

# Dirac and Maxwell Systems in Split Octonions

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## Abstract

The known equivalence of 8-dimensional chiral spinors and vectors, also referred to as triality, is discussed for  $(4 + 4)$ -space. Split octonionic representation of  $SO(4, 4)$  and  $Spin(4, 4)$  groups and the trilinear invariant form are explicitly written and compared with Clifford algebraic matrix representation. It is noted that the complete algebra of split octonionic basis units can be recovered from the Moufang and Malcev relations for the three vector-like elements. Lagrangians on split octonionic fields that generalize Dirac and Maxwell systems are constructed using group invariant forms. It is shown that corresponding equations are related to split octonionic analyticity conditions.

## Keywords

Split Octonions, Triality,  $SO(4, 4)$ , Dirac and Maxwell Equations

## 1. Introduction

Nonassociative algebras, apart from Lie algebras, have never been systematically utilized in physics, only some attempts have been made. Nevertheless, there are some intriguing hints that these kinds of algebras may play an essential role in the ultimate theory, yet to be discovered. Octonions, as an example of such a nonassociative structure, form the largest normed division algebra after the algebras of real, complex and quaternionic numbers [1] [2] [3]. Since their discovery in 1844-1845 there have been various attempts to find appropriate uses for octonions in physics (see reviews [4] [5] [6] [7]). One can point to the possible impact of octonions on: Color symmetry [8] [9]; GUTs [10] [11] [12]; Representation of Clifford algebras [13] [14]; Quantum mechanics [15] [16]; Space-time symmetries [17] [18] [19]; Formulations of wave equations [20] [21] [22]; Quantum Hall effect [23]; Kaluza-Klein program without extra dimensions [24]

[25]; Strings and M-theory [26] [27] [28] [29] [30]; SUSY [31] [32] [33] [34], etc.

Eight-dimensional Euclidean space, in which ordinary octonions reside, possesses certain peculiarities, namely that the vector and the two chiral parts of the spinor are all eight-dimensional objects and there exists a rotation invariant trilinear form in which vectors and chiral spinors act indistinguishably from one another. This property, called triality [35] [36], is usually formulated in terms of spin group automorphisms and symmetry of  $D_4$  Dynkin diagram [13].

Properties of spinors and vectors have been also discussed within the context of split octonions. Unlike ordinary octonions, the split algebra lacks the property of divisibility since it contains zero divisors. On the other hand,  $(4 + 4)$ -space of the split octonions has Minkowskian subspaces. Hence  $SO(8)$  group describing rotational symmetry of the Euclidean space is replaced by its noncompact analog for  $(4 + 4)$ -space, namely  $SO(4,4)$ , in which Lorentz groups  $SO(1,3)$  and  $SO(3,1)$  are contained multiple times as subgroups. This makes the split octonions interesting to study in the context of geometry in physics [37] [38] [39] [40].

In physical applications, split octonions were used to provide possible explanation for the existence of three generations of fermionic elementary particles [41] [42]. In [43] generators of  $SO(8)$  and  $SO(7)$  groups were obtained and have been used to describe the rotational transformation in 7-dimensional space. In [44] [45] real reducible  $16 \times 16$  matrix representation of  $SO(4,4)$  group utilizing the Clifford algebra approach was constructed and it was shown that there are two inequivalent real  $8 \times 8$  irreducible basic spinor representations, potential implementation for 8-dimensional electrodynamics [45] and gravity [46] was also considered. In [47] the basic features of Cartan's triality of  $SO(8)$  and  $SO(4,4)$  was analyzed in the Majorana-Weyl basis, it was shown that the three Majorana-Weyl spacetimes of signatures  $(4 + 4)$ ,  $(8 + 0)$ ,  $(0 + 8)$  are interrelated via the permutation group (signature-triality). In [36] octonionic representation of  $SO(8)$  and triality was discussed, but triality symmetry is also valid in  $(4 + 4)$ -space spanned by the split octonion algebra. Another unique concept associated only with  $(4 + 4)$ -space is 4-ality, it's similar to triality but deals with fourfold symmetry of modified Dynkin diagram  $\tilde{D}_4$  [48].

One of the objectives of this article is to recast results of [35] to  $(4 + 4)$ -space. We also want to describe split octonionic vectorial and spinorial representations of  $SO(4,4)$  group and to construct split octonionic Dirac and Maxwell Lagrangians underlying triality symmetry in this space.

The paper is organized as follows. In Section 2, we present  $16 \times 16$  complex matrix representation of the Clifford algebra  $\mathcal{C}\ell_{4,4}$ . The Section 3 and Section 4 are devoted to vectorial and spinorial matrix representations of  $SO(4,4)$  group respectively. In Section 5, the equivalence of  $(4 + 4)$ -vectors and chiral spinors (triality) is explicitly demonstrated. In Section 6, it is shown that the complete algebra of hypercomplex octonionic basis units can be recovered from the Moufang and Malcev relations. In Section 7, the trilinear form, and the group  $SO(4,4)$  under which it is invariant, is written in terms of split octonions. In

Section 8, split octonionic Lagrangians that can be built by quadratic and trilinear invariant forms are presented. In Section 9, it is shown that an equation similar to the split octonionic analyticity condition can be reduced to the system of the Dirac-Maxwell equations. Finally, Section 10 presents our conclusions.

## 2. Matrix Representation of $\mathcal{Cl}_{4,4}$

Geometric algebra of  $(4 + 4)$ -space is a Clifford algebra over the real number field with a diagonal metric  $g_{\mu\nu}$  (Greek indices, e.g.  $\mu, \nu$  take on the values  $0, 1, \dots, 7$ ) having  $(4, 4)$  signature and is usually denoted as  $\mathcal{Cl}_{4,4}$ . As all Clifford algebras,  $\mathcal{Cl}_{4,4}$  is associative and can be defined through anti-commutation relations:

$$e_\mu e_\nu + e_\nu e_\mu = 2g_{\mu\nu}, \quad (2.1)$$

where  $e_\mu$  are orthogonal basis units of grade-1 vectors.

