta\ell$ in a given *R* and $\delta p'_\ell \delta\ell'$ in another R' one would find

$$\delta p_\ell \delta\ell = \hbar, 2\hbar, \cdots \text{ in } R \Leftrightarrow \delta p'_\ell \delta\ell' = \hbar, 2\hbar, \cdots \text{ in } R'; \tag{2.2}$$

but, in fact, the right hand sides in the respective *R* and R' are clearly indistinguishable because *n* of (2.1) is arbitrary with the *n* and n' are arbitrary themselves, unknown and unknowable. Then *R* and R' are indistinguishable, and thus even interchangeable, because neither *n* nor n' take specific values but rather they symbolize any integer number; a possible way to obtain the covariancy is to formulate the physical laws implementing uncertainty ranges only. This explains why it is not necessary to specify in which coordinate system are defined the uncertainty ranges, primed and unprimed ranges become mere notations. Of course it is possible to acknowledge if δx is defined in any inertial or non-inertial *R*: as all ranges are inherently time dependent, $\delta\dot{x}$ implies non-inertial *R*. Indeed, since $\delta x = \delta x(t)$, it is possible in principle that $\delta\dot{x} = -(n\hbar)\delta p_x^{-2}\delta\dot{p}_x$, any particle possibly delocalized in δx is subjected to a force field $\delta\dot{p}_x$.

However this chance is identically possible for R' too, so that there is now way to distinguish whether the non-inertial character concerns R or R'. Below it is shown that this corollary of (2.1) is nothing else but the equivalence principle concerning a non-inertial reference system.

Eventually deserves attention the chance of rewriting (2.1) also as

$$\delta\mathbf{p}\cdot\delta\mathbf{x}=n\hbar=\delta\varepsilon\delta t, \tag{2.3}$$

where the scalar at the left hand side generalizes $\delta p_\ell \delta\ell$ of (2.1) to an arbitrary number of space dimensions; in other words, any model based on (2.1) admits extra space dimensions besides to the time coordinate hidden inside δt. It is easy to show that (2.3) is a corollary of the operative definition of space time [3]

$$\frac{\hbar G}{c^2}\quad\frac{length^3}{time}. \tag{2.4}$$

Equations (2.1) suggest the necessity of replacing systematically the deterministic dynamical variables with the respective uncertainty ranges. In fact (2.1) exclude leaves out the physical meaning of local dynamical variables, replaced instead by the respective uncertainty ranges: the reason is similar to that suggests trusting on the significance of the whole error bar, *i.e.* δx, rather than to the random measurement values, *i.e.* x, falling in the bar. Thus once accepting (2.1), deserve consideration ratios like $\delta x/\delta t$ rather than $\partial x/\partial t$, which are by definition local values in infinitesimal ranges.

- On the one hand it prevents the idea of instantaneous effects. Writing $a=\partial v/\partial t$ means that v changes by dv in an infinitesimal time range dt; *i.e.* it implies the instantaneous effect of a force on the motion of a body. Despite both changes are infinitesimal it seems more appropriate, and physically more realistic, to think that a finite change of velocity δv, in principle arbitrary and thus even arbitrarily small, needs an appropriate time range δt arbitrary as well to occur.
- On the other hand the classical derivative also conflicts with the Heisenberg principle in the case where are concerned conjugate variables, e.g. $\partial p_x/\partial x$; instead is admissible $\delta p_x=-\left(\hbar/\delta x^2\right)\delta x$ with both ranges in principle finite. So, being the range sizes arbitrary, the numerical worth of this last position is still compliant with the concept of derivative without however requiring in principle the local determinism of both dynamical variables.

To exemplify how (2.1) help formulating advantageously the physical problems, define the energy $\varepsilon=\ell F$; the actual physical meaning of this dimensional way to introduce a deterministic value of ε via local definitions of $|F|$ and $|\ell|$ appears calculating $\delta\varepsilon$ in agreement with (2.1), thus in agreement with the idea that the uncertainty ranges and not the deterministic local values of dynamical variables have actual physical meaning. In fact $\delta\varepsilon$ consists of two addends $F\delta\ell+\ell\delta F$ that are in turn uncertainty ranges themselves, yet with different physical meaning. Consider the force $\mathbf{F}$ acting against a surface A, and let $\boldsymbol{\ell}$ be a vector normal to the surface; then $\delta\varepsilon$ reads

$$\delta\varepsilon = \mathbf{F}\cdot\delta\boldsymbol{\ell} + \frac{\boldsymbol{\ell}}{\ell_0}\cdot\delta\mathbf{F}' \quad \mathbf{F}' = \mathbf{F}\ell_0,$$

being $\ell_0 = |\boldsymbol{\ell}|$ a constant length. The first addend concerns the work $\delta\varepsilon'$ of the component $\mathbf{F}\cdot\delta\boldsymbol{\ell}$ to displace A by $\delta\ell$. The second addend indicates a range of forces, having for example equal strength of modulus $|\mathbf{F}|\ell_0$ but different orientations with respect to the surface unit vector ℓ/ℓ_0; the surface appears at rest. For example A could be the inner surface of a rigid container on which impact gas molecules with random orientations described by all possible $\mathbf{F}'_j$ included in $\delta\mathbf{F}'$. In fact therefore $\mathbf{F}$ and $\mathbf{F}'_j$ have different physical meaning, because the latter have been introduced independently of the former as random values within an arbitrary range of forces. It is clear eventually that $\delta\boldsymbol{\ell}\cdot\delta\mathbf{F} = \delta\varepsilon$ is the most general situation where both contributions are allowed to occur. These three chances are contextually possible with but hidden in the mere deterministic meaning of $F\ell$.

Are now sketched five examples to show the consistency of the present model with: 1) classical physics, 2) relativity, 3) quantum physics.

1) Rewrite (2.1) as

$$v_x\delta p_x = \delta\varepsilon \quad v_x = \frac{\delta x}{\delta t}, \tag{2.5}$$

where $\hbar$ does no longer appear explicitly. Multiply both sides by an arbitrary mass *m*; regarding classically the mass as a constant, one finds
$m\delta\varepsilon = \delta(m\varepsilon) = mv_x\delta p_x$ and thus

$$\delta(m\varepsilon) = p_x\delta p_x = \delta\left(p_x^2/2\right) \quad p_x = mv_x \quad m = const.$$

Moreover $\delta\left(m\varepsilon - p_x^2/2\right) = \delta\left(m\varepsilon - p_x^2/2 + const\right) = 0$, because of course $\delta(const) = 0$, yields

$$\varepsilon = p_x^2/2m + const' \quad const = \frac{const'}{m} \tag{2.6}$$

i.e. the classical kinetic energy of a free particle an arbitrary constant apart, determinable through the initial boundary condition of the specific problem. Also, since according to (1.90)

$$-m\varphi = G\frac{m\overline{m}}{\ell} \;\Rightarrow\; -m\varphi = G\frac{m\overline{m}}{\delta\ell},$$

(2.1) yields

$$-m\varphi = G\frac{m\overline{m}}{\delta\ell} = G\frac{m\overline{m}}{\delta t\delta\varepsilon/\delta p_\ell} = G\frac{m\overline{m}}{v_\ell\delta t} \quad v_\ell = \frac{\delta\varepsilon}{\delta p_\ell};$$

here appears explicitly the finite propagation rate of any interaction, because actually all ranges of (2.1) are inherently related to δt.

2) The right and left hand sides of the first (2.1) yield respectively

$$\delta\varepsilon = n\hbar\omega \quad \omega = \frac{1}{\delta t} \quad \delta p_\ell = \frac{nh}{2\pi\delta\ell} \quad \text{i.e.} \quad \delta p_\ell = \frac{h}{\lambda} \quad 2\pi\delta\ell = n\lambda. \tag{2.7}$$

The concepts of quantization and uncertainty ranges are interconnected. Note indeed the different positions

$$2\pi\delta\ell = n\lambda \quad 2\pi\ell = \lambda:$$

in the former case the range $\delta\ell$ allows a range $\delta\lambda$ of steady circular wavelengths, *i.e.* several quantized $n_i\lambda_i$ such that $n_1 \le n_i \le n_2$ and corresponding $\lambda_1 \le \lambda_i \le \lambda_2$ require $\ell_1 \le \ell_i \le \ell_2$. The second position admits of course one steady λ only for a unique ℓ; the chance of various $\lambda_i = n_i\lambda$ requires formally differentiating both sides, which in turn means replacing both local values of ℓ and λ with the respective uncertainty ranges linked by *n*. Analogous reasoning holds of course for $\delta\varepsilon$ of the first (2.7). In fact the second (2.8) was already found in (1.21), as one equality of the chain reads $p = \hbar/\ell$; therefore $p = h/2\pi\ell$ could be guessed as $p = h\lambda$ along with $2\pi\ell = \lambda$. However the quantization should be purposely hypothesized, instead of being evident consequence of (2.1). It is interesting to note that the quantum number *n* of wave mechanics is actually the number of allowed states.

3) Consider that (2.1) yields

$$\frac{\delta\varepsilon}{\delta p_x} = \frac{\delta x}{\delta t} = v_x; \tag{2.8}$$

multiplying both sides by the ratio ε/p_x, one finds

$$\frac{\varepsilon}{p_x}\frac{\delta\varepsilon}{\delta p_x} = \frac{\delta(\varepsilon)^2}{\delta(p_x)^2} = \frac{\varepsilon}{p_x}v_x. \tag{2.9}$$

Put now in general by dimensional reasons the right hand side as $c^2 - v_x'^2$, being v_x' a velocity to be determined. Then (2.9) yields two equations

$$\frac{\varepsilon}{p_x}v_x = c^2 - v_x'^2 \quad \text{and} \quad \delta\left(\varepsilon^2\right) = \delta\left(p_x\right)^2\left(c^2 - v_x'^2\right)$$
$$\Rightarrow \delta\left(\varepsilon^2 - \left(p_x c\right)^2\right) = -\frac{v_x'^2}{c^2}\delta\left(p_x c\right)^2. \tag{2.10}$$

It is in principle possible that the right hand side of (2.9) is a constant; putting thus $v_x' = 0$, this particular case yields

$$\frac{\varepsilon}{p_x}v = c^2 \quad \delta\left(\varepsilon^2 - \left(p_x c\right)^2\right) = 0 \Rightarrow p_x = \frac{\varepsilon v_x}{c^2} \quad \varepsilon^2 - \left(p_x c\right)^2 = const. \tag{2.11}$$

The comparison with (1.29) shows the physical meaning of this well known result; the notation at the right hand regards a particle moving along the *x*-axis so that $p_x = p$ and $v_x = v$. Merging both (2.11) one finds of course with simplified notation for a massive particle

$$p = \frac{mv}{\beta} \quad \varepsilon = \frac{mc^2}{\beta} \quad \beta = \sqrt{1 - \frac{v^2}{c^2}}, \tag{2.12}$$

which has an interesting corollary: $v \to c$ implies diverging energy and momentum, whereas both uncertainty range sizes δp and $\delta\varepsilon$ enclosing such divergent values must tend themselves to ∞; but then (2.1) requires $\delta x \to 0$ and $\delta t \to 0$ for the particle to have a finite number of quantum states. However one particle with an infinite number of allowed states cannot have definable physical properties; so, as time *t* flows in its allowed range δt just as *x* varies in its space range δx, then $\delta t \to 0$ implies asymptotic vanishing of time flow for matter moving at $v \to c$.

Consider now also the more general case where the right hand side of (2.9) is not constant, in which case (2.11) needs to be corrected via $v'_x \neq 0$ according to (2.10). A relevant example of why this correction should be significant is at the Planck scale and when $v \approx c$ *i.e.* $\beta \to 0$. Rewrite then (2.9) in order to find again (2.11) as a particular case. Multiply both sides of the first (2.10) by the momentum p'^2; with the last notation one finds

$$(p'v')^2 = (p'c)^2 - \varepsilon p'^2 \frac{v}{p}. \tag{2.13}$$

Put then

$$(p'v')^2 = \varepsilon^2 - (m'_0 c^2)^2, \tag{2.14}$$

so that

$$\varepsilon^2 - (m'_0 c^2)^2 = (p'c)^2 - \varepsilon p'^2 \frac{v}{p}. \tag{2.15}$$

By definition $v' \leq c$: thus (2.14) agrees with $(p'v')^2 \leq \varepsilon^2$, in fact it is the standard (2.11). Also, (2.9) fulfills $\varepsilon^2 \to (m'_0 c^2)^2$ for $v' \to 0$ and $(p'c)^2 = \varepsilon^2 - (m'_0 c^2)^2$ for $v' \to c$; moreover in this limit (1.10) requires $v \to 0$, in which case $\varepsilon p'^2 / p \to 0$ as well while (2.15) and (2.13) coincide. Therefore (2.14) is reasonable. Eventually p' is defined by m'_0, so that (2.15) has still the standard form of the special relativity for $v \to 0$. The result

$$\varepsilon^2 = (p'c)^2 + (m'_0 c^2)^2 - \zeta \varepsilon p'^2 \qquad \zeta = \frac{v}{p} = \frac{\beta}{m'_0} \ll 1. \tag{2.16}$$

is the known equation of quantum gravity that solves cosmological paradoxes [4]. In this respect remind now (1.28), according which

$$\epsilon' = \epsilon + \epsilon_0 \qquad \epsilon' = \epsilon - \epsilon_0 \qquad \epsilon^2 = \epsilon' \epsilon'' + \epsilon_0^2$$

yields

$$\epsilon_0^2 = \epsilon_0'^2 - \epsilon' \epsilon'' \frac{\epsilon}{\epsilon_0'''} \Rightarrow \epsilon^2 = \epsilon' \epsilon'' + \epsilon_0'^2 - \epsilon' \epsilon'' \frac{\epsilon}{\epsilon_0'''}.$$

indeed there is no reason that ϵ_0 be actually a constant; this position has been introduced in (1.29) to introduce the usual energy equation of the standard special relativity. Yet, at least in principle, the last equation holds even though

$$\epsilon_0^2 = \epsilon_0'^2 + \epsilon' \epsilon'' - \epsilon' \epsilon'' \frac{\epsilon}{\epsilon_0'''} \Rightarrow \epsilon_0^2 - \epsilon_0'^2 = \epsilon' \epsilon'' \left(1 - \frac{\epsilon}{\epsilon_0'''}\right):$$

as a matter of fact the last implication has been quoted here because it fits (2.15). The physical meaning of (2.15) has been explained; however now further considerations are stimulated by this new way to find the same result. Write indeed

$$\delta(\epsilon_0^2) = \epsilon_0^2 - \epsilon_0'^2 = \left(\delta(mc^2)\right)^2 \qquad \frac{\epsilon}{\epsilon_0'''} = \frac{v^2}{c^2} \Rightarrow \delta(\epsilon_0^2) = \epsilon' \epsilon'' \left(1 - \frac{v^2}{c^2}\right)$$

whence

$$\epsilon'\epsilon'' = \frac{\delta\left(\epsilon_0^2\right)^2}{1-\frac{v^2}{c^2}} \equiv \epsilon_0^2\left(\frac{\delta\epsilon_0}{\beta}\right)^2:$$

this way of defining $\epsilon'\epsilon''$ is analogous to that of (2.12), despite appears here $\delta\varepsilon_0$ instead of ε_0. This is the reason why this quantum correction to the classical energy equation of standard special relativity is small, being due to a small δm, likewise as $\epsilon'\epsilon''/\epsilon_0^2$ corresponding to $\delta\epsilon_0/\beta$. Eventually note the chance of writing

$$\epsilon^2 = \left(p'''c\right)^2 + \epsilon_0'^2 - \epsilon'\epsilon''\frac{\epsilon}{\epsilon_0'''} \quad \left(p'''c\right)^2 = \epsilon'\epsilon''\left(1-\frac{\epsilon}{\epsilon_0'''}\right).$$

i.e. once more the standard form of the special relativity. This is a well well known statement, according which the general relativity admits a particular reference frame where holds *locally* the special relativity. Anyway, it is worth emphasizing once more how the dimensional approach (1.28) is extensible very simply and shortly to include even the quantum gravity.

