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# A Family of Global Attractors for the Generalized Kirchhoff-Beam Equations

**Guoguang Lin, Boshi Chen**

Department of Mathematics, Yunnan University, Kunming, China
Email: gglin@ynu.edu.cn

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## Abstract

In this paper, we discuss the existence and uniqueness of global solutions, the existence of the family of global attractors and its dimension estimation for generalized Beam-Kirchhoff equation under initial conditions and boundary conditions, using the previous research results for reference. Firstly, the existence of bounded absorption set is proved by using a prior estimation, then the existence and uniqueness of the global solution of the problem is proved by using the classical Galerkin's method. Finally, Housdorff dimension and fractal dimension of the family of global attractors are estimated by linear variational method and generalized Sobolev-Lieb-Thirring inequality.

## Keywords

Beam-Kirchhoff Equation, Galerkin's Method, Family of Global Attractors, Housdorff Dimension, Fractal Dimension

## 1. Introduction

In this paper, we study the initial boundary value problem of the following generalized Beam-Kirchhoff equation:

$$u_{tt}+\beta(\Delta)^{2m}u_t+\alpha(\Delta)^{2m}u+\left(\gamma M\left(\left\|D^m u\right\|_p^p\right)+\delta N\left(\left\|u_t\right\|^2\right)\right)(\Delta)^{2m}u=f(x), \quad (1)$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,2,\cdots,2m-1,x\in\partial\Omega,t>0, \quad (2)$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega\subset R^n, \quad (3)$$

$$\alpha>0,\beta>0,\gamma>0,\delta>0, \quad (4)$$

$m>1$ is a positive integer, $\Omega$ is a bounded region in $R^n$ with a smooth boundary, $\partial\Omega$ denoted by the boundary, $f(x)$ is the external force term.

$\beta(\Delta)^{2m}u_t$ is the strongly damped term, $\alpha\Delta^{2m}u$ is the beam term. $\gamma M\left(\|D^m u\|_p^p\right)(\Delta)^{2m}u$ is the *kichhoff* term, $\delta N\left(\|u_t\|^2\right)(\Delta)^{2m}u$ is the nonlinear source term.

The Kirchhoff type equation was first proposed by Kirchhoff as an existence of the nonlinear wave equation for free vibration of elastic strings. The equation has great application in many fields, such as non-Newtonian mechanics, cosmology and astrophysics, plasma problems and elasticity theory, so the study of this kind of equations has a profound practical significance.

In addition, attractor is the key subject of infinite dimensional dynamic system research.

The long-term dynamic characteristics of a system are always dominated by its own attractors, and the shape of the attractors can directly determine the type of dynamic characteristics. Therefore, the attractors are an important index to describe the progressive behavior of the dynamic system at $t\to\infty$. Global attractor is the main research object of autonomous system. In the past 20 or 30 years, autonomous system has been widely studied, and the results have been very mature both in basic theory and practical application [1] [2] [3].

In 1883, Kirchhoff [4] first proposed the wave equation that changes with time

$$\rho h\frac{\partial^2 u}{\partial t^2}-\left\{P_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial t}\right)^2\mathrm{d}x\right\}\frac{\partial^2 u}{\partial x^2}=0, 0<x<L, t\geq 0.$$

where, $h$ represents the cross-sectional area of the stretched string, $E$ is the Young coefficient, $L$ represents the length of the string, $P_0$ represents the initial axial tension, $\rho$ represents the mass density of the string, and $u=u(x,t)$ is the transverse displacement in space $x$ and time coordinates $t$. The expansion model of the equation in higher dimensional space is as follows:

$$u_{tt}-M\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)\Delta u=f(x,u), x\in\Omega\subseteq R^N.$$

where, $u$ denotes the vibration displacement of the string, $f(x,u)$ denotes the external force, $M(s)=as+b, a,b>0$. The characteristic of this equation is that it contains non-local terms Kirchhoff terms, so it is called a Kirchhoff-type equation.

E. hoenriques, D. Borito and J. Hoale studied the initial boundary value problem of the following nonlinear Kirchhoff equation in [5]:

$$\begin{cases} u_{tt}-M\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)\Delta u+\delta u_t=0,\\ u(0)=u_0, u_t(0)=u_1, x\in[0,L], t\in[0,\infty). \end{cases}$$

the only stable solution of the equation is obtained by Galerkin's method.

Lin Chen, Wei, Wang and Guoguang Lin [6] studied the initial boundary value problems of the following class of higher-order Kirchhoff equations:

$$u_{tt}+(-\Delta)^m u_t+\phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m u+g(u)=f(x).$$

They prove the existence of bounded absorption sets and global solutions by a prior estimation and Glerkin's method, and prove the existence of the family of global attractors by using the method of uniform compactness. Then they estimate the upper bounds of Housdorff dimension and fractal dimension.