Basis unit  $e_\mu$  can be represented as  $\Gamma_\mu$ -matrix. To obtain an exact form of the  $\Gamma_\mu$ -matrices for  $\mathcal{Cl}_{4,4}$ , we can take the  $\mathcal{Cl}_{8,0}$  generating matrices  $A_\mu$  described in [35] and multiply four of them by complex imaginary unit  $i$ ,

$$\begin{aligned} \Gamma_\mu &= A_\mu, \quad (\mu = 0, 1, 2, 3) \\ \Gamma_\nu &= iA_\nu, \quad (\nu = 4, 5, 6, 7) \end{aligned} \quad (2.2)$$

This changes the Euclidean metric into the split metric of  $(4 + 4)$ -space. Here we use labeling and ordering of 16-dimensional Hermitian  $A_\mu$  matrices that differs from the one in [35],

$$A_\mu = \begin{pmatrix} 0 & \alpha_\mu \\ \alpha_\mu^\dagger & 0 \end{pmatrix}, \quad (2.3)$$

where the 8-dimensional  $\alpha_\mu$  matrices are:

$$\begin{aligned} \alpha_0 &= \begin{pmatrix} -1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -1 \end{pmatrix}, & \alpha_1 &= \begin{pmatrix} i & & & & & & & \\ & i & & & & & & \\ & & i & & & & & \\ & & & i & & & & \\ & & & & i & & & \\ & & & & & i & & \\ & & & & & & i & \\ & & & & & & & i \end{pmatrix}, \\ \alpha_2 &= \begin{pmatrix} & 1 & & & & & & \\ 1 & & & & & & & \\ & & & & -1 & & & \\ & & & & & -1 & & \\ & & -1 & & & & & \\ & & & -1 & & & & \\ & & & & & & & 1 \end{pmatrix}, & \alpha_3 &= \begin{pmatrix} & -i & & & & & & \\ i & & & & & & & \\ & & & & i & & & \\ & & & & & i & & \\ & & -i & & & & & \\ & & & -i & & & & \\ & & & & & & & -i \end{pmatrix}, \end{aligned}$$

$$\alpha_4 = \begin{pmatrix} & 1 & & \\ & & 1 & \\ 1 & & & \\ & & & -1 \\ & 1 & & \\ & & & -1 \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \alpha_5 = \begin{pmatrix} & i & & \\ & & i & \\ -i & & & \\ & & & -i \\ & -i & & \\ & & & -i \\ & & i & \\ & & & i \end{pmatrix},$$

$$\alpha_6 = \begin{pmatrix} & 1 & & \\ & & 1 & \\ 1 & & & \\ & & & 1 \\ & 1 & & \\ & & & 1 \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \alpha_7 = \begin{pmatrix} & -i & & \\ & & -i & \\ i & & & -i \\ & & & -i \\ & i & & \\ & & & -i \\ & & i & \\ & & & i \end{pmatrix}.$$

We see that four of the  $\alpha_\mu$  matrices, and thus four corresponding  $A_\mu$  matrices, are imaginary and the rest four are real. For obtaining  $(4 + 4)$ -space algebra from that Euclidean version, we could have chosen any four of the eight generators to be multiplied by the complex imaginary unit  $i$ . Choosing the imaginary  $A_\mu$  matrices would have resulted in a real representation of  $\mathcal{C}\ell_{4,4}$ , which is indeed *algebra isomorphic* to the ring of  $16 \times 16$  real matrices [13]. However, in the complex representation defined above (2.2), some calculations are easier and closer to those provided for Euclidean 8-space in [35].

### 3. Vectors in $(4 + 4)$ -Space

Let us take  $x$  to be a real vector in  $(4 + 4)$ -space whose components are labeled as  $x_\mu$ . Object that *transforms like a vector* is represented by a matrix

$$x = \sum_{\beta=0}^7 x_\beta \Gamma_\beta, \tag{3.1}$$

where  $\Gamma_\beta$  -matrices are defined in (2.2). The vectors of  $(4 + 4)$ -space have the property that

$$x^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2, \tag{3.2}$$

where we assume that the right-hand side is multiplied by the  $(16 \times 16)$  identity matrix.

The similarity transformations

$$x' = L_{\mu\nu}(\mathcal{G}) x L_{\mu\nu}^{-1}(\mathcal{G}), \tag{3.3}$$

where

$$L_{\mu\nu}(\mathcal{G}) = \exp\left(-\frac{1}{2} \mathcal{G} \Gamma_\mu \Gamma_\nu\right), \tag{3.4}$$

result in rotations and boosts of the vector  $x$ . This represents the  $SO(4,4)$  group under which the quadratic form (3.2) is invariant. Transformations of  $x$  under  $L_{\mu\nu}$  can be divided into two types: one comprising 2 copies of  $SO(4)$  in two maximal anisotropic subspaces and another comprising 16 copies of  $SO(1,1)$  that mix these two maximal anisotropic subspaces in isotropic planes. The former type of transformations is compact and they are realized when either  $\mu, \nu = 0, 1, 2, 3$  or  $\mu, \nu = 4, 5, 6, 7$ . The letters are Lorentz-like non-compact boosts, *i.e.* hyperbolic transformations and are realized when  $\mu = 0, 1, 2, 3$  and  $\nu = 4, 5, 6, 7$ , or *vice versa*.