4) Consider (1.35) that reads owing to (2.7)

$$2\pi\ell_{bh} = 4\pi\frac{m_{bh}G}{c^2} = n\lambda_{bh};$$

next, multiplying this result by ℓ_{bh} one finds

$$2\pi\ell_{bh}^2 = 8\pi\left(\frac{m_{bh}G}{c^2}\right)^2 = n\lambda_{bh}\ell_{bh}.$$

Eventually, since

$$\frac{2\pi\ell_{bh}^2}{2\ell_{Pl}^2} = n\lambda_{bh}\frac{2m_{bh}G}{c^2}\frac{c^3}{\hbar G} = \frac{n\lambda_{bh}2m_{bh}c}{\hbar} = n\frac{2\lambda_{bh}}{\lambda\!\!\!^{-}_{Cbh}} \quad \lambda\!\!\!^{-}_{Cbh} = \frac{\hbar}{m_{bh}c},$$

the result is

$$S_{BH} = \frac{4\pi\ell_{bh}^2}{4\ell_{Pl}^2} = \frac{A_{bh}}{4\ell_{Pl}^2} = n\frac{2\lambda_{bh}}{\lambda\!\!\!^{-}_{Cbh}}. \tag{2.17}$$

The physical meaning of S_{BH}, which results quantized when expressed via $\lambda_{bh}/\lambda\!\!\!^{-}_{Cbh}$, will be concerned later.

5) Consider the angular momentum $\mathbf{M} = \delta\mathbf{r}\times\delta\mathbf{p}$ and its component $\mathrm{M}_u = \delta\mathbf{r}\times\delta\mathbf{p}\cdot\mathbf{u}$ along an arbitrary direction defined by the unit vector $\mathbf{u}$. Implement $(\mathbf{u}\times\delta\mathbf{r})\cdot\delta\mathbf{p} = \delta\mathbf{w}\cdot\delta\mathbf{p} = \pm\delta w\delta p_w$ having put $\delta\mathbf{w} = \mathbf{u}\times\delta\mathbf{r}$ and $\delta w = |\delta\mathbf{w}|$. Thus $l = 0$ if $\delta\mathbf{w}\perp\delta\mathbf{p}$, otherwise $\delta w\delta p_w = n\hbar$ so that $l = 0, \pm integer$ owing to (2.1) *i.e.*

$$\mathrm{M}_u = 0, \pm l\hbar. \tag{2.18}$$

This is the only possible result: changing $\mathbf{u}$ to find a further component $M_{u'}$ would trivially mean repeating the same, unique information. Although l agrees with the quantum numbers of standard wave mechanics, it takes here the physical meaning of numbers of allowed quantum states of M_u, if any. Next,

M_u is enough to infer M^2. If it is true that $\langle \mathrm{M}_x^2 \rangle = \langle \mathrm{M}_y^2 \rangle = \langle \mathrm{M}_z^2 \rangle$ in a range of arbitrary values of $-L \le l \le L$, then

$$\langle \mathrm{M}_i^2 \rangle = \hbar^2 \sum_i \frac{l_i^2}{2L+1} \Rightarrow \langle \mathrm{M}^2 \rangle = \langle \mathrm{M}_x^2 \rangle + \langle \mathrm{M}_y^2 \rangle + \langle \mathrm{M}_z^2 \rangle = L(L+1)\hbar^2; \quad (2.19)$$

i.e. M^2 is inferred from M_u only. This procedure shows explicitly the statistical meaning of M^2.

Note that the same result is obtained summing all *l* states of angular momentum component in a given interval of allowed states from -*L* to *L*. One finds

$$\sum_{l=-L}^{l=L} l = 0 \quad 2\sum_{l=0}^{l=L} l = L(L+1), \quad (2.20)$$

which shows that the quantum numbers are numbers of allowed states in the conceptual frame of (2.1).

To highlight the significance of the second (2.20) and introduce the relativistic concept of spin in this conceptual frame, write the identities

$$(L+1)L = \mathrm{M}^2 \quad \Rightarrow \quad \mathrm{M}^2 + s^2 = (L+s)^2 \quad s = \frac{1}{2}, \quad (2.21)$$

whereas the same reasoning for $L+1$ yields

$$(L+1)^2 - (L+1) + s^2 = (L+1-s)^2.$$

However, since L symbolizes arbitrary integers, what holds for L must also hold for $L+1$. So it must be true that

$$L = 0, 1, 2, \cdots \quad \Rightarrow \quad \mathrm{M}^2 + s^2 = J^2 \quad J^2 = (L \pm s)^2 : \quad (2.22)$$

in other words $\mathrm{M}^2 + s^2 = (L \pm s)^2$ means $\mathrm{M}^2 = L(L \pm 1)$ with the right hand side calculable once with L or admissibly also with $L-1$. As anyway J^2 is an allowed square angular momentum in $\hbar^2$ units, because (2.21) is an identity, then J^2 has actual physical meaning. An analogous reasoning holds for $s = n_s/2$ with n_s arbitrary integer.

Since both L and $L+1$ represent allowed states of angular momentum, (2.22) is an admissible generalization of (2.20); accordingly a further indicator of the quantum states accessible to the system is *s*, which of course represents quantum angular states of half integer values $s\hbar/2$ of the spin component to be added or subtracted to $\pm l\hbar$. Thus (2.22) implies, likewise the steps (2.18) to (2.20),

$$(L \pm s)^2 \quad \Rightarrow \quad J^2 = \mathrm{M}^2 + 2\mathbf{L} \cdot \mathbf{S} + S(S+1) \quad s^2 = S(S+1). \quad (2.23)$$

Equation (2.23) keeps its own validity even though $L = 0$. To infer the spin of particles, are enough a few elementary considerations on the standard orbital angular momentum.

2.2. The Dynamical Mass

According to (2.12) it is possible to write

$$m_{dyn} = \frac{m_0}{\beta} \quad m_{dyn}c^2 = \epsilon_{kin} \quad m_{dyn}v = p_{kin} \quad \beta = \sqrt{1 - \frac{v^2}{c^2}}. \quad (2.24)$$

The question rises then: do these ways to define ε_{kin} and p_{kin} have mere formal meaning or have themselves physical significance and implications? This question has general character, yet is particularly interesting in the present model where not only dynamical variables but also their combinations are proven useful to provide physical information: e.g. (1.2) shows that ℓ is related to m via the constant factor G/c^2; this holds similarly for t, while m is also linked to t via the constant c^3/G. In principle there is no reason to exclude that the dependence of t and ℓ state of motion of a body of matter can be extended also to the concept of dynamical mass. It is necessary thus to verify the physical meaning and the possible implications of all (2.24), to demonstrate that these definitions are not merely formal. Start from the differentials of the first (2.24); owing to (1.65) regarded with both signs for generality, and thus not with the meaning of gravitational potential only as assumed in the steps (1.93) to (1.100), the classical differentials yield

$$\delta m_{dyn} = m_0 \frac{v/c^2}{1-v^2/c^2}\delta v = \frac{m_0 \delta\varphi}{\beta^2 c^2} \Rightarrow \delta m_{dyn} c^2 = \frac{m_0 \delta\varphi}{\beta^2} = \delta\epsilon_{kin}. \quad (2.25)$$

Follow now some significant implications.

1) Owing to (1.113), (2.25) yield

$$\delta\varepsilon_{kin} - \frac{v^2}{c^2}\delta\epsilon_{kin} = m_0\delta\varphi \Rightarrow \delta\left(\epsilon_{kin} - m_0\varphi\right) = \delta\left(T-U\right) = \delta\mathcal{L}$$

whence

$$\varphi = \frac{1}{2}\frac{\delta\mathcal{L}}{\delta m_{dyn}}. \quad (2.26)$$

2) Is also interesting the corollary of (1.65)

$$\frac{\delta\mathcal{L}}{\delta\varepsilon_{kin}} = \frac{2\varphi}{c^2} = \frac{2mG}{\delta\ell'' c^2} \Rightarrow \delta\ell''' = \frac{2mG}{c^2} \qquad \delta\ell''' = \delta\ell'' \frac{\delta\mathcal{L}}{\delta\epsilon_{kin}}: \quad (2.27)$$

here $\delta\ell'''$ yields once more (1.35).

3) Consider now the reciprocal of m_{dyn} and calculate

$$\delta\left(\frac{1}{m_{dyn}}\right) = \frac{\delta\beta}{m_0} = -\frac{1}{m_0}\beta\frac{\delta\beta}{\beta} = -\frac{1}{m_0}\beta\delta\log\beta = -\frac{\beta_0}{m_0}\frac{\beta}{\beta_0}\log\frac{\beta}{\beta_0} \to \frac{\beta_0}{m_0}S$$
$$S = -\frac{\beta}{\beta_0}\log\frac{\beta}{\beta_0}. \quad (2.28)$$

Let v_j be the velocities defining the various β_j of the respective momenta and energies (2.24); then (2.28) turns out to be the j-th allowed state S_j of a given physical system represented by the respective β_j, *i.e.*

$$\frac{\delta\beta}{m_0} = \frac{\beta_0}{m_0}S \Rightarrow \frac{\delta\beta_j}{m_0} = \frac{\beta_0}{m_0}S_j \quad (2.29)$$

Then, summing over all states of the system, all S_j just found define the dimensionless entropy S_{sys} of the system, *i.e.*

$$\sum_j \delta m_{dyn_j}^{-1} = S_{sys}. \quad (2.30)$$

4) The first (2.25) reads

$$\frac{\delta m_{dyn}}{m_{dyn}}=\delta\log m_{dyn}=\log\frac{m_{dyn}}{m_0'}=\frac{\delta\varphi}{\beta c^2}\ \Rightarrow\ \frac{m_{dyn}}{m_0'}=\exp\left(\frac{\delta\varphi}{\beta c^2}\right)$$

and yields

$$\frac{m_{dyn}}{m_0'}=\frac{m_{dyn}\pm m_0''}{m_0''}=\frac{m_{dyn}}{m_0''}\pm 1\ \Rightarrow\ \frac{m_{dyn}}{m_0''}=\exp\left(\frac{m_0\delta\varphi}{\beta m_0 c^2}\right)\mp 1\quad \frac{1}{m_0'}-\frac{1}{m_0''}=\pm\frac{1}{m_{dyn}} \tag{2.31}$$

and thus

$$\frac{m_0''}{m_{dyn}}=\frac{1}{\exp\left(\frac{m_0\delta\varphi}{\beta m_0 c^2}\right)\mp 1}.$$

Hence, regarding this equation as in the steps (2.28) to (2.29), the result is

$$\frac{m_0''\delta\varphi_j}{m_{dyn}}=\frac{\varepsilon_j}{\exp\left(\frac{m_0\delta\varphi_j}{\beta m_0 c^2}\right)\mp 1}=\frac{\varepsilon_j}{\exp\left(\frac{m_0\delta\varphi_j}{\beta m_0 c^2}\right)\mp 1}=\frac{\varepsilon_j}{\exp\left(\frac{\delta\varepsilon_j}{E}\right)\mp 1}\quad \frac{\delta\varepsilon_j}{E}=\frac{m_0\delta\varphi_j}{\beta m_0 c^2}, \tag{2.32}$$

$$\mathcal{E}=\sum_j\frac{\varepsilon_j}{\exp\left(\frac{\delta\varepsilon_j}{E}\right)\mp 1}\qquad \delta\varepsilon_j=\varepsilon_j-\varepsilon_0=m_0\varphi_j-m_0\varphi_0. \tag{2.33}$$

Also this result is easily recognizable in agreement with (1.107). The link of (2.32) to (1.107) is straightforward; the fact that these results have been inferred with different reasoning supports their validity.

5) To better understand the last (1.29) consider (2.11) written as follows

$$\epsilon^2=(pc)^2+\left(m_0c^2\right)^2\ \Rightarrow\ \epsilon-\frac{(pc)^2}{\epsilon}=\frac{\left(m_0c^2\right)^2}{\epsilon}=\delta\varepsilon$$

which therefore yields

$$\delta\varepsilon=\frac{\left(m_0c^2\right)^2}{\varepsilon}=\frac{n\hbar}{\delta t}=m_0c^2\beta\ \Rightarrow\ m_0c^2\beta\delta t=n\hbar;$$

regarding once more $n\hbar$ as referred to one of the allowed states of a physical system, in which case $\beta\to\beta_j$ and $n\hbar\to n_j\hbar$, this result reads

$$\sum_j n_j\hbar=\sum_j m_0c^2\beta_j\delta t\ \Rightarrow\ \sum_j n_j\hbar=\int m_0c^2\sqrt{1-\frac{v^2}{c^2}}dt=-\mathcal{L}. \tag{2.34}$$

The right hand side is related to the Lagrangian of a relativistic free particle.

These results show that actually the definitions in (2.24) have their own physical meaning.

2.3. Thermodynamic Implications

Equations (2.1) are compliant with the basic concepts of thermodynamics. As (1.101), (1.107) and (2.33) have shown the chance of obtaining thermodynamic results, here some further considerations appear appropriate to explain why the link between (2.1) and thermodynamics is in fact more profound.