Yuhuai Liao, Guoguang Lin, Jie Liu [7] studied the initial boundary value problem of the Beam-Kirchhoff equation in order to study the global stability of the model

$$u_{tt}+\beta(\Delta)^{2m}u_t+M\left(\left\|D^m u\right\|_p^p\right)u_t+\alpha(\Delta)^{2m}u+N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u=f(x),$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,2,\cdots,2m-1,x\in\partial\Omega,t>0,$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega\subset R^n.$$

The existence of the family of global attractors and upper bounds of Housdorff dimension and fractal dimension are proved by proper hypothesis of nonlinear terms.

More results on the existence of attractors in mathematical and physical models can be seen in detail [8]-[15].

On the basis of previous studies, the existence of global solutions and the family of global attractors for Beam-Kirchhoff equations under boundary conditions will be studied.

For illustrative purposes, the following Spaces and symbols are defined:

$$H=L^2(\Omega),D=\nabla,E_k=H^{2m+k}(\Omega)\times H^k(\Omega),(k=1,2,\cdots,2m);$$

$$E_0=H^{2m}(\Omega)\times H(\Omega),(u,v)=\int_\Omega u(x)v(x)\mathrm{d}x;$$

And assume $\gamma M\left(\left\|D^m u\right\|_p^p\right)(\Delta)^{2m}u$, $\delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}u$ that the following conditions are met:

$$M\left(\left\|D^m u\right\|_p^p\right)\in C^2([0,+\infty),R),1<\sigma_0\le M\left(\left\|D^m u\right\|_p^p\right)\le\sigma_1,$$

(A)
$$\delta_2=\begin{cases}\sigma_0,\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2\ge 0\\ \sigma_1,\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2<0\end{cases}$$

where, $\sigma_0,\sigma_1$ is the positive constants, $k=1,2,\cdots,2m$;

(B) $N\left(\left\|u_t\right\|^2\right)\in C^2([0,+\infty),R),1<A_0\le N\left(\left\|D^m u_t\right\|^2\right)\le A_1,$

where, $A_0,A_1$ is the positive constants.

$$A_2=\begin{cases}A_0,\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2\ge 0\\ A_1,\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2\le 0\end{cases}$$

$$k=1,2,\cdots,2m.$$

$\beta$ is a sufficiently large constant.

## 2. Existence of a Family of Global Attractors

In this part, the existence of bounded absorption set is proved by prior estimation, then the existence and uniqueness of global solution is proved by Galerkin's method, and finally the compactibility of global solution in phase space is verified by Soboleve compact embedding, thus the existence of a family of global attractors is proved.

**Lemma 1.** Assumes that (A), (B) is true, $f(x)\in H$, $(u_0,u_1)\in E_0$, $v=u_t+\varepsilon u$, and is satisfied

$$C_1=\frac{1}{\varepsilon}\|f(x)\|^2,\alpha_1=\min\left\{\theta_1,\frac{\theta_2}{\gamma\delta_0+\delta A_0+\alpha}\right\},\theta_1=\beta\lambda_1^{2m}-3\varepsilon-\varepsilon^2,$$
$$\theta_2=2\gamma\delta_0\varepsilon+2\varepsilon\delta A_0+2\alpha\varepsilon-2\varepsilon^2\beta-\varepsilon^2\lambda_1^{-2m}>0,$$

then the global smooth solution of the initial boundary value problem (1) - (3) is $(u,v)\in E_0$ and $v\in L^2(0,T;H^{2m}(\Omega))$.

So there is a non-negative real number $R_0$ and $t_0$

$$\overline{\lim_{t\to\infty}}\|(u,v)\|_{E_0}^2=\|D^{2m}u\|^2+\|v\|^2\le R_0^2\,(t>t_0). \tag{5}$$

**Proof.** Due to $v=u_t+\varepsilon u$, we take the inner product of both sides of Equation (1) with *v*. We can get

$$\begin{aligned}&\left(u_{tt}+\beta(\Delta)^{2m}u_t+\gamma M\left(\|D^m u\|_p^p\right)(\Delta)^{2m}u+\alpha\Delta^{2m}u+\delta N\left(\|u_t\|^2\right)(\Delta)^{2m}u,v\right)\\&=(f(x),v).\end{aligned} \tag{6}$$