To demonstrate these two different types of  $SO(4,4)$ -transformations, it is sufficient to study them in the tangential space. The space is spanned by Taylor expansion of the transformation matrix (3.4) in the neighborhood of the identity element up to the first order term,

$$L_{\mu\nu}(\mathcal{G}) \approx 1 - \frac{1}{2} \mathcal{G} \Gamma_{\mu} \Gamma_{\nu}. \quad (3.5)$$

Using the fact that

$$L_{\mu\nu}^{-1} = L_{\nu\mu}, \quad (3.6)$$

the Formula (3.3) in the tangential space reduces to

$$x' = \sum_{\alpha} x'_{\alpha} \Gamma_{\alpha} = \sum_{\beta} \left[ x_{\beta} \Gamma_{\beta} - \frac{1}{2} \mathcal{G} x_{\beta} (\Gamma_{\mu} \Gamma_{\nu} \Gamma_{\beta} + \Gamma_{\beta} \Gamma_{\nu} \Gamma_{\mu}) \right]. \quad (3.7)$$

As an example let us consider rotations in  $e_4 \wedge e_5$  plane. For  $\beta \neq 4, 5$  the second term in (3.7) vanishes due to the defining algebraic relation (2.1), so we can write

$$x'_{\beta} = x_{\beta}. \quad (3.8)$$

When  $\beta = 5$  the second term in (3.7) turns into  $\mathcal{G} x_5 \Gamma_4$ , which dictates that

$$x'_4 = x_4 + \mathcal{G} x_5. \quad (3.9)$$

Similarly, for  $\beta = 4$  we get

$$x'_5 = x_5 - \mathcal{G} x_4. \quad (3.10)$$

Since we have opposite sign in front of  $\mathcal{G}$  in these two infinitesimal coordinate transformations, corresponding finite transformations would result in compact rotations:

$$\begin{aligned} x'_4 &= x_4 \cos \mathcal{G} + x_5 \sin \mathcal{G}, \\ x'_5 &= x_5 \cos \mathcal{G} - x_4 \sin \mathcal{G}, \\ x'_{\rho} &= x_{\rho}, \quad (\rho \neq 4, 5) \end{aligned} \quad (3.11)$$

We have similar compact rotations in all anisotropic planes.

Alternatively, the transformations that mix maximal anisotropic subspaces are non-compact. For example, if we apply calculations similar to the previous case to  $\mu = 0$  and  $\nu = 4$ , we would get non-compact rotations of the form:

$$\begin{aligned} x'_0 &= x_0 \cosh \mathcal{G} + x_4 \sinh \mathcal{G}, \\ x'_4 &= x_4 \cosh \mathcal{G} + x_0 \sinh \mathcal{G}, \\ x'_\rho &= x_\rho, \quad (\rho \neq 0, 4) \end{aligned} \tag{3.12}$$

At the end of this section we want to introduce one of the 1680 possible grade-4 elements of  $\mathcal{C}\ell_{4,4}$ ,

$$B = -\Gamma_1 \Gamma_3 \Gamma_5 \Gamma_7, \tag{3.13}$$

which due to the property

$$\Gamma_\mu^T = B \Gamma_\mu B, \quad (\mu = 0, 1, \dots, 7) \tag{3.14}$$

will become useful below.

### 4. Spinors in (4 + 4)-Space

A spinor in the (4 + 4)-space can be represented as a 16-dimensional column vector

$$\eta = \phi + \psi, \tag{4.1}$$

where

$$\phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_7 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } \psi = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \psi_0 \\ \psi_1 \\ \vdots \\ \psi_7 \end{pmatrix} \tag{4.2}$$

are spinors of different chirality. We note that  $\phi$  and  $\psi$  have 8 independent real components each.

The spinor transformations under  $Spin(4,4)$  (double cover of  $SO(4,4)$ ) are described by the same matrix (3.4) that was used for vectors, but the transformation law is different

$$\eta' = L_{\mu\nu}(\mathcal{G})\eta. \tag{4.3}$$

Under this transformation the quantity

$$\eta^T B \eta = \phi^T B \phi + \psi^T B \psi \tag{4.4}$$

is invariant. We prove this in the tangential space using the property of  $B$  matrix (3.14),

$$\begin{aligned} \eta'^T B \eta' &= \eta^T \left( 1 + \frac{1}{2} \mathcal{G} \Gamma_\nu^T \Gamma_\mu^T \right) B \left( 1 + \frac{1}{2} \mathcal{G} \Gamma_\mu \Gamma_\nu \right) \eta \\ &= \eta^T B \left( 1 - \frac{1}{2} \mathcal{G} \Gamma_\mu \Gamma_\mu \right) \left( 1 + \frac{1}{2} \mathcal{G} \Gamma_\mu \Gamma_\nu \right) \eta = \eta^T B \eta. \end{aligned} \tag{4.5}$$

It can be noticed that two terms on the right hand side of the relation (4.4) are conserved independently, meaning that their terms do not mix.

## 5. Triality

The vector  $x$  considered in the Section 3 and two kind of spinors  $\psi$  and  $\phi$  considered in the Section 4 are objects of same dimension in the underlying field  $\mathbb{R}$ . This kind of match between the dimensions of vector and chiral spinors only takes place in 8-dimensional space.

In order to extract another peculiarity of  $(4 + 4)$ -space, which relies on the previous one, let us apply the following linear basis change to the spinor (4.1):

$$\xi = \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi_2 + i\phi_3 \\ \phi_0 - i\phi_1 \\ -\phi_7 - i\phi_6 \\ -\phi_5 + i\phi_4 \\ -\phi_5 - i\phi_4 \\ \phi_7 - i\phi_6 \\ -\phi_0 - i\phi_1 \\ -\phi_2 - i\phi_3 \\ \psi_2 - i\psi_3 \\ -\psi_0 - i\psi_1 \\ -\psi_7 - i\psi_6 \\ -\psi_5 + i\psi_4 \\ \psi_5 + i\psi_4 \\ -\psi_7 + i\psi_6 \\ -\psi_0 + i\psi_1 \\ -\psi_2 - i\psi_3 \end{pmatrix}. \quad (5.1)$$