Multiplying (2.1) by c one finds

$$\delta x\delta\left(p_x c\right)=n\hbar c=\frac{e^2}{\alpha/n}=\delta\varepsilon\delta\left(ct\right)=\delta\varepsilon\delta\ell\quad \ell=ct\quad \alpha=\frac{e^2}{\hbar c}=-\mathcal{L},\qquad(2.35)$$

which can be represented graphically in the Minkowski plane ℓ vs $x=v_x t$; in the present conceptual frame the coordinates are implemented as $\delta\left(ct\right)$ vs δx via the factor $\delta\left(p_x c\right)/\delta\varepsilon=c/v_x\geq 1$.

Since $\delta\ell\geq\delta x$ as it appears in (ZZ0) and (HJG), it follows that $\delta\left(p_x c\right)\geq\delta\varepsilon$ and thus according to (2.11)

$$\delta\left(\frac{\epsilon v_x}{c}\right)\geq\delta\varepsilon\quad \delta\left(p_x c-\varepsilon\right)\geq 0:\qquad(2.36)$$

from a physical point of view these inequalities are equivalent, yet their different analytical forms allow complementary corollaries. The notations emphasize that $\delta\epsilon$ and $\delta\varepsilon$ are in general different energy ranges: the former comes from the definition of δp_x at the left hand side of (2.35), the latter is just that at the right hand side of (2.1). Of course ϵ is any random value corresponding to p_x in its own $\delta\epsilon$, consistently with the second inequality.

Consider therefore separately the general case where $\delta\varepsilon\neq\delta\epsilon$ and that where in particular $\delta\varepsilon=\delta\epsilon$.

- In the case $\delta\varepsilon\neq\delta\epsilon$ write the first inequality (2.36) and its correlation to thermodynamic energies as follows

$$\frac{\epsilon}{c}\delta v_x\geq\delta\varepsilon-\frac{v_x}{c}\delta\epsilon\quad\Leftrightarrow\quad T\delta S\geq\delta W+\delta U_{int}\qquad(2.37)$$

having put

$$\frac{\epsilon}{c}\delta v_x=T\delta S\quad \delta\varepsilon=\delta W\quad -\frac{v_x}{c}\delta\epsilon=\delta U_{int};\qquad(2.38)$$

moreover, writing the last (2.37) as $T\delta S=\xi\delta Q$ via the numerical factor $\xi\geq 1$, one merges the first and second laws

$$\delta Q=\delta W+\delta U_{int}\quad T\delta S\geq\delta Q.\qquad(2.39)$$

In these results U_{int} is the usual internal energy of a thermodynamic system, whereas W is the work performed by or done on the system; Q heat energy that completes the total energy conservation.

To check the correspondences (2.38), consider for simplicity mechanical work only in a reversible transformation where holds the equality symbol; as (2.11) yields

$$\delta\left(p_x c\right)=\frac{v_x}{c}\delta\varepsilon+\varepsilon\delta\frac{v_x}{c}=-\delta U_{int}+T\delta S,$$

then $T\delta S-\delta\left(p_x c\right)=\delta U_{int}$ reads with the proposed correspondences

$$T\delta S-P\delta V=\delta U_{int}\quad \text{because}\quad \delta\left(p_x c\right)=\frac{\delta p_x}{\delta t}c\delta t=F_x\delta\ell=\frac{F_x}{A}A\delta\ell=P\delta V\qquad(2.40)$$

as it is known from the elementary thermodynamics. Note anyway that hold the relationships $v \ge v_x$ and $\delta v \ge \delta v_x$ between the modulus $v = |\mathbf{v}|$ of velocity $\mathbf{v}$ and its component v_x; hence it is possible to write as a function of v instead of v_x not only the initial (2.37)

$$\frac{\epsilon}{c}\delta v \ge \delta\varepsilon - \frac{v}{c}\delta\epsilon \tag{2.41}$$

but also all subsequent equations. Keep the given notation. Some remarks deserve attention.

1) If $\delta\varepsilon = \delta W = 0$ and $\delta\epsilon = 0$, so that $\delta U_{int} = 0$ as well, then $\epsilon = const$ and $T\delta S \ge 0$; this holds for an isolated system.

2) If $\delta\epsilon \ne 0$, then according to (2.38) $\delta U_{int} = -\delta x \delta\epsilon / c\delta t$ yields

$$F_x = -\frac{\delta U_{int}}{\delta x} = \frac{\delta\epsilon}{\delta\ell} \quad \delta\ell = c\delta t;$$

so δU_{int} results defined likewise to the change of potential energy $F_x = -\delta U_{pot} / \delta x$ due to an external force F_x acting on the system. Also, since

$$T \ge \frac{\delta U_{int}}{\delta S} + P\frac{\delta V}{\delta S} \quad T = \left.\frac{\delta U_{int}}{\delta S}\right|_{V=const}, \tag{2.42}$$

it is possible to write the first (2.38) with the help of (GK0) as

$$\begin{gathered} \frac{\epsilon}{c/\delta x}\frac{\delta \dot{x}}{\delta x} = \left.\frac{\delta U_{int}}{\delta S}\right|_{V=const} \delta S \\ i.e. \quad \frac{\epsilon}{v}\frac{\delta \dot{x}}{\delta x} = h\frac{\delta \dot{x}}{\delta x} = \left.\frac{\delta U_{int}}{\delta S}\right|_{V=const} \delta S = T\delta \quad Sv = \frac{c}{\delta x}. \end{gathered} \tag{2.43}$$

3) Consider the following chain of equalities

$$\begin{aligned} \frac{\epsilon}{c}\delta v_x &= \frac{\epsilon}{c}(v_{x2} - v_{x1}) = \frac{\epsilon v_{x2}}{c}\left(1 - \frac{v_{x1}}{v_{x2}}\right) \\ &= \frac{\epsilon v_{x2}}{c}\left(1 + \frac{v_{x1}}{v_{x2}}\right)^{-1}\left(1 - \frac{v_{x1}^2}{v_{x2}^2}\right) = \frac{\epsilon v_{x2}^2}{c}\frac{1 - v_{x1}^2/v_{x2}^2}{v_{x2} + v_{x1}}; \end{aligned} \tag{2.44}$$

owing to (2.38) this chain defines $T\delta S$. Moreover define an analogous equation to account for δQ; introducing an analogous equation as a function of a different v'_{x1} write

$$\begin{aligned} \frac{\epsilon}{c}\delta v_x &= \frac{\epsilon v_{x2}^2}{c}\frac{1}{v_{x2} + v_{x1}}\left(1 - \frac{v_{x1}^2}{v_{x2}^2}\right) = T\delta S \\ \frac{\epsilon}{c}\delta v'_x &= \frac{\epsilon v_{x2}^2}{c}\frac{1}{v_{x2} + v'_{x1}}\left(1 - \frac{v'^2_{x1}}{v_{x2}^2}\right) = \delta Q. \end{aligned} \tag{2.45}$$

As $T\delta S \ge \delta Q$, it is reasonable to put the velocity component $v_{x1} = -v'_{x1}$ with $v'_{x1} \ge 0$ in order that

$$\begin{gathered} \frac{\epsilon v_{x2}^2}{c}\frac{1}{v_{x2} - v'_{x1}}\left(1 - \frac{v'^2_{x1}}{v_{x2}^2}\right) = T\delta S \\ \frac{\epsilon v_{x2}^2}{c}\frac{1}{v_{x2} + v'_{x1}}\left(1 - \frac{v'^2_{x1}}{v_{x2}^2}\right) = \delta Q \quad \frac{\delta Q}{T\delta S} = \frac{v_{x2} - v'_{x1}}{v_{x2} + v'_{x1}}; \end{gathered} \tag{2.46}$$

in this way is surely fulfilled the second law (2.39), whereas the equality sign is fulfilled for $v_{x1} \ll v_{x2}$ e.g. $v_{x1} \to 0$. The energy ratio of the last (2.46) defines

$$\delta S = \left(\frac{v_{x2} + v'_{x1}}{v_{x2} - v'_{x1}} \right) \frac{\delta Q}{T} \quad v_{x2} \ge 0 \quad v'_{x1} \ge 0 \tag{2.47}$$

with the inequalities ensuring that $\delta S \ge \delta Q/T$, this is the usual definition of entropy alternative to $-w \log w$; it clarifies that a reversible heat exchange is that with $v'_{x1} = 0$. Are also definable the velocity components

$$V_{xS} = \frac{\epsilon v_{x2}^2}{cT\delta S} = \frac{v_{x2} - v'_{x1}}{1 - v'^2_{x1}/v_{x2}^2} \quad V_{xQ} = \frac{\epsilon v_{x2}^2}{c\delta Q} = \frac{v_{x2} + v'_{x1}}{1 - v'^2_{x1}/v_{x2}^2} \quad V_{xS} \le V_{xQ} \tag{2.48}$$

via (2.46), which at this point clarify that

$$V_{xQ} = c \quad V_{xS} \le V_{xQ} \tag{2.49}$$

to ensure the second inequality in agreement with $v'_{x1} > 0$. In other words, the entropy thermodynamic condition $T\delta S \ge \delta Q$ requires the velocity relativistic condition $v \le c$, in agreement with (1.51) or its probabilistic formulation (1.60). A subtle connection links thermodynamics, statistics and relativity; precisely this link suggests that these well known concepts are appropriate in this theoretical frame.

2.4. CPT Theorem

Are useful in this section (2.1) and (2.35). Consider first (2.1) and replace $\delta x = x_2 - x_1$ with $-\delta x$ by exchanging x_1 and x_2; in principle this way of rewriting the coordinate uncertainty range does not affect the theoretical formulation of present model, as all ranges are by definition arbitrary and unknown; the positive sign due to $x_2 > x_1$ is merely assumed by uniformity with that of $n\hbar$ and the other ranges. Formally this choice identifies allowed values on the positive side of the x axis, whereas the negative sign means moving all coordinates in δx to the negative side of the *x*-axis. Yet the sign reversal due to the mere exchange of range boundaries cannot be regarded as an unphysical operation; being however $\delta x \delta p_x = n\hbar$, this change of sign compels replacing $\delta p_x \to -\delta p_x$ unless rewriting $-|\delta x||\delta p_x| = -n\hbar$, which trivially means multiplying by -1 both sides and considering the component $-\delta p_x$ of the vector $\delta \mathbf{p}$. These steps are summarized simply writing $(-\delta x)(-\delta p_x) = n\hbar$. From a physical point of view, however, the coordinate sign reversal has an interesting implication on the conjugate momentum component: replacing $\delta x \to -\delta x$ implies implementing a mirror imagine of the momentum uncertainty range, while in turn $\delta p_x \to -\delta p_x$ has an analogous implication for the velocity component. But since $p_x = mv_x$, changing sign to the velocity component means considering $v_x \to -v_x$, *i.e.* $(-\delta x)/\delta t$. This holds whatever v_x might actually be, e.g. even if v_x is defined as $v_x/\beta' = v'_x/\sqrt{1 - v'^2_x/c^2}$. Yet in principle there is also the chance that $v_x \to -v_x$ can be also due to $\delta x/(-\delta t)$: no physical reason excludes that the mirror image of the space range δx correspond to the time mirror change of δt too, as time and space coordinates are both concerned in the same way and physical role in (2.1). In other words, changing contextually $\delta x/\delta t \to (-\delta x)/(-\delta t)$ leaves unchanged both v_x and thus p_x and eventually δp_x. This reasoning triggers however a further consideration about the second equality (2.1): for the same reasons already exposed, the position $\delta t \to -\delta t$ requires in turn changing $\delta\varepsilon \to -\delta\varepsilon$ once keeping unchanged the sign of $n\hbar$ and thus of $\hbar c$ in (1.2). Now the meaning of this requirement is less intuitive than before, as there are no components of vectors with signs to be arbitrarily changed. In the case of Coulomb energy, for example, it is enough to regard $e^2 = e \times e$ and thus $-e \times e$ to fulfill the attractive effect between opposite charges; it is significant that the sign changes just described require the contextual change of charge sign.

- On the one hand these short notes enunciate the CPT theorem: the mirror $\delta x \to -\delta x$ implies $\delta t \to \delta t$ that in turn implies $e \to -e$, which however holds for the Coulomb energy.
- On the other hand the meaning of $-\delta\varepsilon$ must have a general valence as long as related to time reversal whatever its specific nature might be, which suggests regarding these states as the ones of antimatter.

Even the CPT theorem appears as a natural corollary of the "mirror uncertainty": strictly speaking, it is self-evident that the Heisenberg principle cannot be contradicted in the mirror image of the physical reality described by the space and time ranges of values.

Thus (2.1) provides a concurring explanation to the Dirac see, in agreement with the quantum principle of superposition of solutions admissible and even extensible to negative solutions; the Dirac equation admits operations, e.g. complex conjugate and matrix multiplication, which convert a negative solution into a positive solution for a particle traveling in the opposite direction with reversed spin and charge. Anyway, the explanation proposed here does not contradict the Dirac reasoning.