By using the Holder's inequality, the Young's Inequality and the Poincare's Inequality and conditions (A), (B), each item in (6) is processed successively

$$\begin{aligned}(u_{tt},v)&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2-\varepsilon\|v\|^2+\varepsilon^2(u,v)\\&\ge\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2-\frac{2\varepsilon+\varepsilon^2}{2}\|v\|^2-\frac{\varepsilon^2\lambda_1^{-2m}}{2}\|D^{2m}u\|^2.\end{aligned} \tag{7}$$

$$\begin{aligned}\left(\beta(\Delta)^{2m}u_t,v\right)&=\left(\beta\Delta^{2m}(v-\varepsilon u),v\right)=\beta\|D^{2m}v\|^2-\beta\varepsilon\left(\Delta^{2m}u,v\right)\\&\ge\frac{\beta}{4}\|D^{2m}v\|^2+\frac{\beta\lambda_1^{2m}}{2}\|v\|^2-\beta\varepsilon^2\|D^{2m}u\|^2.\end{aligned} \tag{8}$$

$$\begin{aligned}&\left(\gamma M\left(\|D^m u\|_p^p\right)\Delta^{2m}u,v\right)\\&=\gamma M\left(\|D^m u\|_p^p\right)\left(D^{2m}u,D^{2m}u_t\right)+\gamma M\left(\|D^m u\|_p^p\right)\left(D^{2m}u,\varepsilon D^{2m}u\right)\\&\ge\frac{\gamma\delta_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}u\|^2+\gamma\delta_0\varepsilon\|D^{2m}u\|^2.\end{aligned} \tag{9}$$

$$\left(\delta N\left(\|u_t\|^2\right)(\Delta)^{2m}u,v\right)\ge\frac{\delta A_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}u\|^2+\varepsilon\delta A_0\|D^{2m}u\|^2. \tag{10}$$

$$\left(\alpha\Delta^{2m}u,v\right)=\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^{2m}u\|^2+\alpha\varepsilon\|D^{2m}u\|^2. \tag{11}$$

$$(f(x),v)\le\frac{1}{2\varepsilon}\|f(x)\|^2+\frac{\varepsilon}{2}\|v\|^2. \tag{12}$$

Therefore:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+(\gamma\delta_2+\delta A_2+\alpha)\|D^{2m}u\|^2\right)+\left(\beta\lambda_1^{2m}-3\varepsilon-\varepsilon^2\right)\|v\|^2 \\ +\left(2\gamma\delta_0\varepsilon+2\varepsilon\delta A_0+2\alpha\varepsilon-2\varepsilon^2\beta-\varepsilon^2\lambda_1^{-2m}\right)\|D^{2m}u\|^2\le\frac{1}{\varepsilon}\|f(x)\|^2. \tag{13}$$

There are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+(\gamma\delta_0+\delta A_0+\alpha)\|D^{2m}u\|^2\right)+\alpha_1\left(\|v\|^2+(\gamma\delta_0+\delta A_0+\alpha)\|D^{2m}u\|^2\right)\le C_1. \tag{14}$$

It's given by the Gronwall's inequality

$$\|v\|^2+(\gamma\delta_0+\delta A_0+\alpha)\|D^{2m}u\|^2 \\ \le\left(\|v_0\|^2+(\gamma\delta_0+\delta A_0+\alpha)\|D^{2m}u_0\|^2\right)\mathrm{e}^{-\alpha_1 t}+\frac{C_1}{\alpha_1}. \tag{15}$$

Let $L=\min\left(1,(\gamma\delta_0+\delta A_0+\alpha)\right)$, Then we can get

$$\|v\|^2+\|D^{2m}u\|^2\le\frac{\|v_0\|^2+(\gamma\delta_0+\delta A_0+\alpha)\|D^{2m}u_0\|^2}{L}\mathrm{e}^{-\alpha_1 t}+\frac{C_1}{L\alpha_1}, \tag{16}$$

Such that there are non-negative real numbers $R_0$,

$$\overline{\lim_{t\to\infty}}\|(u,v)\|_{E_0}^2=\|D^{2m}u\|^2+\|v\|^2\le\frac{C_1}{L\alpha_1}=R_0,(t>t_0). \tag{17}$$

Lemma 1 is proved.

**Lemma 2.** Assumes that (A), (B) is true, $f(x)\in H$, $(u_0,u_1)\in E_k$, $v=u_t+\varepsilon u$, and is satisfied $C_2=\frac{4\lambda_1^{2m-k}}{\beta}\|f(x)\|^2$, $\alpha_2=\min\left\{\theta_3,\frac{\theta_4}{\gamma\delta_0+\delta A_0+\alpha}\right\}$,

$\theta_3=\frac{\beta\lambda_1^{2m}}{2}-2\varepsilon-2\varepsilon^2>0$, $\theta_4=2\gamma\delta_0\varepsilon-\beta\varepsilon^2-\frac{\varepsilon^2}{\lambda_1^{2m}}+2\varepsilon\delta A_0+2\alpha\varepsilon>0$,

then the global smooth solution of the initial boundary value problem (1) - (3) is $(u,v)\in E_k$ and $v\in L^2\left(0,T;H^{2m+k}(\Omega)\right)$.