In this basis, the invariant quadratic form (4.4) for 8-spinors  $\phi$  and  $\psi$  yields

$$\begin{aligned} \phi^T B \phi &= \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 - \phi_4^2 - \phi_5^2 - \phi_6^2 - \phi_7^2, \\ \psi^T B \psi &= \psi_0^2 + \psi_1^2 + \psi_2^2 + \psi_3^2 - \psi_4^2 - \psi_5^2 - \psi_6^2 - \psi_7^2, \end{aligned} \quad (5.2)$$

which are analogous to the invariant quadratic form for the vector (3.2). Then one can construct a trilinear form

$$\mathcal{F} : \mathbb{R}^8 \times \mathbb{R}^8 \times \mathbb{R}^8 \rightarrow \mathbb{R} \quad (5.3)$$

on  $x$ ,  $\phi$  and  $\psi$ ,

$$\mathcal{F}(\phi, x, \psi) = \phi^T B x \psi, \quad (5.4)$$

which is preserved under simultaneously transforming  $x$  and  $\eta = \phi + \psi$  under the vector (3.3) and spinor (4.3) transformation rules with the same  $L_{\mu\nu}$ . Proof is provided in the tangential space:

$$\begin{aligned} \phi'^T B x' \psi' &= \phi^T L_{\mu\nu}^T B L_{\mu\nu} x L_{\nu\mu} L_{\mu\nu} \psi \\ &= \phi^T \left( 1 + \frac{1}{2} \mathcal{G} \Gamma_\nu^T \Gamma_\mu^T \right) B \left( 1 + \frac{1}{2} \mathcal{G} \Gamma_\mu \Gamma_\nu \right) x \psi = \phi^T B x \psi. \end{aligned} \quad (5.5)$$

Let us look closely at these transformations. For example, the infinitesimal  $L_{01}(\mathcal{G})$  rotations of vector and spinors are:

$$\begin{cases} x'_0 = x_0 - \mathcal{G}x_1 \\ x'_1 = x_1 + \mathcal{G}x_0 \\ x'_2 = x_2 \\ x'_3 = x_3 \\ x'_4 = x_4 \\ x'_5 = x_5 \\ x'_6 = x_6 \\ x'_7 = x_7 \end{cases}, \begin{cases} \phi'_0 = \phi_0 + \frac{1}{2}\mathcal{G}\phi_1 \\ \phi'_1 = \phi_1 - \frac{1}{2}\mathcal{G}\phi_0 \\ \phi'_2 = \phi_2 - \frac{1}{2}\mathcal{G}\phi_3 \\ \phi'_3 = \phi_3 + \frac{1}{2}\mathcal{G}\phi_2 \\ \phi'_4 = \phi_4 - \frac{1}{2}\mathcal{G}\phi_5 \\ \phi'_5 = \phi_5 + \frac{1}{2}\mathcal{G}\phi_4 \\ \phi'_6 = \phi_6 + \frac{1}{2}\mathcal{G}\phi_7 \\ \phi'_7 = \phi_7 - \frac{1}{2}\mathcal{G}\phi_6 \end{cases}, \begin{cases} \psi'_0 = \psi_0 + \frac{1}{2}\mathcal{G}\psi_1 \\ \psi'_1 = \psi_1 - \frac{1}{2}\mathcal{G}\psi_0 \\ \psi'_2 = \psi_2 + \frac{1}{2}\mathcal{G}\psi_3 \\ \psi'_3 = \psi_3 - \frac{1}{2}\mathcal{G}\psi_2 \\ \psi'_4 = \psi_4 + \frac{1}{2}\mathcal{G}\psi_5 \\ \psi'_5 = \psi_5 - \frac{1}{2}\mathcal{G}\psi_4 \\ \psi'_6 = \psi_6 - \frac{1}{2}\mathcal{G}\psi_7 \\ \psi'_7 = \psi_7 + \frac{1}{2}\mathcal{G}\psi_6 \end{cases}. \quad (5.6)$$

As usual one full rotation for a vector  $x$  is only half a rotation for spinors  $\phi$  and  $\psi$ . However, since  $L_{\mu\nu}$  matrices form a group under matrix multiplication, we can construct transformations for  $x$  that exactly reproduce transformations (5.6) of  $\phi$ ,

$$L_{10}\left(\frac{\mathcal{G}}{2}\right)L_{23}\left(\frac{\mathcal{G}}{2}\right)L_{54}\left(\frac{\mathcal{G}}{2}\right)L_{67}\left(\frac{\mathcal{G}}{2}\right) \simeq 1 - \frac{1}{4}\mathcal{G}(\Gamma_1\Gamma_0 + \Gamma_2\Gamma_3 + \Gamma_5\Gamma_4 + \Gamma_6\Gamma_7), \quad (5.7)$$

which results in

$$\begin{cases} x'_0 = x_0 + \frac{1}{2}\mathcal{G}x_1 \\ x'_1 = x_1 - \frac{1}{2}\mathcal{G}x_0 \\ x'_2 = x_2 - \frac{1}{2}\mathcal{G}x_3 \\ x'_3 = x_3 + \frac{1}{2}\mathcal{G}x_2 \\ x'_4 = x_4 - \frac{1}{2}\mathcal{G}x_5 \\ x'_5 = x_5 + \frac{1}{2}\mathcal{G}x_4 \\ x'_6 = x_6 + \frac{1}{2}\mathcal{G}x_7 \\ x'_7 = x_7 - \frac{1}{2}\mathcal{G}x_6 \end{cases}, \begin{cases} \phi'_0 = \phi_0 + \frac{1}{2}\mathcal{G}\phi_1 \\ \phi'_1 = \phi_1 - \frac{1}{2}\mathcal{G}\phi_0 \\ \phi'_2 = \phi_2 + \frac{1}{2}\mathcal{G}\phi_3 \\ \phi'_3 = \phi_3 - \frac{1}{2}\mathcal{G}\phi_2 \\ \phi'_4 = \phi_4 + \frac{1}{2}\mathcal{G}\phi_5 \\ \phi'_5 = \phi_5 - \frac{1}{2}\mathcal{G}\phi_4 \\ \phi'_6 = \phi_6 - \frac{1}{2}\mathcal{G}\phi_7 \\ \phi'_7 = \phi_7 + \frac{1}{2}\mathcal{G}\phi_6 \end{cases}, \begin{cases} \psi'_0 = \psi_0 - \mathcal{G}\psi_1 \\ \psi'_1 = \psi_1 + \mathcal{G}\psi_0 \\ \psi'_2 = \psi_2 \\ \psi'_3 = \psi_3 \\ \psi'_4 = \psi_4 \\ \psi'_5 = \psi_5 \\ \psi'_6 = \psi_6 \\ \psi'_7 = \psi_7 \end{cases}. \quad (5.8)$$

What's peculiar here is the ways in which vector  $x$  and spinors  $\phi$  and  $\psi$  transform have interchanged between the three, namely  $x$  and  $\phi$  appear to behave like a spinors now and  $\psi$  looks like a vector, since full rotation in  $\psi$  gives half a rotation in  $x$  and  $\phi$ . This is the property of the eight dimensional space, which was named as *trinality*, similar to the *duality* for dual vector spaces.