2.5. Quantum Uncertainty and Equivalence Principle

In this model the equivalence principle is not postulated, rather it is a corollary of (2.1); since (2.1) contain explicitly the time dependence of all ranges, it appears in principle rational to formulate the concept of force via that of time dependent uncertainty. Considering for simplicity one space dimension only along with the time, define in general

$$\delta x = \delta x(x,t) \quad \delta p_x = \delta p_x(x,t) \quad n = n(x,t)$$
$$v_x = \frac{\delta x}{\delta t} \quad F_x = \frac{\delta p_x}{\delta t} \quad \delta t = t_2 - t_1; \tag{2.50}$$

if for any physical reason the range sizes change as a function of time, then

$$\delta\dot{x} = \frac{\delta}{\delta t}\delta x \quad \delta x = \frac{n\hbar}{\delta p_x} \quad \Rightarrow \quad \delta\dot{x} = \frac{\hbar}{\delta p_x}\frac{\delta n}{\delta t} - \frac{n\hbar}{\delta p_x^2}\delta\dot{p}_x \quad \delta\dot{p}_x = \frac{\delta}{\delta t}\delta p_x. \tag{2.51}$$

To figure out $\delta\dot{x}$, let be $\delta x = x_2 - x_1$ a range in a flat space time and let then this initial size stretch by keeping fixed its boundary coordinates x_1 and x_2:

postponing the physical explanation about why this stretching should occur, a possible way to describe $\delta\dot{x}$ is to convert the linear range δx into a curved profile, say a circular arc, by keeping fixed its boundaries. As a result, the coordinates x_1 and x_2, whatever they might be, define in this example an initial diameter of size δx strained into one half circumference of radius $\delta x/2$ and length $(2\pi\delta x/2)/2 = \pi\delta x/2$ consequent to this particular kind of space time deformation. Therefore is definable a "stretch coefficient"

$$\Xi = \frac{\delta x_{str} - \delta x_{unstr}}{\delta x_{unstr}} \tag{2.52}$$

given in this specific example by $\pi/2 - 1$. Regardless of the oversimplification deliberately introduced for simplicity only, any $\delta\dot{x}$ implies a corresponding $\delta\dot{p}$ because of (2.1); in general, the chances of shrinking or stretching the range size is in principle compliant with forces of both signs. The analytical form of (2.1) accounts for the conceptual link between deformation rate of space time and rising of a corresponding force; in effect it has been shown [5] that actually (2.51) is nothing else but the Einstein equivalence principle, shortly sketched here for completeness. Equations (2.1) provide themselves a possible hint in this respect; calculate

$$\delta\dot{x} = \frac{\delta}{\delta t}\delta x = \left.\frac{\delta x}{\delta t}\right|_{x_2,t_2} - \left.\frac{\delta x}{\delta t}\right|_{x_1,t_1} = v_x(x_2,t_2) - v_x(x_1,t_1) = \delta v_x \qquad \frac{\delta\dot{x}}{\delta x} = \frac{\delta v_x}{\delta x} = \frac{1}{\delta\tau} \tag{2.53}$$

and analogously

$$\delta\dot{p}_x = \frac{\delta}{\delta t}\delta p_x = \left.\frac{\delta p_x}{\delta t}\right|_{x_2,t_2} - \left.\frac{\delta p_x}{\delta t}\right|_{x_1,t_1} = F_x(x_2,t_2) - F_x(x_1,t_1) = \delta F_x. \tag{2.54}$$

As concerns this last result, it is also true that

$$\delta\dot{p}_x = \frac{\delta}{\delta t}\frac{n\hbar}{\delta x} = \frac{\hbar}{\delta x}\frac{\delta n}{\delta t} - n\hbar\frac{\delta\dot{x}}{\delta x^2} = m\delta\dot{x}\left(\frac{1}{n}\frac{\delta n}{\delta t} - \frac{\delta\dot{x}}{\delta x}\right) = \delta F_x$$
$$\delta n = integer \qquad \delta F_x \gtreqless 0. \tag{2.55}$$

Of course δn concurs itself to the link between $\delta\dot{x}$ and δF_x; is interesting the chance that $\delta F_x = 0$ despite $\delta\dot{x} \neq 0$ simply because $\delta n/(n\delta t) = \delta\dot{x}/\delta x$, in which case $\delta F_x = 0$ means $F_x = const$ and in particular even $F_x = 0$. The inertia principle for a steady motion requires thus $\delta\dot{x} = 0$ and $\delta n = 0$, *i.e.* a steady motion with $\delta\dot{p}_x = 0$. Anyway it is self-evident that the force field inside δx due only to the space time stretching $\delta\dot{x}$ is nothing else but that postulated by the Einstein equivalence principle, found in [5] and summarized as follows: given any $\delta x = x_2 - x_1$, in principle $\delta\dot{x} = x_2 - \dot{x}_1$ implies a force field $\overline{F_x}$ inside δx, having considered for simplicity to $x_1 = x_1(t)$ only. Let be for example x_1 the boundary coordinate related to the position of δx with respect to the origin *O* of an arbitrary reference system *R*, and x_2 that determining the size of δx. During $\delta\dot{x}$ an observer sitting on x_1 would move at rate $\dot{x}_1$ with respect to *O*, *i.e.* he thinks that its displacement is due to F_x; another observer sitting on x_2 still experiences via an appropriate experiment the same

F_x although being at rest, *i.e.* he thinks to be in an external force field, e.g. in particular the gravity field. Yet x_1 and x_2 are boundary coordinates by fundamental assumption completely equivalent, *i.e.* the role of x_1 and x_2 can be exchanged while the conclusions of the respective observers are identical: *i.e.* space time deformation $\delta\dot{x}$ and gravity field are indistinguishable themselves. If the range size δx is small enough, the force field turns into a local force, which shows that (2.1) define in fact a sort of "state equation" of the space time, in agreement with their direct link to the operative definition $\hbar G/c^2$ of space time [3].

Returning to (2.55), the following comments are worth noting:

- If the number of allowed state of the system is such that $\delta n/(n\delta t) \ll \delta\dot{x}/\delta x$, then (2.55) takes the form $\delta\dot{p}_x \approx -n\hbar\delta\dot{x}/\delta x^2$ similar to (1.120); indeed, as $\dot{p}_x = F_x$ and thus $\delta\dot{p}_x = \delta F_x$, one finds

$$\delta F_x \approx -\epsilon\mathcal{R} \quad -\epsilon = n\hbar\frac{\delta\dot{x}}{\delta x} \quad \mathcal{R} \approx \frac{1}{\delta x}. \tag{2.56}$$

As in general $\delta F_x = F_x - F_{0x}$ even though the initial $F_{0x} = 0$, (2.56) implies the rising of a force F_x related to $\mathcal{R}$ in an initially flat space time.

- The Einstein intuition of mass curving the space time is found here in a slightly different form involving the uncertainty ranges: the space time curvature, regarded through the one dimensional ratio $\delta\dot{x}/\delta x$, is mere range size deformation that implies itself an energy mc^2 defined by the presence of m that triggers the strain rate $\delta\dot{x}$, *i.e.*

$$\hbar\frac{\delta\dot{x}}{\delta x} = \pm\frac{\hbar}{\tau}; \tag{2.57}$$

so it is reasonable to expect that

$$\frac{\delta\dot{x}}{\delta x} = \pm m\frac{c^2}{\hbar} = \pm\frac{c}{\lambda_C} = \pm\omega_m \quad mc^2 = \frac{\hbar}{\tau}, \tag{2.58}$$

i.e. the mass m is related to the stretching coefficient (2.52) previously introduced.

- Einstein aimed to describe specifically the gravity, whereas the model so far exposed admits in general the chances of range size shrinking and stretching the space time range sizes; this implies the existence of forces both attractive and repulsive emphasized by the double sign allowed in (2.58), whereas formally m refers to positive or negative states of energy.
- The crucial point to explain without specific hypotheses the reason of such a space deformation about what can cause stretching of an initial flat space time is that is the uncertainty range δx shrinks or stretches simply because (2.1) allow this chance, so that a force field δF_x is expected throughout it. Let therefore $\delta\dot{x}$ and $\delta\dot{p}_x$ imply the space time curvature, then (1.87) indicates that the concept of mass triggers the concept of force in (1.88) and acceleration (1.93) likewise as it appears in (1.89) too.

These remarks hold even considering n unchanged in (2.55) during δt of (2.53); however δn is the added value to the mere geometric view of the gravitational interaction. Instead of describing the gravity through the curvature of space time only, the geometric information must be completed by quantum considerations about the change of number of allowed states inherent the transition from an ideal flat space time without matter and the actual universe filled with matter; in other words, the matter contributes with its own number of allowed quantum states to modify that of the quantum vacuum, *i.e.* $\delta n = n_{matter} - n_{vacuum}$. Considering the former without the latter, Einstein has actually concerned one half only of the whole story. Replacing the deterministic formalism of gravity based on the geometrical tensor metrics with a non-deterministic quantum model based on the uncertainty, appears reasonable the idea that the second addend of (2.55) is related to the cosmological term reluctantly added by Einstein to its original tensor driven formulation: neglecting the δn contribution, *i.e.* the quantum aspect of the gravity, phenomena like entanglement or quantization are obviously "a priori" excluded. In fact this is shown by (2.16) and (2.23), and will be further emphasized in the next section. The proposed idea of replacing the postulate of mere space time curvature with that, more general than (2.52), deductible from the space time uncertainty ranges, sounds as follows:

> "the matter tells the space time how to deform; the space time deformation tells matter how to move and, in doing so, how to change its number of allowed quantum states".

2.6. Quantum Contribution to the Dark Matter

It is instructive to show first that a key result of the general relativity neglecting in fact the term δn of (2.55). If in particular $\delta n \ll n$, then owing to (2.7), (2.53), (2.54) and (1.73) the third equality (2.55) reads approximately

$$-\frac{\delta F_x}{m} \approx \frac{(\delta v_x)^2}{\delta x} \qquad n = const:$$

having neglected the δn term and taken $\delta x > 0$, the sign of δF_x results negative, as it is in fact possible in (2.55). According to (2.1), it means concerning an attractive local force $-F_x$ defined in a range of values $-F_{1x} \le -F_x \le -F_{2x}$ included in the attractive force field $-\delta F_x = -(F_{x2} - F_{x1})$. For example F_x could be the local Newton force $-F_N$ acting on a point $-x_1 \le x \le x_2$; hence $-F_x \equiv -F_N$, whereas the same holds for the force range boundaries $F_{x2} \equiv F_{N2}$ and $F_{x1} \equiv F_{N1}$; this is possible because the range boundaries are arbitrary and unknown by fundamental assumption. So in practice the force field range is a positive range; however, to remind that δF_x is in fact an attractive force field, use the notation $|\delta F_x|$. A typical case to exemplify this physical situation is that of a photon moving radially with respect to a gravity point source. Hence multiply both sides by v_x^2 and rewrite

$$-\frac{\delta F_x}{m} v_x^2 \approx \frac{(v_x \delta v_x)^2}{\delta x} \qquad -\frac{\delta F_x}{m} v_x^2 \delta x \approx (\delta\varphi)^2 \qquad \delta\varphi \approx \pm\sqrt{\frac{-\delta F_x}{m} v_x^2 \delta x} \qquad -\delta F_x > 0;$$

as

$$\frac{\delta\varphi}{c^2}\approx\pm\sqrt{\frac{|\delta F_x|\delta x}{mc^2}\frac{v_x^2}{c^2}}=\pm\sqrt{\frac{|\delta F_x|\delta x}{mc^2}\frac{m_0v_x^2}{m_0c^2}}=\frac{\delta\epsilon}{\epsilon_0}$$
$$\delta\epsilon=\sqrt{\left(m_0v_x^2\right)\left(|\delta F_x|\delta x\right)}\qquad \epsilon_0=\sqrt{\left(mc^2\right)\left(m_0c^2\right)}. \tag{2.59}$$

Next, since owing to (2.7) one infers $\delta\varepsilon=n\hbar\delta\omega$ with $n=constant$ by assumption, (2.59) yields

$$\left.\frac{\delta\varphi}{c^2}\right|_{\delta n=0}\approx\frac{n\hbar\delta\omega}{\hbar\omega_0}=\frac{\delta\omega}{\omega_0}\qquad \epsilon_0=\hbar\omega_0\qquad \delta\epsilon=\hbar\delta\omega \tag{2.60}$$

in agreement with (1.94).

This result of the general relativity can be in fact obtained even without including the change of number δn of allowed quantum states.

Clearly however the incompleteness of a theory based on the geometrical of the space time only appears weak when attempting to describe physical problems where the quantum effects are crucial. To highlight this further point, consider (2.55) in the case where instead $\delta n/n\delta t\gg\delta\dot{x}/\delta x$; thus, owing to (2.53),

$$\frac{\delta F_x}{m}\approx\delta\dot{x}\frac{1}{n'}\frac{\delta n}{\delta t}=\delta v_x\frac{1}{n'}\frac{\delta n}{\delta t};$$

then, as before,

$$\frac{v_x\delta F_x}{m}\approx\delta\varphi\frac{1}{n'}\frac{\delta n}{\delta t}\ \Rightarrow\ \frac{\delta\varphi}{c^2}\approx\frac{\dfrac{v_x\delta F_x}{mc^2}}{\dfrac{1}{n'}\dfrac{\delta n}{\delta t}}.$$

In this case the result is

$$\left.\frac{\delta\varphi}{c^2}\right|_{\delta\dot{x}\to0}\approx\frac{v_x\delta F_x}{mc^2}\frac{\delta t}{\delta n}=\frac{n'}{\delta n}\frac{\delta x\delta F_x}{mc^2}=\frac{n'}{\delta n}\frac{\delta\epsilon'}{\epsilon_0'}\qquad \delta\epsilon'=\delta F_x\delta x\qquad \epsilon'=mc^2. \tag{2.61}$$

- On the one hand the form of $\delta\varphi/c^2$ in (2.61) is similar to that in (2.60); this means that the gravitational effect $\delta\epsilon'/\epsilon_0'$ of $\delta n/(n\delta t)$ is in principle analogous to that of the space time stretching/curvature $\delta\dot{x}/\delta x$.
- On the other hand, as necessarily $\delta n\le n$ because the change δn cannot be greater than n itself, comparing (2.60) and (2.61) one infers

$$\left.\frac{\delta\varphi}{c^2}\right|_{\delta\dot{x}\to0}\approx\left.\frac{\delta\varphi}{c^2}\right|_{\delta n=0}\frac{1}{\delta n}\left(\frac{\delta\epsilon'}{\epsilon_0'}\frac{n'\omega_0}{n\delta\omega}\right) \tag{2.62}$$

Strictly speaking, this conclusion is reasonable because the number of allowed states of a physical system affects its energy, while it is known that in relativity energy and mass are equivalent as concerns their gravitational effects. However the gravitational effect of the mass is easily recognizable once having established that precisely the mass induces the space time curvature, whereas instead the change of the number of states is not a tangible and visible source of gravity likewise the matter; yet $\delta\varphi$ that causes the red shift of (2.60) is the same, $v_x\delta v_x\equiv\delta v_x^2/2$, as that defined in (2.61).

It is therefore interesting to learn more about this topic by considering both concurring effects.