So there is a non-negative real number $R_k$ and $t_k$

$$\overline{\lim_{t\to\infty}}\|(u,v)\|_{E_k}^2=\|D^{2m+k}u\|^2+\|D^k v\|^2\le R_k,(t>t_k). \tag{18}$$

**Proof.** Due to $(-\Delta)^k v=(-\Delta)^k u_t+\varepsilon(-\Delta)^k u$, take the inner product of both sides of the Equation (1) with $(-\Delta)^k v$,

$$\left(u_{tt}+\beta(\Delta)^{2m}u_t+\gamma M\left(\|D^m u\|_p^p\right)(\Delta)^{2m}u+\alpha\Delta^{2m}u+\delta N\left(\|u_t\|^2\right)(\Delta)^{2m}u,(-\Delta)^k v\right) \\ =\left(f(x),(-\Delta)^k v\right). \tag{19}$$

By using Holder's inequality, the Young's Inequality and the Poincare's Inequality and conditions (A), (B), the terms in (19) are obtained

$$\left(u_{tt},(-\Delta)^k v\right)=\left(v_t-\varepsilon u_t,(-\Delta)^k v\right) \\ =\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^k v\|^2-\varepsilon\|D^k v\|^2+\varepsilon^2\left(u,(-\Delta)^k v\right) \\ \ge\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|D^k v\|^2-\left(\varepsilon+\frac{\varepsilon^2}{2}\right)\|D^k v\|^2-\frac{\varepsilon^2}{2\lambda_1^{2m}}\|D^{2m+k}u\|^2. \tag{20}$$

$$\begin{aligned}&\left(\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u,(-\Delta)^k v\right)\\&=\frac{\gamma M\left(\left\|D^m u\right\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\gamma M\left(\left\|D^m u\right\|_p^p\right)\varepsilon\left\|D^{2m+k}u\right\|^2\\&\geq\frac{\gamma\delta_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\gamma\delta_0\varepsilon\left\|D^{2m+k}u\right\|^2.\end{aligned}\tag{21}$$

$$\begin{aligned}&\left(\beta(\Delta)^{2m}u_t,(-\Delta)^k v\right)\\&=\beta\left\|D^{2m+k}v\right\|^2-\beta\varepsilon\left(\Delta^{2m}u,(-\Delta)^k v\right)\\&\geq\beta\left\|D^{2m+k}v\right\|^2-\beta\varepsilon\left\|D^{2m+k}u\right\|.\left\|D^{2m+k}v\right\|\\&\geq\beta\left\|D^{2m+k}v\right\|^2-\frac{\beta\varepsilon^2}{2}\left\|D^{2m+k}u\right\|^2-\frac{\beta}{2}\left\|D^{2m+k}v\right\|^2\\&\geq\frac{\lambda_1^{2m}\beta}{4}\left\|D^k v\right\|^2+\frac{\beta}{4}\left\|D^{2m+k}v\right\|^2-\frac{\beta\varepsilon^2}{2}\left\|D^{2m+k}u\right\|^2.\end{aligned}\tag{22}$$

$$\begin{aligned}&\left(\delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}u,(-\Delta)^k v\right)\\&=\frac{\delta N\left(\left\|u_t\right\|^2\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\varepsilon\delta N\left(\left\|u_t\right\|^2\right)\left\|D^{2m+k}u\right\|^2\\&\geq\frac{\delta A_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\varepsilon\delta A_0\left\|D^{2m+k}u\right\|^2.\end{aligned}\tag{23}$$

$$\left(\alpha\Delta^{2m}u,(-\Delta)^k v\right)=\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\alpha\varepsilon\left\|D^{2m+k}u\right\|^2.\tag{24}$$

$$\left(f(x),(-\Delta)^k v\right)\leq\frac{\beta}{8}\left\|D^{2m+k}v\right\|^2+\frac{4\lambda_1^{2m-k}}{\beta}\left\|f(x)\right\|^2.\tag{25}$$

All kinds of comprehensive can be written as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k v\right\|^2+\left(\gamma\delta_2+\delta A_2+\alpha\right)\left\|D^{2m+k}u\right\|^2\right)+\theta_3\left\|D^k v\right\|^2+\frac{3\beta}{8}\left\|D^{2m+k}v\right\|^2\\&+\theta_4\left\|D^{2m+k}u\right\|^2\leq\frac{4\lambda_1^{2m-k}}{\beta}\left\|f(x)\right\|^2.\end{aligned}\tag{26}$$

Available

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k v\right\|^2+\left(\gamma\delta_2+\delta A_2+\alpha\right)\left\|D^{2m+k}u\right\|^2\right)\\&+\alpha_2\left(\left\|D^k v\right\|^2+\left(\gamma\delta_2+\delta A_2+\alpha\right)\left\|D^{2m+k}u\right\|^2\right)\leq C_2.\end{aligned}\tag{27}$$