For the completeness let us also write out boost-like non-compact transformations, which are only realized in anisotropic spaces, for example the trans-



formations generated by  $L_{04}(\mathcal{G})$ ,

$$\begin{cases} x'_0 = x_0 + \mathcal{G}x_4 \\ x'_1 = x_1 \\ x'_2 = x_2 \\ x'_3 = x_3 \\ x'_4 = x_4 + \mathcal{G}x_0 \\ x'_5 = x_5 \\ x'_6 = x_6 \\ x'_7 = x_7 \end{cases}, \begin{cases} \phi'_0 = \phi_0 - \frac{1}{2}\mathcal{G}\phi_4 \\ \phi'_1 = \phi_1 - \frac{1}{2}\phi_5\mathcal{G} \\ \phi'_2 = \phi_2 - \frac{1}{2}\mathcal{G}\phi_6 \\ \phi'_3 = \phi_3 - \frac{1}{2}\mathcal{G}\phi_7 \\ \phi'_4 = \phi_4 - \frac{1}{2}\phi_0\mathcal{G} \\ \phi'_5 = \phi_5 - \frac{1}{2}\mathcal{G}\phi_1 \\ \phi'_6 = \phi_6 - \frac{1}{2}\phi_2\mathcal{G} \\ \phi'_7 = \phi_7 - \frac{1}{2}\mathcal{G}\phi_3 \end{cases}, \begin{cases} \psi'_0 = \psi_0 - \frac{1}{2}\mathcal{G}\psi_4 \\ \psi'_1 = \psi_1 + \frac{1}{2}\mathcal{G}\psi_5 \\ \psi'_2 = \psi_2 + \frac{1}{2}\mathcal{G}\psi_6 \\ \psi'_3 = \psi_3 + \frac{1}{2}\mathcal{G}\psi_7 \\ \psi'_4 = \psi_4 - \frac{1}{2}\mathcal{G}\psi_0 \\ \psi'_5 = \psi_5 + \frac{1}{2}\mathcal{G}\psi_1 \\ \psi'_6 = \psi_6 + \frac{1}{2}\mathcal{G}\psi_2 \\ \psi'_7 = \psi_7 + \frac{1}{2}\mathcal{G}\psi_3 \end{cases}. \tag{5.9}$$

We see that, similar to the compact case, the hyperbolic transformation of one of the three objects (vector and two chiral spinors) in the isotropic plane  $e_0 \wedge e_4$  generates spinorial transformations of other two objects in corresponding four isotropic planes,  $e_0 \wedge e_4$ ,  $e_1 \wedge e_5$ ,  $e_2 \wedge e_6$  and  $e_3 \wedge e_7$ . Again, it is possible to replicate transformations of  $x$  in one of the spinors which would trially swap their behavior.

### 6. Split Octonions

It is known that spinors and vectors of  $(4 + 4)$ -space, considered in Section 3 and Section 4, can also be represented using split octonions instead of matrices [44] [45] [46] [47]. Split octonions  $\mathbb{O}'$  form non-associative algebra with the property of alternativity. The algebra can be defined through the algebraic relations:

$$\begin{aligned} I^2 = 1, \quad j_n I = J_n, \quad j_m j_n = -\delta_{mn} + \sum_{\ell} \varepsilon_{\ell mn} j_{\ell}, \quad (\ell, m, n, = 1, 2, 3) \\ J_m J_n = \delta_{mn} + \sum_{\ell} \varepsilon_{\ell mn} j_{\ell}, \quad J_m j_n = \delta_{mn} I - \sum_{\ell} \varepsilon_{\ell mn} J_{\ell}. \end{aligned} \tag{6.1}$$

From the above relations and the alternativity property one can extract the entire multiplication table for basis units

	1	$j_1$	$j_2$	$j_3$	$I$	$J_1$	$J_2$	$J_3$
1	1	$j_1$	$j_2$	$j_3$	$I$	$J_1$	$J_2$	$J_3$
$j_1$	$j_1$	-1	$j_3$	$-j_2$	$J_1$	$-I$	$-J_3$	$J_2$
$j_2$	$j_2$	$-j_3$	-1	$j_1$	$J_2$	$J_3$	$-I$	$-J_1$
$j_3$	$j_3$	$j_2$	$-j_1$	-1	$J_3$	$-J_2$	$J_1$	$-I$
$I$	$I$	$-J_1$	$-J_2$	$-J_3$	1	$-j_1$	$-j_2$	$-j_3$
$J_1$	$J_1$	$I$	$-J_3$	$J_2$	$j_1$	1	$j_3$	$-j_2$
$J_2$	$J_2$	$J_3$	$I$	$-J_1$	$j_2$	$-j_3$	1	$j_1$
$J_3$	$J_3$	$-J_2$	$J_1$	$I$	$j_3$	$j_2$	$-j_1$	1

(6.2)

Now we want to show that complete algebra of the seven hypercomplex basis

units of the split octonions follows from the Moufang and Malcev relations written for only three vector-like split octonionic elements  $J_n$ . It is known that the anti-commuting basis units of octonions and split octonions,  $xy = -yx$ , are Moufang loops [49]. The algebra formed by them is not associative but instead is alternative, *i.e.* the associator

$$\mathcal{A}(x, y, z) = \frac{1}{2}((xy)z - x(yz)) \tag{6.3}$$

is totally antisymmetric

$$\mathcal{A}(x, y, z) = -\mathcal{A}(y, x, z) = -\mathcal{A}(x, z, y). \tag{6.4}$$