As (2.55) reads

$$\delta a_x = \frac{\delta F_x}{m} = v_x\left(v_n - v_x\right)\delta x \quad v_x = \frac{\delta \dot{x}}{\delta x} \quad v_n = \frac{\delta n}{n\delta t},$$

then it follows

$$\frac{\delta a_x}{v_x} = \delta v_x = \left(v_n - v_x\right)\delta x \quad v_n - v_x = \frac{c/\ell_0}{\left(c/\ell_0\right)\delta x + v_{x0}}$$
$$\Rightarrow \delta v_x = \frac{\left(c/\ell_0\right)\delta x}{\left(c/\ell_0\right)\delta x + v_{x0}} \quad \delta\varepsilon = h\left(v_n - v_x\right), \tag{2.63}$$

being ℓ_0 and v_{x0} dimensional parameters. The second position is reasonable because δv_x must be defined in order that $0 \le \delta v_x \le c$, which in effect is fulfilled for $\delta x = 0$ and $\delta x \to \infty$. Regarding in particular $\delta v_x = v_{per} - v_{centr}$ related to the peripheral velocity of a rotating body of size $\delta x = r$ around its center, then with $\delta v_x = v_{per}$ and $v_{centr} = 0$ this result reads

$$v_{per} = \frac{\left(v_{lim}/\ell_0\right)r}{\left(v_{lim}/\ell_0\right)r + v_{r0}} \quad v_{lim}/\ell_0 = const \quad v_{r0} = const'. \tag{2.64}$$

This simple function fits well the rotation velocity of spiral galaxies as a function of the increasing distance from their rotation center [6]. Of course the best fit parameters ℓ_0 and v_{r0} must be determined case by case to match the various observed curves; in other words v_{lim} replacing c depends on the mass, size and evolution of specific galaxies, whereas c of (2.63) has been implemented because in principle neither v_x nor its variability range δv_x can trespass this threshold rate. Works is in progress in this respect. However such best fit calculation is not the crucial point; the essential facts are: 1) that the quantum term v_n has a physical meaning analogous to the relativistic space time deformation rate v_x of (2.61) and 2) that just for this reason both addends δF_x of (2.55) account for the observed velocity profile without invoking the presence of dark matter. Also, $h\left(v_n - v_x\right)$ is one quantum of dark matter.

As a closing remark, a further interesting property of (2.55) is due to the chance of writing

$$m\left(\frac{1}{n}\frac{\delta n}{\delta t} - \frac{\delta \dot{x}}{\delta x}\right) = \frac{\delta F_x}{\delta \dot{x}}$$

whence

$$\frac{\delta}{\delta t}m\left(\frac{1}{n}\frac{\delta n}{\delta t} - \frac{\delta \dot{x}}{\delta x}\right) = \frac{\delta}{\delta t}\frac{\delta F_x}{\delta \dot{x}} \quad \frac{m\delta \dot{x}}{\delta x}\left(\frac{1}{n}\frac{\delta n}{\delta t} - \frac{\delta \dot{x}}{\delta x}\right) = \frac{\delta F_x}{\delta x}.$$

If

$$\frac{\delta}{\delta t}m\left(\frac{1}{n}\frac{\delta n}{\delta t} - \frac{\delta \dot{x}}{\delta x}\right) = m\frac{\delta \dot{x}}{\delta x}\left(\frac{1}{n}\frac{\delta n}{\delta t} - \frac{\delta \dot{x}}{\delta x}\right) \Rightarrow \frac{\delta}{\delta t}\frac{\delta F_x}{\delta \dot{x}} = \frac{\delta F_x}{\delta x},$$

then

$$\frac{\delta \dot{x}}{\delta x} = \frac{1}{Y}\frac{\delta Y}{\delta t} \quad Y = m\left(\frac{1}{n}\frac{\delta n}{\delta t} - \frac{\delta \dot{x}}{\delta x}\right);$$

therefore, since $\delta Y/Y = \delta \log Y = \log Y'' - \log Y'$, the result is

$$Y'' = Y' \exp(v_x t) \quad v_x = \frac{\delta \dot{x}}{\delta x} = \frac{\delta v_x}{\delta x} = \frac{1}{\delta t}.$$

Note eventually that

$$Y = \frac{mass}{time} = \frac{momentum}{length} = \frac{force}{velocity} \Rightarrow \frac{\delta p_x}{\delta x} = \frac{n\hbar}{\delta x^2} \quad Y = \frac{n\hbar/\delta t}{\delta x^2},$$

i.e. Y is related via (1.75) to the De Broglie momentum.

2.7. The Quantum Concept of Force

Since (2.1) contain explicitly the time dependence of all ranges, it appears natural to formulate the concept of force via that of time dependent uncertainty; in fact, this means changing the quantum states of a particle delocalized in δx before and after stretching of its δx during δt. This generates a force that cannot depend upon δp_x only. This point is explained writing (2.55) as follows

$$\delta F_x = \frac{n\hbar}{\delta x}\left(-\frac{\delta \dot{x}}{\delta x} + \frac{\delta n/n}{\delta t}\right): \tag{2.65}$$

from a macroscopic point of view the space time geometry is enough to account satisfactorily for several experimental observations; yet it is reasonable to pay attention also to the quantum contribution, crucial for a full understanding of the concept of force and required to merge relativity and quantum theory.

Examine these hints writing first, according to (2.54) and (2.53),

$$\delta \dot{p}_x = \delta F_x = F_{x2} - F_{x1} = -n\hbar \frac{\delta v_x}{\delta x^2} \quad \delta v_x = v_{x2} - v_{x1} \quad \delta n = 0; \tag{2.66}$$

in turn $F_{x2} - F_{x1} = -n\hbar (v_{x2} - v_{x1})/\delta x^2$ reads indistinguishably either

$$F_{x2} = -n\hbar \frac{v_{x2}}{\delta x^2} \quad F_{x1} = -n\hbar \frac{v_{x1}}{\delta x^2} \tag{2.67}$$

or

$$F_{x2} = n\hbar \frac{v_{x1}}{\delta x^2} \quad F_{x1} = n\hbar \frac{v_{x2}}{\delta x^2}, \tag{2.68}$$

depending on whether the final and initial states are defined by the indexes of F_x or v_x.

First of all, whatever the signs of velocity components v_{x2} and v_{x1} might be, one infers two chances in defining the force component F_x resulting from the time deformation rate of the range $\delta x = \delta x(x,t)$ summarized in general by the form

$$F_x = \pm \frac{\mathcal{K}}{\delta x^2} \quad \mathcal{K} = |n\hbar v_x|; \tag{2.69}$$

then, taking the minus sign and regarding in particular v_x as a constant in order to write $\hbar v_x = Gm'\overline{m}$ as in (1.114), F_x takes the form (1.120). However (2.1) allow writing $\delta x = \delta\varepsilon\delta t/\delta p_x$, so that $\delta\varepsilon/\delta p_x = v_x$ yields $\delta x = v_x \delta t$; (2.69) turns into

$$F_x = \pm \frac{\mathcal{K}}{(v_x \delta t)^2} \quad v_x = \frac{\delta \varepsilon}{\delta p_x}. \tag{2.70}$$

Hence (2.1) bypasses the necessity of introducing purposely the propagation time of the gravitational interaction, as done in the steps (1.117) to (1.119), because their form implies inherently the link of δx and δt. As the interaction propagates at rate *c*, then in this case it is convenient to write explicitly

$$F_G = -\frac{\mathcal{K}_G}{(c\delta t)^2} \quad \delta p_x = \frac{\delta \varepsilon}{c} \quad \mathcal{K}_G = n\hbar c = Gm'\overline{m}. \tag{2.71}$$

Then (2.1) account for the propagation time of any interaction, because they anyway require the time range inherently related to the space range size, but also allow to find two invariants of the special relativity along with the Newton law.

In this context, where $p_x = \varepsilon v_x / c^2$ is a quantum result obtained in (HH1) and (2.9) as a straightforward corollary of (2.1) only, thus bypassing relativistic considerations about flat space time and curved space time, it is possible to write with vector notation

$$\mathbf{F} = \frac{\delta \mathbf{p}}{\delta t} = \frac{\delta\left(\mathbf{v}\epsilon/c^2\right)}{\delta t} = \frac{\mathbf{v}}{c^2}\frac{\delta \epsilon}{\delta t} + \frac{\epsilon}{c^2}\frac{\delta \mathbf{v}}{\delta t} = \mathbf{v}\frac{\delta m_\epsilon}{\delta t} + m_\epsilon \frac{\delta \mathbf{v}}{\delta t} \quad m_\epsilon = \frac{\epsilon}{c^2}: \tag{2.72}$$

having written all dynamical variables via their uncertainty ranges, and not via their local coordinates implied by the symbols ∂. Note that the second addend at the right hand side is obvious, it is just $m_\epsilon a$; the first addend is more interesting because it shows that the definition of $\mathbf{F}$ implies an additional mass m'_ϵ inherent $\delta m_\epsilon = m'_\epsilon - m_\epsilon$ during the time range δt. This term additional to the classical m_ϵ defining $\mathbf{F}$ is just that exchanged via force carriers between the bodies interacting according to $\mathbf{F}$, being thus δt the time necessary for m'_ϵ to propagate the force; by definition $\delta m_\epsilon = \delta \epsilon / c^2$ shows that the carriers can even be massless. This generalizes the mere $F = ma$ of the classical mechanics and fits the idea of finite propagation time of any interaction mediated by force carrier.

Consider now the left hand side of (2.55) that reads according to (1.29), (2.55), (2.58) and (2.65)

$$F_x = -\frac{\delta U}{\delta x} \quad \delta U = n\hbar \frac{\delta \dot{x}}{\delta x} - \frac{\hbar \delta n}{\delta t} = \pm \frac{n\hbar}{\tau} - \frac{\hbar \delta n}{\delta t} = \pm \frac{n\hbar}{\delta t}\left(\frac{|\delta v|\delta t}{\delta x} \mp \frac{|\delta n|}{n}\right) \tag{2.73}$$

$$\mathbf{F} = -\nabla U.$$

If a potential function actually exists, then the key parameters defining F_x are: 1) the general minus sign in the first (2.73), due to the initial dependence of δp_x upon δx in (2.55) according to (2.1), and 2) the double sign of τ due to the specific kind of deformation rate $\delta \dot{x}$ of the space time range δx depending on whether $\delta \dot{x} \lesseqgtr 0$ in (2.58). This in turn implies $\delta U \lesseqgtr 0$ itself in the second (2.73). In fact from a physical point of view the double signs of δU anyway fulfills its standard definition in the first (2.73); although in the case of the gravity is of interest the negative potential, for sake of generality both signs in principle compatible with (2.55) are still considered from now on.

Nevertheless, whatever in particular δU might be, replacing $\hbar$ via the Planck mass $m_{Pl}=\sqrt{\hbar c/G}$ one finds owing to (GD3) and (2.57)

$$F_x=\pm\frac{n\hbar}{\tau\delta x}-\frac{\hbar\delta n}{\delta x\delta t}=\pm\frac{n\hbar}{\tau\delta x}\left(1\mp\frac{\tau}{\delta t}\frac{\delta n}{n}\right)=\pm\frac{nGm_{Pl}^2}{c\tau\delta x}\left(1\mp\frac{\tau}{\delta t}\frac{\delta n}{n}\right)\qquad(2.74)$$

having put

$$\frac{n\hbar}{\tau}=\frac{nGm_{Pl}^2}{c\tau}=\frac{Gm^2}{\ell}\qquad m^2=nm_{Pl}^2\qquad \ell=c\tau;\qquad(2.75)$$

hence, by comparison with (2.73),

$$F_x=\pm\frac{Gm^2}{\delta x'^2}\left(1\mp\frac{\tau}{\delta t}\frac{\delta n}{n}\right)\quad \delta x'^2=\ell\delta x\quad \delta U=\pm\frac{Gm^2}{\ell}\left(1\mp\frac{\tau}{\delta t}\frac{\delta n}{n}\right)\quad \frac{\tau}{\delta t}=\frac{\delta x}{\delta\dot{x}\delta t}.\qquad(2.76)$$

This result is acknowledged thinking $\delta t\to\infty$ and/or $n\to\infty$, so that the second addend in parenthesis becomes negligible with respect to 1 for large times and finite change δn of the number of gravitational states; then at $\delta t/\tau\gg\delta n/n$, force and potential take the standard form

$$F_x\approx\pm\frac{Gm^2}{\delta x'^2}\equiv\pm\frac{Gm_1m_2}{\delta x'^2}=\pm\frac{Gm_1m_2}{\left(\delta\varepsilon'\delta t'/\delta p_x'\right)^2}=\pm\frac{Gm_1m_2}{\left(v_x'\delta t'\right)^2}$$
$$v_x'=\frac{\delta\varepsilon'}{\delta p_x'}\qquad \delta U\approx\pm\frac{Gm^2}{\ell}.\qquad(2.77)$$

The chance of rewriting $m^2=m_1m_2$ is due to the arbitrariness of *m*, *i.e.* any values of m_1 and m_2 arbitrary themselves are certainly compatible with *m*. Clearly the primed notation $\delta x'$ is irrelevant, being instead essential that F_x results expressed via the ratio of two masses m_1m_2 over a square range $\delta x'^2$. This classical limit case with the minus sign corresponds therefore to the Newton law; yet even in this approximation are significant: 1) the fact that $\hbar$ does no longer appear in the equation, replaced by *G*, 2) the chance suggested by the positive sign concerned later, and 3) the fact that the propagation time $\delta t'$ appears explicitly in (2.76) and implicitly in (2.77) via $v_x'\delta t'$ defining $\delta x'$.

Is easily identifiable the meaning of $\delta\dot{x}/\delta x$ even at the cosmological level: the first addend in the parenthesis is nothing else by the Hubble law, as it appears replacing the generalized coordinate *x* with the radius *r* of the universe. This term has been calculated in (2.53) and compares with (1.42) where

$$v=\frac{\delta\dot{x}}{\delta x}\quad\Leftrightarrow\quad\left(G\frac{\ell_0^2}{c^2}\right)\frac{\rho-\rho_0'}{\tau}=\omega''':\qquad(2.78)$$

the left hand side concerns space time ranges, the same holds for the right hand side that concerns a frequency inferred from and related to *G* only. This agrees with the idea of cosmological valence of (2.78).