And that is given by Gronwall's inequality

$$\begin{aligned}&\left\|D^k v\right\|^2+\left(\gamma\delta_2+\delta A_2+\alpha\right)\left\|D^{2m+k}u\right\|^2\\&\leq\left(\left\|D^k v_0\right\|^2+\left(\gamma\delta_2+\delta A_2+\alpha\right)\left\|D^{2m+k}u_0\right\|^2\right)\mathrm{e}^{-\alpha_2 t}+\frac{C_2}{\alpha_2}.\end{aligned}\tag{28}$$

Let $L_1=\min\left(1,\left(\gamma\delta_2+\delta A_2+\alpha\right)\right)$, then

$$\left\|(u,v)\right\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^k v\right\|^2\leq\frac{\left\|D^{2m+k}u_0\right\|^2+\left\|D^k v_0\right\|^2}{L_1}\mathrm{e}^{-\alpha_2(t-t_0)}+\frac{C_2}{L_1\alpha_2}.\tag{29}$$

So there is a non-negative real number $R_k$ and $t_k$ that make

$$\overline{\lim_{t\to\infty}}\|(u,v)\|_{E_k}^2=\|D^{2m+k}u\|^2+\|D^k v\|^2\le R_k=\frac{C_2}{L_1\alpha_2},(t>t_k). \tag{30}$$

Lemma 2 is proved.

**Theorem 3.** (Existence and uniqueness of solutions) Assumes that (A), (B) is true, $f(x)\in H$, $(u_0,v_0)\in E_0$, then the initial boundary value problem (1) - (3) has an unique smooth solution $(u,v)\in L^\infty([0,+\infty);E_k)$, $v\in L^2(0,T;H^{2m+k}(\Omega))$.

**Proof.** Galerkin's finite element method is used to prove the existence of global solution.

**The first step is to construct the approximate solution**

Let $u''=\frac{\partial^2 u}{\partial t^2}$.

Take the sequence $w_1,w_2,\cdots,w_m,\cdots,w_i\in H^{2m}$, $w_1,w_2,\cdots,w_m$ is linearly independent. Linear combinations of $w_i$ are dense in $H^{2m}$. The approximate solution is $u_m(t)=\sum_{i=1}^{m}g_{im}(t)w_i$ in problem (1).

Where $g_{im}(t)$ is determined by the following conditions

$$\begin{aligned}&\left(u_m''+\beta(\Delta)^{2m}u_m'+\gamma M\left(\|D^m u\|_p^p\right)(\Delta)^{2m}u_m+\alpha\Delta^{2m}u_m+\delta N\left(\|u_t\|^2\right)(\Delta)^{2m}u_m,w_j\right)\\&=\left(f(x),w_j\right),1\le j\le m,\end{aligned} \tag{31}$$

The nonlinear system of ordinary differential Equations (31) satisfies the initial condition

$u_{m0}(0)=u_{m0},u_{m1}(0)=u_{m1}$. When $m\to+\infty$, $(u_{m0},u_{m1})\to(u_0,u_1)$ in $H^{2m}$. We know from the basic theory of ordinary differentiation that approximate solutions exist on $(0,t_m)$.

**The second step is prior estimation**

Multiply both ends of Equation (31) by $g'_{jm}(t)+\varepsilon g_{jm}(t)$, and sum over *j* to get

$$\begin{aligned}&\left(u_m''+\beta(\Delta)^{2m}u_m'+\gamma M\left(\|D^m u\|_p^p\right)(\Delta)^{2m}u_m+\alpha\Delta^{2m}u_m+\delta N\left(\|D^m u_t\|^2\right)(\Delta)^{2m}u_m,v_m\right)\\&=\left(f(x),v_m\right),1\le j\le m,\end{aligned} \tag{32}$$

A prior estimate of the solution in $E_0$ is obtained from lemma 1:

$$\overline{\lim_{t\to\infty}}\|(u_m,v_m)\|_{E_0}^2=\|D^{2m}u_m\|^2+\|v_m\|^2\le\frac{C_1}{L\alpha_1}=R_0,(t>t_0).$$

And that's given by lemma 2:

$$\overline{\lim_{t\to\infty}}\|(u_m,v_m)\|_{E_k}^2=\|D^{2m+k}u_m\|^2+\|D^k v_m\|^2\le R_k,(t>t_k),$$

so $(u_m,v_m)$ is bounded in $L^\infty([0,+\infty];E_k)$.

**The third step is limit process**

Danford-Pttes theorem tells us that space $L^\infty([0,+\infty];H^{2m+k}\times H^k)$ is conjugate to $L^{\infty}\left([0,+\infty];H^{-2m-k}\times H^{-k}\right)$. Choosing subcolumn $u_{\mu}$ from the sequence $u_m$ causes $\left(u_{\mu},v_{\mu}\right)\to(u,v)$ to converge weakly * in $L^{\infty}\left([0,+\infty)\right)$.