Consequently, any two units  $x$  and  $y$  generate an associative subalgebra and obey the following mild associative laws:

$$(xy)y = xy^2, \quad x(xy) = x^2y, \quad (xy)x = x(yx). \tag{6.5}$$

The octonionic basis units also satisfy the flexible Moufang identities:

$$(xy)(zx) = x(yz)x, \quad (zyz)x = z(y(zx)), \quad x(yzy) = ((xy)z)y. \tag{6.6}$$

In the algebra we have the following relationship between the associator,

$$\mathcal{A}(x, y, z) = \frac{1}{3}([\![x, [y, z]]\!] + [\![y, [z, x]]\!] + [\![z, [x, y]]\!]), \tag{6.7}$$

and the commutator

$$[x, y] = \frac{1}{2}(xy - yx). \tag{6.8}$$

Since the hypercomplex octonionic basis units anti-commute, their commutator can always be replaced by the simple product,  $[x, y] = xy$ .

It is also known that basis units of octonions and split octonions form the Malcev algebra (see, for example [50] [51]). Due to non-associativity, commutator algebra of split octonionic units is non-Lie and instead of satisfying the Jacobi identity, they satisfy the Malcev relation:

$$(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y, \tag{6.9}$$

or equivalently

$$\mathcal{J}(x, y, (xz)) = \mathcal{J}(x, y, z)x, \tag{6.10}$$

where

$$\mathcal{J}(x, y, z) = \frac{1}{3}((xy)z + (yz)x + (zx)y) \tag{6.11}$$

is so-called Jacobiator of  $x$ ,  $y$  and  $z$ . Indeed, using anti-commutativity of elements, we find:

$$\begin{aligned} 3\mathcal{J}(x, y, (xz)) &= (xy)(xz) + (y(xz))x + ((xz)x)y \\ &= ((xy)z)x + ((yz)x)x + ((zx)x)y + (y(xz))x + ((xz)x)y \tag{6.12} \\ &= ((xy)z + (yz)x + (zx)y)x = 3\mathcal{J}(x, y, z)x. \end{aligned}$$

In Malcev's algebra two types of products are defined: bilinear  $xy = -yx$  and

trilinear  $\mathcal{J}(x, y, z)$ , which can be expressed using bilinear products as:

$$\mathcal{J}(x, y, z) = \frac{1}{3}(x(yz) + y(zx) + z(xy)) = -\mathcal{J}(y, x, z) = \mathcal{J}(x, z, y). \tag{6.13}$$

We also have identities containing four and five elements of the algebra:

$$\begin{aligned} \mathcal{J}(xy, z, w) + \mathcal{J}(yz, x, w) + \mathcal{J}(zx, y, w) &= 0, \\ \mathcal{J}(x, y, zw) &= \mathcal{J}(x, y, z)w + z\mathcal{J}(x, y, w), \\ \mathcal{J}(x, y, \mathcal{J}(z, u, v)) &= \mathcal{J}(\mathcal{J}(x, y, z), u, v) + \mathcal{J}(z, \mathcal{J}(x, y, u), v) \\ &\quad + \mathcal{J}(z, u, \mathcal{J}(x, y, v)). \end{aligned} \tag{6.14}$$

So, one can generate a complete basis of the split octonions by the multiplication and distribution laws of only three vector-like elements  $J_n$ . Indeed, we can define pseudo-vector like basis units of the split octonions  $j_n$  by the commutators (or simple binary products) of  $J_n$ ,

$$j_n = \frac{1}{2} \sum_m \sum_k \varepsilon_{nmk} J_m J_k, \tag{6.15}$$

where  $\varepsilon_{nmk}$  is the totally antisymmetric unit tensor. Also using Moufang identities for  $J_1, J_2$  and  $J_3$  we can identify the seventh basis unit  $I$  with the Jacobiator,

$$J_1 j_1 = J_2 j_2 = J_3 j_3 = -\mathcal{J}(J_1, J_2, J_3) = I. \tag{6.16}$$

As a result, from the Moufang and Malcev relations we can recover the complete algebra of all seven hypercomplex split octonionic basis units (6.1). The non-vanishing associators of these basis units are:

$$\begin{aligned} \mathcal{A}(J_n, J_m, J_k) &= \delta_{nm} J_k - \delta_{nk} J_m, \quad \mathcal{A}(j_n, j_m, J_k) = -\varepsilon_{nmk} I - \delta_{nk} J_m + \delta_{mk} J_n, \\ \mathcal{A}(j_n, j_m, I) &= \sum_k \varepsilon_{nmk} J_k, \quad \mathcal{A}(j_n, J_m, I) = -\sum_k \varepsilon_{nmk} j_k, \\ \mathcal{A}(J_n, J_m, J_k) &= -\varepsilon_{nmk} I, \quad \mathcal{A}(J_n, J_m, I) = \sum_k \varepsilon_{nmk} J_k. \end{aligned} \tag{6.17}$$

General split octonion  $x \in \mathbb{O}'$  over the field of real numbers and its conjugate are

$$\begin{aligned} x &= x_0 + Ix_4 + \sum_n (j_n x_n + J_n x_{4+n}), \\ \bar{x} &= x_0 - Ix_4 - \sum_n (j_n x_n + J_n x_{4+n}), \end{aligned} \tag{6.18}$$

where  $n = 1, 2, 3$  and  $x_0, x_1, \dots, x_7 \in \mathbb{R}$ . Quadratic form  $\mathcal{Q}: \mathbb{O}' \rightarrow \mathbb{R}$  is defined as multiplication of  $x \in \mathbb{O}'$  with its conjugate

$$\mathcal{Q}(x) = \bar{x}x. \tag{6.19}$$

The quadratic form cannot be used to construct a norm since it's not positive definite and also evaluates to zero for nonzero split octonions. Symmetric and non-degenerate bilinear form  $\langle \cdot, \cdot \rangle: \mathbb{O}' \times \mathbb{O}' \rightarrow \mathbb{R}$  is defined in terms of the quadratic form as

$$\langle x, y \rangle = \frac{1}{2} \mathcal{Q}(x + y) - \frac{1}{2} \mathcal{Q}(x) - \frac{1}{2} \mathcal{Q}(y). \tag{6.20}$$