2.8. Gravitational Waves

To shorten as much as possible the exposition of this section, let us start from the early Einstein result of energy loss rate of an orbiting system of masses m_1 and m_2 that generate gravitational waves

$$-\frac{\delta E}{\delta t}=2\pi G\frac{r^4}{c^5}\omega^6\mu^2 \quad \frac{1}{\mu}=\frac{1}{m_1}+\frac{1}{m_2} \quad 2\pi\approx 32/5 \quad \omega^2\delta\ell^3=m_{tot}G; \tag{2.79}$$

the original integration factor 32/5 has been replaced here by the more significant 2π for physical reasons clarified below; the difference between these numerical values is less than 2% only. So (2.79) reads

$$-\frac{\delta E}{\delta t}=2\pi G\frac{\omega^6\delta\ell^4}{c^5}\mu^2=2\pi\frac{G}{c^3}\mu^2\left(\frac{\omega^3\delta\ell^2}{c}\right)^2=2\pi\frac{G}{c^3}(\mu a)^2=2\pi c\frac{(\mu a)^2}{F_{Pl}}, \tag{2.80}$$

where $\omega^3\delta\ell^2/c$ has been regarded as modulus of acceleration by dimensional reasons: it is the centripetal acceleration $\omega^2\delta\ell$ times the dimensionless tangential velocity $\omega\delta\ell/c$. To identify this term, implement the last (2.79) inferred in (1.5); then $\omega^3\delta\ell^2/c=m_{tot}G\omega/(c\delta\ell)$ yields

$$-a=\frac{\omega^3\delta\ell^2}{c}=\frac{m_{tot}G\omega}{c\delta\ell}=\frac{m_{tot}G}{(c/\omega)\delta\ell}=\frac{m_{tot}G}{\delta\ell'\delta\ell} \tag{2.81}$$

so that

$$\mu a=-\frac{\mu m_{tot}G}{\delta\ell\delta\ell'}=-G\frac{m_1m_2}{\delta\ell''^2} \quad \delta\ell''^2=\delta\ell\delta\ell' \tag{2.82}$$

i.e. μa is nothing else but the Newton law. Then write

$$-\delta E=2\pi\delta\ell\frac{(\mu a)^2}{F_{Pl}} \quad \delta\ell=c\delta t, \tag{2.83}$$

being δt the time range necessary to propagate the gravitational interaction at rate c between m_1 and m_2 through $\delta\ell$. The modulus of acceleration a has been acknowledged by dimensional reasons, being indeed the classical $\omega^2\delta\ell$ times the tangential velocity $\omega\delta\ell$. In fact (2.83) is simply a rewrite of the initial (2.79).

However new information is suggested by a possible way to regard (2.83) according to (2.7), *i.e.*

$$-\delta E=n\lambda_{gw}\frac{(\mu a)^2}{F_{Pl}} \quad 2\pi\delta\ell=n\lambda_{gw}$$
$$\Rightarrow\ -\frac{W}{W_{Pl}}=n\left(\frac{\mu a}{F_{Pl}}\right)^2=n\frac{\mu a_x}{F_{Pl}}\frac{\mu a_y}{F_{Pl}} \quad W=\frac{\delta E}{\delta t}. \tag{2.84}$$

The second equation is the quantization condition concerning the number n of steady waves allowed along a circumference of radius $\delta\ell$; the notation emphasizes the gravitational wavelengths $n\lambda_{gw}$ allowed around a gravity center $\delta\ell$ apart, which escape out of the orbiting system in lack of a black body-like constrain. So the energy loss δE corresponds to the emission of a number n of quantum energy $\lambda_{gw}(\mu a)^2/F_{Pl}$. Both W and μa are referred to the respective Planck quantities, *i.e.* they are expressed in Planck units.

Reverting the steps from (2.84) back to (2.79) one realizes that the formula of gravitational waves has mere Newtonian root. However, new information is provided by (2.84).

If this way of reading δE is correct, then the fate of gravitational systems is not, sooner or later, their unavoidable collapse: likewise two electromagnetic systems, one gravitational system decays and emits quanta of gravitational wave that in principle can excite another resonant gravitational system. The universe would appear accordingly much more interconnected, via resonant light and gravitational waves.

Eventually it is worth comparing (2.84) and (1.63); owing to the first (1.1)

$$-\frac{W}{W_{Pl}} = n\left(\frac{\mu a}{F_{Pl}}\right)^2 \Leftrightarrow -\frac{a}{\ell}\frac{(c\ell)^2}{F_{Pl}G} = \frac{(ma)^2}{F_{Pl}^2}. \quad (2.85)$$

Apart from n, the quantization due to (2.1), the right hand sides of these equations correspond well. Whatever the left hand side of (1.63) might be, it comes directly from the first (1.4), whose classical differential corresponds then to one quantum of gravitational wave energy.

2.9. Black Hole

By comparing (1.105) and (1.94), one infers via (2.7)

$$\varphi = \frac{mG\ell}{\ell_0^2} = \frac{n\lambda mG}{2\pi\ell_0^2}$$

and thus, via (1.65)

$$2\pi\ell_0^2 = \frac{n\lambda mG}{\varphi} = \frac{n\lambda mG}{v^2/2};$$

hence

$$\frac{2\pi\ell_0^2}{2\ell'^2} = \frac{n\lambda mG}{v^2\ell'^2}$$

and eventually, by identifying $\ell_0 = \ell_{bh}$ and $\ell' = \ell_{Pl}$ in order to express the black hole radius in Planck units,

$$\frac{4\pi\ell_{bh}^2}{4\ell_{Pl}^2} = \frac{\epsilon}{\epsilon'} = S_{bh} \quad \ell_0 = \ell_{bh} \quad \ell' = \ell_{Pl} \quad \frac{\epsilon}{\epsilon'} = \frac{nm'mG\lambda/\ell'^2}{m'v^2}. \quad (2.86)$$

This result is the dimensionless Beckenstein-Hawking surface entropy, which of course results expressible as the ratio of two energies. Consider now (1.68), whose physical meaning is extrapolated from that of (1.64). However it is appropriate to introduce a further way to get (1.68), more directly related to first principles and thus more significant from a physical point of view.

Equations (2.86) and (1.97) suggest an actual chance of obtaining the physical features of the black hole independently of the Einstein field equations and Schwarzshild metrics, by following instead a quantum approach; according to (1.69), $\delta\ell_{bh}$ is simply expression of the maximum gravitational potential consistent for any given m with the finite value of c, which is essential also in this respect. The ability of this model to bypass the idea of black hole similar to an event horizon around a singularity, along with its ability of accounting correctly and straightforwardly for the Hawking surface entropy, suggests instead the more sensible concept of an extremely compact body consisting of particles packed at the minimum lattice spacing physically admissible. This is a possible meaning of the macroscopic $\delta\ell''_{min}$ resulting in (1.69); for example is reasonable in this respect a body of matter with lattice spacing given by the Compton length $\hbar/mc$ of the constitutive particles, defined by their individual mass m contributing to the whole m_{bh} of (1.69).

To check the implications of this idea, start from (1.61) and (TZL) to define the following length

$$\Lambda = \ell_{dyn} + \bar{\lambda}_{dyn} = \xi_1 \frac{m_{dyn}G}{c^2} + \xi_2 \frac{\hbar}{m_{dyn}c} \quad \beta = \sqrt{1-\frac{v^2}{c^2}} \quad m_{dyn} = \frac{m_0}{\beta}: \tag{2.87}$$

from a physical point of view Λ is a linear combination of the length ℓ_{dyn} of (1.61) traveled by a corpuscle of mass m_{dyn} and its Compton wavelength $\bar{\lambda}_{dyn}$, according to the wave/corpuscle behavior of a quantum particle, ξ_1 and ξ_2 are the constant coefficients of the linear combination. Regard β as a free parameter determinable looking for the minimum of Λ as a function of v^2/c^2.

Trivial considerations show that Λ_{min} corresponds to a specific value of $v = v_{bh}$ such that

$$\beta_{min} = \pm\sqrt{\frac{\xi_1}{\xi_2}\frac{m_0}{m_{Pl}}} = \pm\sqrt{\frac{\xi_1}{\xi_2}\frac{m_0c^2}{m_{Pl}c^2}}$$
$$\Rightarrow \frac{v_{bh}^2}{c^2} + \frac{\xi_1}{\xi_2}\frac{\left(m_0c^2\right)^2}{\left(m_{Pl}c^2\right)^2} = 1 \quad \Lambda\left(\beta_{min}\right) = \frac{2m_{Pl}G}{c^2}\sqrt{\xi_1\xi_2}, \tag{2.88}$$

i.e. m_0 defining m_{dyn} is expressed in Plank mass units $m_{Pl} = \sqrt{\hbar c/G}$. The first (2.88) requires equal signs of ξ_1 and ξ_2, which however must be positive for both coefficients to define Λ in (2.87). The resulting β_{min} is compliant with the existence of states of negative energy $-m_0c^2$ and also consistent with

$$\frac{v_v^2}{v_c^2} + \frac{m^2}{m_{Pl}^2} = \frac{\left(h\nu_v\right)^2}{\left(h\nu_c\right)^2} + \frac{\left(mc^2\right)^2}{\epsilon_{Pl}^2} = 1$$
$$\nu_v = \frac{v_{bh}}{\ell_0} \quad \nu_c = \frac{c}{\ell_0} \quad \xi_1 = \frac{1}{\xi_2} \quad m = \frac{m_0}{\xi_2} < m_{Pl}; \tag{2.89}$$

owing to the coefficient $\xi_2 \gtrless 1$ in principle arbitrary, the last inequality holds whatever m_0 of (2.87) might be. So the condition $\xi_1\xi_2 = 1$ allows two results:

- On the one hand (2.89) merges quantum and relativistic concepts of energy and emphasizes the probabilistic meaning of the mass/energy equivalence and corpuscle/wave behavior, in such a way that the sum of these probabilities yields the certainty of the actual behavior of the particle compliant with both properties.
- On the other hand, replacing the positive solution β_{min} in (2.87) one finds

$$\Lambda_{min} = \Lambda\left(\beta_{min}\right) = \frac{2m_{Pl}G}{c^2} = 2\ell_{Pl} = 2\frac{\bar{\lambda}}{m_{Pl}c}; \tag{2.90}$$

thus, multiplying both sides of this result by an arbitrary constant factor $\xi \ge 1$, one finds again (1.68)

$$\ell_{bh} = \frac{2m_{bh}G}{c^2} \quad m_{bh} = \xi m_{Pl} \quad \ell_{bh} = \xi\Lambda_{min}. \tag{2.91}$$

The scale factor ξ defines the macroscopic length ℓ_{bh} as a function of the macroscopic mass m_{bh}. It is not surprising that (1.68) is found here scaling the minimum of the function (2.87). This result, already found in (1.70), and evident here with the same physical meaning of ξ, confirms the validity of (2.87).

Now ℓ_{bh} is no longer lucky extrapolation of classical escape velocity (1.67) towards the limit c of (1.68) to escape the gravity field at the threshold distance $\delta\ell''$ from the gravity center; rather (2.91) is rooted on the basic principle of superposition of states, corpuscular and wavelike, typical of the quantum physics. This appears appropriate at the Planck and macroscopic scales, both uniquely described by the arbitrary values of a single parameter ξ. Since according to (1.4)

$$V_{bh} = \frac{32\pi}{3}\left(\frac{m_{bh}G}{c^2}\right)^3 \quad \rho_{bh} = \frac{3}{32\pi}\frac{c^6}{m_{bh}^2G^3} = \frac{1}{4\pi G}\frac{3}{8}\left(\frac{c^3}{m_{bh}G}\right)^2 = \frac{3}{8}\frac{\omega_{bh}^2}{4\pi G}$$
$$\omega_{bh} = \frac{c^3}{m_{bh}G}, \tag{2.92}$$

it is possible to infer a Poisson-like gravity equation

$$4\pi G\rho_{bh} = \left(\frac{1}{\mathcal{T}}\right)^2 = -\nabla\cdot\mathbf{g} = \nabla^2\varphi \quad \frac{1}{\mathcal{T}} = \pm\sqrt{\frac{3}{8}\frac{c^3}{m_{bh}G}} \quad \mathbf{g} = -\nabla\varphi \tag{2.93}$$

being $\mathcal{T}^{-1}$ frequency. The last (2.92) also yields the event horizon energy

$$\frac{h}{\mathcal{T}} = 2\pi\frac{\hbar}{\mathcal{T}} = 2\pi\epsilon_{bh} = \sqrt{\frac{3}{8}}\frac{\hbar c^3}{m_{bh}G} \quad \Rightarrow \quad \epsilon_{bh} = \sqrt{\frac{3}{8}}\frac{\hbar\omega_{bh}}{2\pi}$$
$$\Rightarrow \quad \epsilon_{bh} = \sqrt{\frac{3}{8}}\hbar\frac{a_{bh}}{2\pi c} \quad a_{bh} = \omega_{bh}c, \tag{2.94}$$

where the last equality expresses ω_{bh} via the black hole surface acceleration a_{bh}. The last (2.94) is acknowledged thinking to the Hawking and Unruh temperatures, both having the form $kT = \hbar a/2\pi c$; the agreement is reasonable, a numerical factor $\sqrt{8/3} \approx 1.63$ apart due to this quantum approach. Also, (2.94) defines the black hole energy per unit surface

$$\sigma_{bh} = \frac{\epsilon_{bh}}{4\pi\ell_{bh}^2} = \sqrt{\frac{3}{8}}\frac{\hbar c}{4\pi^2}\frac{1}{\ell_{bh}^3}. \tag{2.95}$$

The further idea that follows up on the steps from (2.87) to (2.91) is to identify the space range Λ_{min} with the Compton length $\lambda\!\!\!^{-}_{\sim}$ of a mass $m_{\sim}$ plus that $\lambda\!\!\!^{-}_{\odot}$ of a further mass $m_{\odot}$; so (2.90) is rewritten as

$$\Lambda_{min} = \xi_{\sim}\lambda\!\!\!^{-}_{\sim} + \xi_{\odot}\ell_{\odot} = \frac{\hbar}{m_{\sim}c} + \frac{m_{\odot}G}{c^2} \quad \frac{\hbar}{m_{\sim}c} = \frac{m_{\odot}G}{c^2}$$
$$\lambda\!\!\!^{-}_{\sim} + \ell_{\odot} = 2\ell_{Pl} \quad m_{\sim} = m_{\odot} = m_{Pl}. \tag{2.96}$$

The masses defined in this way fulfill

$$\left(\xi\xi_{\sim}\right)\lambda\!\!\!^{-}_{\sim}+\left(\xi\xi_{\odot}\right)\ell_{\odot}=\ell_{bh}. \tag{2.97}$$

The fact that $\lambda\!\!\!^{-}_{\sim}$ and $\ell_{\odot}$, both necessarily positive, are of the order of or smaller than the Planck length, suggests that $m_{\sim}$ and $m_{\odot}$ could actually be two virtual particles at an average distance of $2\ell_{Pl}$; the probabilistic meaning of (2.89) justifies this idea, which in fact concerns the minimum β_{min} of (2.87) and does not exclude quantum fluctuations around this minimum. The scheme hitherto proposed consists of three conditions

$$\Lambda_{min}=2\frac{m_{\odot}G}{c^2}\quad \hbar c=Gm_{\sim}m_{\odot}=\frac{e^2}{\alpha}\quad \Lambda_{min}=2\frac{\hbar}{m_{\sim}c}, \tag{2.98}$$

which are checked right now:

- On the one hand, when considering $m_{\odot}$ only, $m_{bh}=\xi m_{\odot}$ agrees with (2.91).
- On the other hand, dividing the $\hbar c$ term by $-\delta\ell$, (1.73) implies the Newtonian/Coulomb forms

$$\frac{\hbar c}{-\delta\ell}=\epsilon_N=-G\frac{m_{\sim}m_{\odot}}{|\delta\ell|}\quad |\delta\ell|\sim 2\ell_{Pl}=2\frac{\hbar}{m_{Pl}c} \tag{2.99}$$

when considering both $m_{\sim}m_{\odot}$; the second option holds for charged virtual particles.