It is known from *Rellich-Kohπ paiiiob* theorem that $E_k$ compact embedded in $E_0$, and $\left(u_{\mu},v_{\mu}\right)\to(u,v)$ converges strongly almost everywhere in $E_0$.

In (31) let $m=\mu$ and take the limit, for fixed *j* and $\mu\geq j$

$$\left(u''_{\mu},w_j\right)=\left(v_{\mu},w_j\right)-\left(\varepsilon u_{\mu},w_j\right)\to\left(v,w_j\right)-\left(\varepsilon u,w_j\right). \tag{33}$$

Weakly * converges in $L^{\infty}[0,+\infty)$.

$\left(\beta(\Delta)^{2m}u'_m,w_j\right)\to\beta\left((\Delta)^{2m}u',w_j\right)$ converges weakly * in $L^{\infty}\left([0,+\infty)\right)$.

$\left(\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u_{\mu},w_j\right)\to\gamma M\left(\left\|D^m u\right\|_p^p\right)\left(\Delta^{2m}u,w_j\right)$ converges weakly * in $L^{\infty}\left([0,+\infty)\right)$.

In the space $L^{\infty}\left([0,+\infty];H^{2m+k}\times H^{k}\right)$, $u''_{\mu}$ weakly converges to $u''$, $u'_{\mu}$ weakly converges to $u'$, and there is

$$\lim_{\mu\to\infty}\int\int_0^T\left|u'_{\mu}-u'\right|\mathrm{d}x\mathrm{d}t=0,$$

So $\left(\delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}u_{\mu},w_j\right)\to\left(\delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}u,w_j\right)$ converges weakly * in $L^{\infty}\left([0,+\infty)\right)$.

$\left(\alpha\Delta^{2m}u_{\mu},w_j\right)\to\left(\alpha\Delta^{2m}u,w_j\right)$ converges weakly * in $L^{\infty}\left([0,+\infty)\right)$.

Derivable

$$\begin{aligned}&\left(u_{tt}+\beta(\Delta)^{2m}u_t+\gamma M\left(\left\|D^m u\right\|_p^p\right)(\Delta)^{2m}u+\alpha\Delta^{2m}u+\delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}u,w_j\right)\\&=\left(f(x),w_j\right),\ j=1,2,\cdots,m.\end{aligned} \tag{34}$$

Therefore, the existence of the weak solution of problems (1) - (3) is obtained. The existence is proved, and the uniqueness of the solution below is obtained.

Let $u,v$ be two solutions to the problem, and let $w=u-v$ have $w(0)=0$, $w_t(0)=0$

$$\begin{aligned}&w_{tt}+\beta(\Delta)^{2m}w_t+\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u-\gamma M\left(\left\|D^m v\right\|_p^p\right)\Delta^{2m}v+\alpha\Delta^{2m}w\\&+\delta N\left(\left\|D^m u_t\right\|^2\right)(\Delta)^{2m}u-\delta N\left(\left\|D^m v_t\right\|^2\right)(\Delta)^{2m}v=0.\end{aligned} \tag{35}$$

- We take the inner product of $w_t$ in both sides of this equation,

$$\begin{aligned}&\left(w_{tt}+\beta(\Delta)^{2m}w_t+\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u-\gamma M\left(\left\|D^m v\right\|_p^p\right)\Delta^{2m}v+\alpha\Delta^{2m}w\right.\\&\left.+\delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}u-\delta N\left(\left\|v_t\right\|^2\right)(\Delta)^{2m}v,w_t\right)=0.\end{aligned} \tag{36}$$

- Each item is processed successively

$$\left(w_{tt},w_t\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|w_t\right\|^2, \tag{37}$$

$$\left(\beta(\Delta)^{2m}w_t,w_t\right)=\beta\left\|D^{2m}w_t\right\|^2, \tag{38}$$

$$\left(\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u-\gamma M\left(\left\|D^m v\right\|_p^p\right)\Delta^{2m}v,w_t\right). \tag{39}$$

From lemma 1, lemma 2 and differential mean value theorem and Young's inequality

$$\begin{aligned}&\left(\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u-\gamma M\left(\left\|D^m v\right\|_p^p\right)\Delta^{2m}v,w_t\right)\\&=\left(\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u-\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}v+\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}v-\gamma M\left(\left\|D^m v\right\|_p^p\right)\Delta^{2m}v,w_t\right)\\&\ge\frac{1}{2}\gamma M\left(\left\|D^m u\right\|_p^p\right)\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^m w\right\|^2-\gamma\left\|M'(\xi)\right\|\left(\left\|D^m u\right\|^{p-1}+\left\|D^m v\right\|^{p-1}\right)\left\|D^m w\right\|\cdot\left\|\Delta^m v\right\|\cdot\left\|\Delta^m w_t\right\|\\&\ge\frac{\gamma\delta_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\Delta^m w\right\|^2-\frac{B_1}{2}\left\|D^m w\right\|^2-\frac{B_2}{2}\left\|\Delta^m w_t\right\|^2,\end{aligned} \tag{40}$$

where $\xi=\theta\left\|D^m u\right\|_p^p+(1-\theta)\left\|D^m v\right\|_p^p,\theta\in(0,1)$.