Explicitly it is

$$\langle x, y \rangle = \frac{1}{2}(\bar{x}y + y\bar{x}) = \sum_{n=0}^3 (x_n y_n - x_{4+n} y_{4+n}). \tag{6.21}$$

At the end of this section we define split octonionic gradients:

$$\begin{aligned} \partial &= \frac{1}{2}(\partial_0 + I\partial_4) + \frac{1}{2}\sum_{n=0}^3 (j_n \partial_n + J_n \partial_{4+n}), \\ \bar{\partial} &= \frac{1}{2}(\partial_0 - I\partial_4) - \frac{1}{2}\sum_{n=0}^3 (j_n \partial_n + J_n \partial_{4+n}), \end{aligned} \tag{6.22}$$

where  $\partial_n$  is a partial differentiation operator with respect to  $x_n$ . They are defined in such a way that they mimic properties of regular derivative for  $\mathbb{R} \rightarrow \mathbb{R}$  functions and Wirtinger derivatives for  $\mathbb{C} \rightarrow \mathbb{C}$  functions, namely

$$\begin{aligned} \partial x &= \bar{\partial} \bar{x} = 1, \\ \bar{\partial} x &= \partial \bar{x} = 0. \end{aligned} \tag{6.23}$$

But these properties do not extend to higher order terms in  $x$  and  $\bar{x}$  [52], since they already fail for quaternionic derivatives [53].

### 7. Split Octonions and Triality

Now let us express the triality of  $(4 + 4)$ -space in terms of split octonions. We can write split octonionic representation of the  $(4 + 4)$ -space vector and chiral spinors, (3.1) and (4.2), as

$$\begin{aligned} \phi &= \phi_0 + \phi_1 j_1 + \phi_2 j_2 + \phi_3 j_3 + \phi_4 I + \phi_5 J_1 + \phi_6 J_2 + \phi_7 J_3, \\ x &= x_0 + x_1 j_1 + x_2 j_2 + x_3 j_3 + x_4 I + x_5 J_1 + x_6 J_2 + x_7 J_3, \\ \psi &= \psi_0 + \psi_1 j_1 + \psi_2 j_2 + \psi_3 j_3 + \psi_4 I + \psi_5 J_1 + \psi_6 J_2 + \psi_7 J_3. \end{aligned} \tag{7.1}$$

Note that unlike the Clifford algebraic representation of spinors and vectors (considered in Section 3 and Section 4), where they are represented by different type of objects, here they are a same type of object. Furthermore, the invariants constructed by the split octonionic vector and spinors (7.1), can also be written identically to each other, namely

$$\mathcal{Q}(\phi) = \bar{\phi}\phi, \quad \mathcal{Q}(x) = \bar{x}x, \quad \mathcal{Q}(\psi) = \bar{\psi}\psi. \tag{7.2}$$

These expressions respect the fact that they evaluate to same quadratic forms (3.2) and (5.2) and are interchangeable in the trilinear form (5.4) as we have seen above.

Trilinear form (5.4) represented with split octonions can be written using bilinear form (6.21) as [54],

$$\mathcal{F}(\phi, x, \psi) = \langle \bar{\phi}, x\psi \rangle. \tag{7.3}$$

Split octonionic vectorial and spinorial representations of  $SO(4,4)$  can be constructed similarly to octonionic representations of  $SO(8)$  as described in [36]. Let us denote two distinct basis units of split octonions by  $u$  and  $v$  and define an object:

$$T_{uv} = \begin{cases} u \cos\left(\frac{\mathcal{G}}{2}\right) + v \sin\left(\frac{\mathcal{G}}{2}\right), & u\bar{u} = v\bar{v} \\ u \cosh\left(\frac{\mathcal{G}}{2}\right) + v \sinh\left(\frac{\mathcal{G}}{2}\right), & u\bar{u} = -v\bar{v} \end{cases} \quad (7.4)$$

Then any group transformation can be constructed by compositions of transformations

$$x' = T_{uv}(uxu)T_{uv} \quad (7.5)$$

for  $SO(4,4)$  and

$$\phi' = (\phi u)\bar{T}_{uv}, \quad (7.6)$$

$$\psi' = \bar{T}_{uv}(u\psi), \quad (7.7)$$

for  $Spin(4,4)$ . For example, the transformation  $L_{01}(\mathcal{G})$ , whose infinitesimal version is written out in (5.6), is achieved by taking  $u=1$  and  $v=j_1$  in (7.5), (7.6) and (7.7), since 1 and  $j_1$  are the first two units of split octonions. Ordering of the rest of the basis units is same as in the multiplication table (6.2).