- Eventually, when considering $m_{\sim}$ only, one finds owing to (1.73)

$$2\frac{\hbar}{m_{\sim}c}=\Lambda_{min}\quad\Rightarrow\quad 2\frac{D_{\sim}}{c}\delta t=\Lambda_{min}\delta t, \tag{2.100}$$

which surprisingly yields

$$2D_{\sim}\delta t=\delta\overline{\ell}^2\quad \delta\overline{\ell}^2=\Lambda_{min}\left(c\delta t\right). \tag{2.101}$$

The second equation defines a mean value of the square range $\delta\ell\left(c\delta t\right)$ to concern the fact that the mass $m_{\sim}$ does not travel at speed c but at an effective speed reduced by averaging with the path Λ_{min}. Is remarkable that the statistical meaning of the Brownian motion (1.44) holds even in this case where are reasonably expected virtual particles; furthermore, it would be even more interesting if in particular $m_{\sim}$ and $m_{\odot}$ would be oppositely charged, e.g. in connection with the Hawking vacuum polarization and radiation beyond the black hole event horizon.

Note that (2.98) and (2.99) suggest that both $m_{\sim}$ and $m_{\odot}$ can be effectively charged, because e^2 corresponding to $m_{\sim}m_{\odot}$ does not seem accidental. If this idea is true, the black hole should be regarded as a space time region of pure energy whose vacuum fluctuations are precisely the virtual particles $m_{\sim}$ and $m_{\odot}$.

These results should be conceivably related to the extreme features of the black holes; the right hand side of (2.91), obtained through the Compton and corpuscular lengths of m, implies the minimum distance physically definable for each one of the two particles. Therefore a macroscopic black hole is a collection of virtual $m_{\sim}$ and $m_{\odot}$ waves and corpuscles oppositely charged, tightly bound by a Newton-like/Coulomb-like potential allowing the reasonable lattice packing with mean spacing $2\ell_{\odot} = 2\lambda\!\!\!^{-}_{\sim}$ physically admissible by Compton lengths inside V_{bh} of (2.92).

Note anyway that no singularity appears in this model once having bypassed the Schwarzschild deterministic metrics via the quantum approach. This suggests merging (2.96) and (2.97) as follows:

$$\ell_{bh} = \frac{m_{\sim} + m_{\odot}}{m_{\odot}} \frac{m_{bh} G}{c^2} \begin{array}{l} \nearrow m_{\odot} c^2 = m_{bh} \dfrac{(m_{\sim} + m_{\odot}) G}{\ell_{bh}} \\ \searrow \dfrac{\xi m_{\sim} G}{\ell_{bh}} = \varphi_{max} = \dfrac{c^2}{2} \end{array} . \quad (2.102)$$

This result consists actually of identities, it is mere way of rewriting (2.91) to emphasize the actual physical meaning of $m_{\odot}$ and $m_{\sim}$; the former accounts for the rest energy of the black hole, the latter for the total gravitational potential φ_{max}. In other words the black hole is pure $m_{\odot} c^2$ mass energy plus $\varphi_{max} m_{\sim}$ wave energy in agreement with the first (2.89) and (2.98).

Eventually consider that the factor 2 in (2.90) has an interesting interpretation simply rewriting, without any specific hypothesis,

$$\frac{c^2}{m_{bh} G} = \frac{2}{\ell_{bh}} = \frac{1}{(\xi \xi_{\odot}) \ell_{\odot}} + \frac{1}{(\xi \xi_{\sim}) \lambda\!\!\!^{-}_{\sim}} = \mathcal{R}_s, \quad (2.103)$$

and that in (2.92) and (2.93) appears the reciprocal time $c^3 / m_{bh} G$; this result suggest rewriting identically (2.103) multiplying both sided by c/ξ, with ξ of (2.91), in which case one obtains

$$\frac{1}{(\xi \xi_{\odot}/c) \ell_{\odot}} + \frac{1}{(\xi \xi_{\sim}/c) \lambda\!\!\!^{-}_{\sim}} = \mathcal{R}_t \iff \frac{1}{\mathcal{T}_{\odot}} + \frac{1}{\mathcal{T}_{\sim}} = \frac{1}{\mathcal{T}} = \frac{c^3}{m_{bh} G} \quad \mathcal{R}_t = \frac{c}{\xi} \mathcal{R}_s. \quad (2.104)$$

In fact, further ideas about how to regard the reciprocals of space time coordinates are deductible by implementing the black hole equation only to calculate also the Einstein gravitational light beam bending. Start uniquely from the initial (2.91): squaring both sides

$$\ell_{bh}^2 = \frac{4 m_{bh} G}{c^2} \frac{m_{bh} G}{c^2}$$

it reads

$$\ell_{bh} = \left(\frac{4 m_{bh} G}{\ell_{bh} c^2} \right) \left(\frac{m_{bh} G}{c^2} \right). \quad (2.105)$$

Multiply both sides by a dimensionless parameter ξ_1 and introduce in (2.105) a further parameter ξ_2, both arbitrary, to find

$$\xi_1 \ell_{bh} = \left(\frac{4 (\xi_1 m_{bh}) G}{(\xi_2 \ell_{bh}) c^2} \right) \left(\frac{(\xi_2 m_{bh}) G}{c^2} \right) \quad \xi_1 > 1 \quad \xi_2 > 2\xi_1 > 1:$$

the ratio $\xi_1 m_{bh} / \xi_2 \ell_{bh}$ smaller than the initial m_{bh} / ℓ_{bh} means that the related term does no longer concern the pertinent mass/radius ratio of a black hole event horizon, *i.e.* both sides implement lengths beyond the event horizon. In other words, the parameters ξ_1 and ξ_2 convert the black hole radius and mass into a different non-black hole scenario; write then

$$\ell' = \left(\frac{4m'G}{c^2\ell''}\right)\left(\frac{m''G}{c^2}\right) \quad \ell' = \xi_1\ell_{bh} \quad m' = \xi_1 m_{bh} \quad \ell'' = \xi_2\ell_{bh} \quad m'' = \xi_2 m_{bh}. \quad (2.106)$$

In (2.106) appear two lengths, ℓ' and $m''G/c^2$, both outside the initial event horizon of the black hole; it is compliant with the idea of describing the curved space time around the black hole of mass m_{bh}, at a distance ℓ'' from its center; the first factor implies in this respect $\xi_2 > 2\xi_1$ to have $\ell''/m' > \ell_{bh}/m_{bh}$. The bending effect of a photon passing at ℓ'' from the gravity center corresponds to and is described by the local space time curvature. Approximating this curvature as an arc δs of circumference of radius ℓ, the tangents at the boundaries of δs define and angle $\delta\phi = \delta s/\ell$ on a Euclidean plane. Yet define $\delta\phi$ by calculating δs and ℓ via the space time curvatures $\mathcal{R}'$ and $\mathcal{R}''$ deductible from (2.106) itself, *i.e.* $\mathcal{R}' \approx 1/\ell'$ and $\mathcal{R}'' \approx c^2/m''G$. Then the geometrical $\delta s/\ell$ of a flat space circumference is actually given in the curved space time by $\mathcal{R}''/\mathcal{R}'$, *i.e.* by $\ell'c^2/m''G$: once more and again in agreement with the Laplace-like (1.85), the curvature radii are the reciprocals of the respective lengths. More specifically, a glance to (2.106) shows that in effect

$$\frac{4m'G}{c^2\ell''} = \frac{1/\ell'}{1\Big/\dfrac{m''G}{c^2}}$$

must be regarded according to

$$\delta\phi\big|_{Eucl} = \frac{\delta s}{\ell'} \equiv \frac{1/\ell'}{1/\delta s} \Rightarrow \delta\phi\big|_{non\text{-}Eucl} = \frac{\delta\mathcal{S}_{curv}}{\mathcal{R}_{curv}} \quad \delta\mathcal{S}_{curv} = \frac{1}{\ell'} \quad \mathcal{R}_{curv} = \frac{1}{\dfrac{m''G}{c^2}}$$

and therefore

$$\delta\phi \approx \frac{4m'G}{c^2\ell''} \quad (2.107)$$

for a mass m' passing at a minimum distance r'' from the gravity source.

Returning to (2.103) and (2.104), their possible explanation assumes that $c^2/m_{bh}G = F_{Pl}/m_{bh}c^2$ corresponds again to the Laplace curvature $\mathcal{R}_s$ of a black hole sphere of radius ℓ_{bh}, as if the event horizon would actually be a shell with its own characteristic surface tension (2.95). If so, then $\mathcal{R}_t$ extrapolates the meaning of space curvature (2.103) to the corresponding time curvature $c^3/m_{bh}G$ of (2.104) still related to m_{bh}. In this non-relativistic quantum model, the non-Riemann but Laplace-like space and time curvatures imply inherently each other; in other words (2.103) and (2.104) are not hypotheses but non-metric ways of rewriting (2.97) according to (2.96), which however are consistent with (2.86) and (2.94). Throughout this paper the deterministic Einstein metrics has been systematically waived, whereas in analogy with (1.120) now the result of (2.91) $2/\ell_{bh} = c^2/m_{bh}G$ is

$$F_{bh} = m_{bh} c^2 \mathcal{R}_s. \tag{2.108}$$

Moreover owing to (2.92) the energy density η_{bh} inside a black hole is, as anticipated in (1.2),

$$\eta_{bh} = \frac{3}{8}\frac{(\omega c)^2}{4\pi G} = \frac{3}{8}\frac{a^2}{4\pi G},$$

to which corresponds a surface pressure reminiscent of that found in (1.89).

To take a step forward, implement now the results of the Section 3.2 and regard (2.91) according to (2.97), obtained by up scaling (2.96) via the parameter ξ in order to infer (2.102); therefore implement

$$\ell_{bh} = \frac{m_{dyn} G}{c^2} + \frac{m_{dyn} G}{c^2}$$

with the right hand side conceptually equivalent to the first (2.96). Rewrite thus (2.91) according to (2.87) and (2.24); *i.e.*, owing to $(\xi\xi_\sim)\lambda\!\!\!^{-}_\sim$ and $(\xi\xi_\odot)\ell_\odot$ of (2.97), write identically by analogy

$$\ell_{bh} = \frac{m_0 \beta G}{c^2} + \frac{m_{dyn} G}{c^2} \Rightarrow \Lambda_{bh} = \frac{m_{dyn} G}{c^2} + \sqrt{\frac{(m_{dyn} G)^2}{c^4}} \tag{2.109}$$

that in turn reads

$$\Lambda_{bh} = \frac{m_{dyn} G}{c^2} + \sqrt{\frac{(m_0 G)^2}{c^4}\left(1 - \frac{v^2}{c^2}\right)} = \frac{m_{dyn} G}{c^2} + \sqrt{\left(\frac{m_0 G}{c^2}\right)^2 - \frac{(m_0 G)^2 v^2}{c^6}}. \tag{2.110}$$

This form of Λ_{bh} consists of two addends, in the second of which the dynamical character of m_{dyn} has been resolved in order to introduce the square root where appears rest mass m_0 only; this aims to highlight the dynamical meaning of the second addend (2.109) as follows

$$\left(\frac{m_0 G}{c^3}\right)^2 v^2 = \frac{Q^2}{F_{Pl}} + \frac{J^2}{(m_0 c)^2} \quad \Rightarrow \quad \Lambda_{bh} = \frac{m_{dyn} G}{c^2} + \sqrt{\left(\frac{m_0 G}{c^2}\right)^2 - \frac{Q^2}{F_{Pl}} - \frac{J^2}{(m_0 c)^2}}. \tag{2.111}$$

The first addend of Λ_{bh} is a dynamic mass quantity, likewise the terms of the square root where the dynamic meaning is made explicit. For brevity, the step from (2.110) to (2.111) is in fact guessed and sketched only: the aim is to show that even the generalization from (2.91) to the charged and rotating black hole is straightforward, elementary and embedded in the general frame hitherto described without additional hypotheses.

This section has aimed to be as self-consistent as possible, while being compliant with two crucial literature concepts, Equations (2.86) and (2.94). However the more in-depth analysis carried out in this section has shown that the black hole condition is actually probabilistic: (2.88) leads to (2.90), which in turn scales to (2.91) at a macroscopic level. This means that the black hole condition is compatible in principle with a non-black hole excited quantum state. Clearly this quantum property has to do with the ability of excited black hole quantum states to evaporate.

3. Closing Remarks

The strategy of any model aimed to be as self-contained as possible is: 1) to check step by step, wherever possible, the validity of the results as they are achieved and 2) to emphasize a recognizable physical meaning of the corollaries of these results. This goal is in turn achievable in various ways.

- The sought approach should reveal unexpected links between results apparently non-correlated. In fact relevant examples found in this model are the heat Equation (1.76) obtainable along with the De Broglie momentum (1.75), or the statistical distribution (1.107) obtainable in the same conceptual context of the gravitational red shift Equation (1.105). Attention deserve in this respect also the Brownian displacement (1.44) and the third Kepler law (1.46), while the way to infer (1.28) is closer to the classical physics than expected. This holds even for the "naive" Newton law.

Consider the third (1.2) with the appropriate sign of *a*, formally obtainable multiplying both sides by $-m$ to find the Newtonian force, and its classical differential

$$F_N = -G\frac{m^2}{\ell^2} \qquad \delta m = \frac{c^2}{G}\delta\ell. \tag{3.1}$$

The differential of F_N reads

$$\begin{aligned}\delta F_N &= -\frac{2mG}{\ell^2}\delta m + \frac{2m^2G}{\ell^3}\delta\ell = \frac{2mG}{\ell^2}\delta m - \frac{2m^2G}{\ell^3}\frac{G}{c^2}\delta m \\ &= \frac{2mG}{c^2}\left(\frac{Gm}{c^2\ell} - 1\right)\frac{\delta\left(mc^2\right)}{\ell^2};\end{aligned} \tag{3.2}$$

it vanishes for

$$\ell^* = \frac{Gm}{c^2} \to \delta F^* = 0 \qquad \frac{\delta m}{\ell^{*2}} = \frac{c^2}{G^2}\delta\left(mc^2\right) \qquad F^* = \frac{c^4}{G}.$$

The right hand side of (3.2) contains three interesting factors

$$\delta F_N = -\frac{\ell_{bh}}{\ell}Z\frac{\delta\epsilon}{\ell} \quad \Rightarrow \quad \ell_{bh} = \frac{2mG}{c^2} \qquad \epsilon = mc^2 \qquad Z = 1 - \frac{G}{c^2\ell}: \tag{3.3}$$

the first one is again (1.35), the second one is the rest energy (1.29), the third one reads

$$Z = 1 - \frac{V_G^2}{c^2} = \beta_G^2 \quad V_G^2 = \frac{G}{\ell} \Rightarrow \delta F = \frac{\ell_{bh}}{\ell/\beta_G}\frac{\delta\epsilon}{\ell/\beta_G} \quad \ell = \ell_{pr}\beta_G \quad \delta F = \frac{\ell_{bh}\delta\epsilon}{\ell_{pr}^2}. \tag{3.4}$$

The first position, possible by dimensional reasons, implies the last equation written in terms of the proper length ℓ_{pr}. Elementary manipulations of (3.1) reveal a hidden content, showing that the simple Newton law consists actually of three key results of special relativity.