Similarly,

$$\begin{aligned}&\left(\delta N\left(\left\|u_t\right\|^2\right)\Delta^{2m}u-\delta N\left(\left\|v_t\right\|^2\right)\Delta^{2m}v,w_t\right)\\&=\left(\delta N\left(\left\|u_t\right\|^2\right)\Delta^{2m}u-\delta N\left(\left\|u_t\right\|^2\right)\Delta^{2m}v+\delta N\left(\left\|u_t\right\|^2\right)\Delta^{2m}v-\delta N\left(\left\|v_t\right\|^2\right)\Delta^{2m}v,w_t\right)\\&\ge\frac{\delta N\left(\left\|u_t\right\|^2\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2-\left\|\delta\left(N'(\varsigma)\right)\right\|_\infty\left(\left\|u_t\right\|+\left\|v_t\right\|\right)\left\|w_t\right\|\left\|D^{2m}v\right\|\left\|D^{2m}w_t\right\|\\&\ge\frac{\delta A_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2-\frac{B_3}{2}\left\|w_t\right\|^2-\frac{B_4}{2}\left\|D^{2m}w_t\right\|^2,\end{aligned} \tag{41}$$

where $\varsigma=\theta'\left\|u_t\right\|^2+(1-\theta')\left\|v_t\right\|^2,\theta'\in(0,1)$.

$$\left(\alpha\Delta^{2m}w,w_t\right)=\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2. \tag{42}$$

Substitute (37) - (42) into Equation (36)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|w_t\right\|^2+\left(\frac{\gamma\delta_0}{2}+\frac{A_0\delta}{2}+\alpha\right)\left\|D^{2m}w\right\|^2\right]-B_1\left\|D^m w\right\|^2-B_3\left\|w_t\right\|^2\\&+\left(2\beta-B_4-B_2\right)\left\|D^{2m}w_t\right\|^2\le 0.\end{aligned} \tag{43}$$

Since $\beta$ is a sufficiently large number, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|w_t\right\|^2+\left(\frac{\gamma\delta_0}{2}+\frac{A_0\delta}{2}+\alpha\right)\left\|D^{2m}w\right\|^2\right]-B_1\left\|D^m w\right\|^2-B_3\left\|w_t\right\|^2\le 0. \tag{44}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|w_t\right\|^2+\left(\frac{\gamma\delta_0}{2}+\frac{A_0\delta}{2}+\alpha\right)\left\|D^{2m}w\right\|^2\right]\\&\le B_1\left\|D^m w\right\|^2+B_3\left\|w_t\right\|^2\le B_1\lambda_1^{-m}\left\|D^{2m}w\right\|^2+B_3\left\|w_t\right\|^2.\end{aligned} \tag{45}$$

Let $C_8=\max\left\{B_1\lambda_1^{-m},B_3\right\}$,

Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|w_t\right\|^2+\left(\frac{\gamma\delta_0}{2}+\frac{A_0\delta}{2}+\alpha\right)\left\|D^{2m}w\right\|^2\right]\\&\le C_8\left(\left\|w_t\right\|^2+\left(\frac{\gamma\delta_0}{2}+\frac{A_0\delta}{2}+\alpha\right)\left\|D^{2m}w\right\|^2\right).\end{aligned} \tag{46}$$

From the Gronwall's inequality

$$\begin{aligned}&\left\|w_t\right\|^2+\left(\frac{\gamma\delta_0}{2}+\frac{A_0\delta}{2}+\alpha\right)\left\|D^{2m}w\right\|^2\\&\le\left[\left\|w_t(0)\right\|^2+\left(\frac{\gamma\delta_0}{2}+\frac{A_0\delta}{2}+\alpha\right)\left\|D^{2m}w(0)\right\|^2\right]\mathrm{e}^{\alpha_3 t}=0.\end{aligned}\tag{47}$$

Then $w(t)=0,u=v$, thus uniqueness is proved.

Theorem 3 is proved.