### 8. Split Octonionic Field Theories

Using the expression of trilinear form (7.3) we can construct a Lagrangian by replacing  $x \in \mathbb{O}'$  with the derivative defined in (6.22) that act to the right,

$$\mathcal{L} = \langle \bar{\phi}, \bar{\partial}\psi \rangle. \quad (8.1)$$

By stationarizing the action integral

$$S = \int d^8x \mathcal{L}, \quad (8.2)$$

we get right-analyticity and left-analyticity conditions on  $\phi$  and  $\psi$

$$\begin{cases} \phi\bar{\partial} = 0, \\ \bar{\partial}\psi = 0, \end{cases} \quad (8.3)$$

These equations represent the generalized Cauchy-Riemann and Cauchy-Riemann-Fueter [55] conditions for split octonions. Split octonionic conjugation of the first equation in (8.3) turns the system into

$$\begin{cases} \bar{\bar{\partial}}\bar{\phi} = 0, \\ \bar{\partial}\psi = 0, \end{cases} \quad (8.4)$$

Adding quadratic terms in  $\phi$  and  $\psi$  to the Lagrangian

$$\mathcal{L} = \langle \bar{\phi}, \bar{\partial}\psi \rangle + \frac{1}{2}\lambda_1 \langle \phi, \phi \rangle + \frac{1}{2}\lambda_2 \langle \psi, \psi \rangle \quad (8.5)$$

results in mixing of the these split octonionic fields at the equation level,

$$\begin{cases} \bar{\bar{\partial}}\bar{\phi} = \lambda_2\psi, \\ \bar{\partial}\psi = -\lambda_1\bar{\phi}. \end{cases} \quad (8.6)$$

If we take  $\lambda_2 = 0$  and use the property of alternativity, then equations reduce to eight independent Klein-Gordon like equations for  $(4 + 4)$ -space

$$\langle \bar{\partial}, \bar{\partial} \rangle \psi = 0. \tag{8.7}$$

### 9. Dirac and Maxwell Equations

We define a new gradient operator  $D$  in terms of  $\partial$  as

$$D = I\partial I, \tag{9.1}$$

which has the opposite sign for imaginary parts  $j_n$  and  $J_n$

$$D = \frac{1}{2}(\partial_0 + I\partial_4) - \frac{1}{2}\sum_n (j_n\partial_n + J_n\partial_{4+n}). \tag{9.2}$$

We also consider fields (the case of split quaternions see in [56] [57]):

$$A = C_0 + j_1A_1 + j_2A_2 + j_3A_3 + IA_0 + J_1C_1 + J_2C_2 + J_3C_3, \tag{9.3}$$

$$F = \bar{D}A. \tag{9.4}$$

If we take the quadratic Lagrangian defined above (8.5) with parameters  $\lambda_2 = 0$  and  $\lambda_1 = -1$ , set  $\phi = \bar{F}$  and  $\psi = A$  and use  $D$  instead of  $\partial$  we get

$$\mathcal{L} = \langle F, \bar{D}A \rangle - \frac{1}{2}\langle \bar{F}, \bar{F} \rangle. \tag{9.5}$$

Using the definition of  $F$  in terms of  $A$  (9.4) and the fact that  $\langle \bar{F}, \bar{F} \rangle = \langle F, F \rangle$ , the Lagrangian simplifies to

$$\mathcal{L} = \frac{1}{4}\langle F, F \rangle. \tag{9.6}$$

Equation of motion for this Lagrangian is

$$\langle \bar{D}, \bar{D} \rangle A = 0. \tag{9.7}$$

In the limit when  $D \rightarrow \mathcal{D} = \frac{1}{2}(-j_1\partial_x - j_2\partial_y - j_3\partial_z + I\partial_t)$  the equation (9.7) reduces to free dyonic Maxwell equations in Minkowski space

$$\langle \bar{\mathcal{D}}, \bar{\mathcal{D}} \rangle A = 0, \tag{9.8}$$

where  $A_n$  and  $C_n$  for  $n = 0, 1, 2, 3$  are electromagnetic and dyonic 4-potentials, split octonionic form of which was first introduced in [58] [59].

For the following Lagrangian with mass parameter  $m$

$$\mathcal{L} = \langle \bar{\phi}, \bar{D}\psi \rangle - \frac{1}{2}m\langle \bar{\phi}, J_3\psi \rangle = \left\langle \bar{\phi}, \left( \bar{D} - \frac{1}{2}mJ_3 \right) \psi \right\rangle, \tag{9.9}$$

two independent equations of motion result after stationarizing the corresponding action

$$\begin{cases} \left( \bar{D} - \frac{1}{2}J_3m \right) \bar{\phi} = 0, \\ \left( \bar{D} - \frac{1}{2}J_3m \right) \psi = 0, \end{cases} \tag{9.10}$$

second of which reduces to Dirac equation in the limit  $D \rightarrow \mathcal{D}$ .

Lagrangian for a single field  $\psi$  obtained by setting  $\phi = \bar{\psi}J_3$  and taking the limit  $D \rightarrow \mathcal{D}$

$$\mathcal{L} = \left\langle -J_3\psi, \left( \vec{\mathcal{E}} - \frac{1}{2}J_3m \right)\psi \right\rangle \quad (9.11)$$

is equivalent to Dirac Lagrangian and consequently resulting equation of motion

$$\vec{\mathcal{E}}\psi = \frac{1}{2}J_3m\psi, \quad (9.12)$$

is equivalent to the regular Dirac equation.

## 10. Summary and Concluding Remarks

In this paper, the equivalence of 8-dimensional spinors and vectors is discussed for  $(4 + 4)$ -space within the context of the algebra of the split octonions. It is shown that the complete algebra of hypercomplex split octonionic basis units can be recovered from the Moufang and Malcev relations for the three vector-like elements. The trilinear form, together with  $SO(4,4)$  and  $Spin(4,4)$  group transformations, under which it is invariant, is represented using split octonions. It is shown that, unlike matrix cases, this representation respects the triality symmetry. Subsequently Lagrangians on split octonionic spinorial and vectorial fields were constructed using the group invariant quadratic and trilinear forms. Split octonionic analyticity conditions were obtained by stationarizing action corresponding to the simplest trilinear Lagrangian. It is shown that similar Lagrangians correspond to the system that reduces to the free Dirac and dyonic Maxwell equations when extra four spacetime dimensions are removed. It is worth noting that the trilinear relation is exactly of the form used in supersymmetric theories (see, for example [33] [34]), so it is only natural that the overall symmetry of such models is given by triality algebras.

## Authors' Contributions

Both M. G. and A. G. contributed to the final version of the manuscript. M. G. supervised the project.

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## Availability of Data and Materials

The source code for symbolic computations used during the current study along with calculations in Jupyter notebooks are publicly available at:

<https://github.com/EQUINOX24/SplitOct>.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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