This highlights that regarding the general relativity as a conceptual world far from and alternative to the classical physics is a superficial and misleading attitude.

Even the crucial concept of statistical basis of the reality shown by the probabilistic meaning of (1.60), a typically relativistic formula, has been inferred in

(1.51) via classical reasoning.

- The key point is not the classical or relativistic or quantum approach; is crucial instead that the starting point of any reasoning is sufficiently inclusive to provide a wide variety of outcomes, even serendipitous, yet all clearly recognizable and controllable. In fact, nothing can be more general than the systematic dimensional analysis of the physical dynamical variables. With this kind of approach in mind, the black hole does not imply any singularity, as it is found in (1.35), (1.68), (1.97) and once more in (3.3). Also, the analysis of the allowed quantum states of a system must prove capable of not containing concepts to be validated themselves; in fact (2.65) surrogates the elusive idea of dark matter, as shown in (2.62) and mostly in (2.64).
- In agreement with the fact that the space coordinates δx are actually generalized coordinates, note that $\delta\dot{x}/\delta x$ is also representative of the Hubble law; it appears more evident replacing *x*, which merely symbolizes any space coordinate, with the universe radius $r_u = r_u(t)$. Indeed the rate at which the universe boundary expands is

$$\delta x \Leftrightarrow r_u \Rightarrow \frac{\delta\dot{x}}{\delta x} \equiv \frac{\dot{r}_u}{r_u} \equiv \omega''' ; \tag{3.5}$$

here has been quoted also (1.42), completely analogous to (2.53) although obtained via dimensional considerations on G only.

- The textbook [1] implements the deterministic metric of relativity to calculate the transformation properties of the three components of angular momentum. As of course the reasoning is correct, what is conflicting with the quantum theory is the starting point, *i.e.* the space time metrics itself. For this reason the present model bypasses systematically the deterministic metrics to implement instead an approach free of the implications that make the standard relativity difficult to reconcile with the quantum physics. Even classical assumptions appear more reliable in this respect, e.g. the third (1.43) to find (1.51) and the relativistic Doppler shift (1.58). Moreover the quantum Equation (2.1) allows inferring (2.23) *i.e.* the spin of particles.
- The mere definition (1.27) of β implies with the help of (1.27) and (2.74)

$$1 = \frac{v^2}{c^2} + \beta^2 = \frac{v^2}{c^2} + \frac{\tau^2}{t^2} = \frac{2\varphi}{c^2} + \frac{\tau^2 h^2}{t^2 h^2} = \frac{2mG}{\ell c^2} + \frac{\epsilon_t^2}{\epsilon_\tau^2} = \frac{\ell_{bh}}{\ell} + \frac{\epsilon_t^2}{\epsilon_\tau^2} \tag{3.6}$$
$$\tau < t \quad \epsilon_\tau > \epsilon_t \quad \ell_{bh} < \ell$$

One expects that this chain of equations is sensible because: 1) the proper time τ fulfills the first condition of time t dilation, 2) the definition of energy h/t inversely proportional to t is coherent with the second condition and 3) the minimum length ℓ_{bh} consistent with a given m with respect to any other ℓ agrees with the third condition. If so, then (3.6) is corollary of these three basic requirements consistent with the initial Lorentz form at the left hand side. However is in principle reasonable also the reversed interpretation of this chain, *i.e.*: actually is precisely the probabilistic character of this result, regarded as a fundamental statement, that requires itself the three conditions (3.6). Either option is acceptable, yet the second one is more attracting: in fact it gives the relativistic results the probabilistic meaning, without which deterministic relativity and quantum physics would remain irreconcilable. In other words, requiring the time dilation and thus the length contraction as well to allow the first inequality (3.6), means introducing a relativistic model fully compliant with the quantum Equation (2.1), whereas the form (1.59) of the Lorentz factor is functional to this aim; this consideration holds also for (2.89) and (1.28) itself. This explains why in [3] [5] have been contextually obtained quantum and relativistic results in a systematic way. Trying to introduce the quantization in the relativistic results is a secondary problem; the main point is to introduce the concept of probability as an alternative to the metrics, the quantization is next introduced via (2.1).

- Let m_1 and m_2 be the masses of two particles $\delta\ell$ apart, *i.e.* $\delta\ell = \ell - \ell_0$ is the space range through which propagates the gravitational interaction at rate *c*. Owing to (2.73) and (1.5) consider the classical positions

$$\mathbf{F} = -\nabla U \quad \Rightarrow \quad \mu a_\ell = -\frac{\partial U}{\partial \ell} \quad \omega^2 \delta\ell^3 = mG \tag{3.7}$$

to examine the cases where a_ℓ is given by

$$a' = \omega^2 \delta\ell \quad a'' = \frac{\omega^3 \delta\ell^2}{c} \quad a''' = \frac{\omega^4 \delta\ell^3}{c^2} \quad \cdots;$$

all definitions fulfill the dimensional property of acceleration, yet they differ substantially in describing a two body orbiting system. For simplicity is considered uniform motion for which holds the last (3.7).

1) In the first case a' is the centripetal acceleration and yields

$$a' = \frac{mG}{\delta\ell^3}\delta\ell \quad \Rightarrow \quad F'_N = -\frac{m\mu G}{\delta\ell^2}$$
$$U' = -\frac{\mu G}{\delta\ell} = -m\omega^2\delta\ell^2 \quad T = -\frac{U}{2} = \frac{1}{2}m\omega^2\delta\ell^2.$$

2) $a'' = a'\omega\delta\ell/c$ has been already concerned in (2.81), it yields directly the Newton law (2.82).

3) Is particularly interesting the third case, $a''' = a''\omega\delta\ell/c$ not considered before but allowed because it is still consistent with the meaning of acceleration and thus acceptable in the present conceptual frame where the properties of physical systems are determined by the physical dimensions of their dynamical variables. Thus

$$a''' = \frac{\omega^4 \delta\ell^3}{c^2} = \frac{(mG)^2}{\delta\ell^6}\frac{\delta\ell^3}{c^2}$$

yields

$$a''' = \frac{(mG)^2}{c^2\delta\ell^3} \quad \Rightarrow \quad a''' = \frac{m^2 G}{\delta\ell^2}\frac{G}{c^2\delta\ell} \quad U''' = -\frac{1}{2}\frac{(mG)^2}{c^2\delta\ell^2}$$

however, since $G/(c^2\delta\ell)$ has physical dimensions of $mass^{-1}$, the resulting terms of this reasoning are

$$F_N'''=\frac{m^2G}{\delta\ell^2}\quad U'''=-\frac{1}{2}\frac{(mG)^2}{c^2\delta\ell^2}=-\frac{const}{\delta\ell^2}\quad F_N'''a'''=m'''a'''\quad m'''=\frac{c^2\delta\ell}{G}: \quad (3.8)$$

i.e. one finds again the Newton law, whereas the potential energy is that indicated, in addition to the mere $-const/\delta\ell$. The fact that also $U=-const/\delta\ell^2$ is compatible with the Newtonian force, has crucial implications.

- The interesting consideration is that all kinds of accelerations stimulated by (2.80) have their identifiable physical meaning; in particular it appear also the non-Newtonian potential U'''. It is emphasized in [2] that the perihelion precession is easily deductible with elementary methods via a central potential having the form $const/r^2$, which however is not expected in the standard classical gravity. The fact that U''' fits the present approach implies the chance of calculating the other result of the general relativity. For example, it is shown in the quoted textbook that the potential form $-const/\delta\ell^2$ allows calculating the perihelion precession of planets with elementary calculus; however the Authors remark that this potential form cannot be justified in the frame of the Newton law, so that the relativistic approach is required for such calculation. However, the quantum approach just highlighted that this non-Newtonian potential is obtainable. For this reason, the problem of planet precession has not been concerned in this paper: it is already solved in the quoted textbook.
- The key requirement to unite quantum physics and relativity is not the quantization, but to abandon the deterministic metrics. The link merging classical physics and special relativity is simple; yet even the general relativity is a simple generalization of the quantum uncertainty, as it appears for example in (2.16) and (2.86). The most significant results are the elementary (2.84) obtained from (2.79) and (2.107) merely obtainable from (1.97) only. This justifies why the Riemann curvature, deterministic, has been waived while however still inferring (2.107) and (2.111).

On the one hand the equivalence principle concerns the impossibility of discriminating non-inertial R and gravity field, precisely because the uncertainty requires the physical equivalence of the range boundaries x_1 and x_2 of δx before and after $\delta\dot{x}$; in other words the force field F_x rising within $\delta\dot{x}$ cannot discriminate $\dot{x}_1$ with x_2 at rest or x_1 at rest with $\dot{x}_2$, both concurring indistinguishably to $\delta\dot{x}$ driven $\delta\dot{p}_x$.

On the other hand, however, the range boundaries cannot explicitly affect physical information implied by their replacement of the local coordinates; in effect they are arbitrary and unknowable as required by (0F1). But this implies that since two different ranges $\delta\dot{x}$ and $\delta\dot{x}'$ fulfill identically the equivalence principle regardless of the masses therein possibly delocalized, then both ranges must be compliant with a unique conclusion: the amount of mass does not appear explicitly in this reasoning, *i.e.* all masses behave in the same way in a gravity field. For the same reason are illusory concepts like behavior of gravitational mass in δx and inertial mass in $\delta x'$ or *vice-versa*, which would contradict the equivalence principle itself as formulated here in the frame of the quantum uncertainty. Uncertainty ranges and local coordinates are physically different from the local coordinates as usually intended: the former have physical meaning, the latter do not. Consider by analogy the error bar characterizing a series of measurements of any physical amount: the total scattering of all results that defines the confidence interval of the measurement process has physical reliability, not the single outputs of the measurement process. Likewise the uncertainty ranges have actual physical meaning instead of the possible local values allowed in the ranges, here regarded as random and unknowable and unpredictable. In effect it is acknowledged that the coordinates do not exist "a priori" in nature, being rather artifices introduced to describe physical events, and thus should not play any role to formulate fundamental physical laws. This is the reason why in fact the uncertainty ranges, and not the local coordinates, have physical meaning in this model.

- The gravitational red shift is inferred from (2.7) thinking a photon of wavelength λ and frequency ν traveling in the vacuum along the radial direction with respect to a gravitational energy field. As $c = \lambda\nu$ implies $\delta\lambda/\lambda = -\delta\nu/\nu$, let the gravitational potential φ change by $\delta\varphi$ in the space interval corresponding to $\delta\lambda$ be

$$\frac{\delta\varphi}{c^2} = -\frac{\delta(m_0\varphi)}{m_0c^2} = -\frac{\delta\varepsilon}{\varepsilon_0} = -\frac{h\delta\nu}{\varepsilon_0} = \frac{c}{\lambda}\frac{\delta\lambda}{\lambda} = \nu\frac{\delta\lambda}{\lambda} \quad \varepsilon_0 = m_0c^2$$

$$\varepsilon = h\nu = m_0\varphi \quad \nu = \frac{c}{\lambda};$$

then, in agreement with (1.94)

$$\frac{\delta\varphi}{c^2} = \frac{\delta\lambda}{\lambda} = -\frac{\delta\nu}{\nu} = \frac{\nu_2 - \nu_1}{\nu}. \tag{3.9}$$

Since owing to (1.8) $\varphi \propto velocity^2$ by dimensional reason, then $\delta\varphi$ related to $\delta\lambda$ can be nothing else but the gravitational red shift.

This result has been mentioned as it legitimates the quantum/relativistic link $h\nu = m_0\varphi$, which is not self-evident, and justifies the interest of finding again in (3.9) a result already inferred in (1.105). Moreover (3.9) allows a possible interpretation of the gravitational red shift reminiscent of the rotational/linear frame-dragging of the Lense-Thirring effect: the wavelength stretching $\delta\lambda$ is actually guessable as a space stretching induced by the presence of a gravity field, analogous to that of (2.52), which in turn extends the local wavelength electromagnetic wave passing locally. Then also the Doppler effect has an immediate and intuitive explanation: if a luminous solid body travels towards an observer and "compresses" the space ahead it, then the light emitted that crosses this region of space appears blue-shifted to the observer; the contrary occurs if the body stretches the space behind it, in which case the light is red-shifted for the observer.

4. Conclusion

The main purpose of this model is to produce ideas, not numbers. Despite the dimensional analysis does not include numerical coefficients, its validity has been checked throughout the paper. It is convenient to start from (1.1) and (1.6) than from the elementary dynamical variables to infer physical laws in a self-consistent and self-contained conceptual frame. The other conclusion inferred in this paper is that the deterministic metrics of the standard general relativity can be successfully bypassed, whereas an alternative quantum approach is easily implementable.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Landau, L. and Lifchitz, E. (1986) Theorie du Champ. MIR, Moscow.

[2] Landau, L. and Lifchitz, E. (1989) Mecanique. MIR, Moscow.

[3] Tosto, S. (2016) Quantum and Relativistic Corollaries of an Operative Definition of Space Time. *Journal of Advances in Physics*, **11**, 3048.

[4] Amelino-Camelia, G. (2002) Space-Time Quantum Solves Three Experimental Paradoxes. *Physics Letter B*, **528**, 181-187. https://doi.org/10.1016/S0370-2693(02)01223-6

[5] Tosto, S. (2021) An "*ab Initio*" Model for Quantum Theory and Relativity. *Journal of Applied Mathematics and Physics*, **9**, 1718-1807. https://doi.org/10.4236/jamp.2021.97114

[6] Corbelli, E. and Salucci, P. (2000) The Extended Rotation Curve and the Dark Matter Halo of M33. *Monthly Notices of the Royal Astronomical Society*, **311**, 441.