**Theorem 4.** The global smooth solution of problem (1) - (3) satisfies lemma 1, lemma 2, and theorem 3.

Then the initial boundary value problem (1) - (3) has a family of global attractors

$$A_k=\omega(D_{0k})=\bigcap_{\tau\ge0}\overline{\bigcup_{t\ge\tau}S(t)D_{0k}},k=1,2,\cdots,2m,$$

where:

$D_{0k}=\left\{(u,v)\in E_k:\left\|(u,v)\right\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^k v\right\|^2\le R_k^2+R_2^2\right\}$ is a bounded absorbing set in $E_k$ and satisfies the following conditions:

1) $S(t)A_k=A_k,t>0$;

2) $\lim_{t\to\infty}dist(S(t)D_k,A_k)=0$ (where $\forall D_k\subset E_k$ and is a bounded set), where

$$dist(S(t)D_k,A_k)=\sup_{x\in D_k}\inf_{y\in A_k}\left\|S(t)x-y\right\|_{E_k}.$$

$S(t)$ is the solution semigroup generated by the problem (1) - (3).

**Proof.** According to theorem 3, there exists a solution semigroup $S(t):E_k\to E_k$ of the problem (1) - (3).

According to lemma 2, we can obtain

$$\left\|S(t)(u_0,v_0)\right\|_{E_k}^2=\left\|u\right\|_{H_0^{2m+k}(\Omega)}^2+\left\|v\right\|_{H_0^k(\Omega)}^2\le\left\|u_0\right\|_{H_0^{2m+k}(\Omega)}^2+\left\|v_0\right\|_{H_0^k(\Omega)}^2\le R_k^2.$$

This indicates that $\{S(t)\}(t\ge0)$ is uniformly bounded on $E_k$.

Furthermore, for any $(u_0,v_0)\in E_k$, when $t\ge\max\{t_1,t_k\}$, we have

$$\left\|S(t)(u_0,v_0)\right\|_{E_k}^2=\left\|u\right\|_{H_0^{2m+k}(\Omega)}^2+\left\|v\right\|_{H_0^k(\Omega)}^2\le R_k^2+R_0^2.$$

Therefore,

$$D_{0k}=\left\{(u,v)\in E_k:\left\|(u,v)\right\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^k v\right\|^2\le R_k^2+R_0^2\right\}$$

is a bounded absorbing set for semigroup $S(t)$.

According to the Rellich-Kondrachov's theorem, $E_k$ is compactly embedded into $E_0$, so the bounded set in $E_k$ is the compact set in $E_0$. Therefore, solution semigroup $S(t)$ is a completely continuous operator, thus the family of global attractors $A_k$ of solution semigroup $S(t)$ is obtained. where $A_k=\omega(D_{0k})=\bigcap_{\tau\ge0}\overline{\bigcup_{t\ge\tau}S(t)D_{0k}}$.

Theorem 4 is proved.

## 3. Dimension Estimation for the Family of Global Attractors

In this section, we first linearize the equation to a first order variational equation and prove that solution semigroups $S(t)$ is uniformly differentiable on $E_k$. Finally, we estimate the upper bounds of the Hausdorff dimension and the Fractal dimension by using the generalized Sobolev-Lieb-Thirring inequality.

If we linearize the equation, we get

$$U_{tt}+\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}U+\gamma pM'\left(\left\|D^m u\right\|_p^p\right)\int_\Omega\left|D^m u\right|^{p-2}\cdot D^m u\cdot D^m U\mathrm{d}x\cdot\Delta^{2m}u$$
$$+\beta(\Delta)^{2m}U_t+\delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}U+2\delta N'\left(\left\|u_t\right\|^2\right)\int_\Omega u_t\cdot U_t\mathrm{d}x\cdot(\Delta)^{2m}u \tag{48}$$
$$+\alpha\Delta^{2m}U=0,$$

$$U(x,t)=0,\frac{\partial^i U}{\partial V^i}=0,i=1,2,\cdots,2m-1,x\in\partial\Omega,t>0, \tag{49}$$

$$U(x,0)=\xi,U_t(x,0)=\eta. \tag{50}$$

And for any $(u_0,v_0)\in E_k$, the initial value problem (1) - (3) has a solution $(u(t),v(t))=S(t)(u_0,v_0)$.

**Lemma 5.** Initial value problem (1) - (3) exists a family of attractors $A_k$. $A_k$ is bounded in $E_k$. The solution semigroup $S(t)$ determined by the initial value problem (1) - (3) is uniformly differentiable on the compact invariant set $A_k$. Its derivative is defined as $F(t,u_0)\xi=U(t)$. $U(t)$ is the solution of the linear initial value problem (48) - (50). There is a bounded operator $F:(\xi,\eta)\to(U(t),U_t(t))$.

And let

$$\phi_0=(u_0,u_1)\in E_k,\overline{\phi_0}=(u_0+\xi,u_1+\eta)\in E_k,$$

then

$$\sup\frac{\left\|S(t)\overline{\phi_0}-S(t)\phi_0-\left(U(t),U_t(t)\right)^{\mathrm{T}}\right\|^2}{\left\|(\xi,\eta)\right\|^2}\to 0\ \left((\xi,\eta)\to 0\right). \tag{51}$$

**Proof.** $\phi_0=(u_0,u_1)\in E_k,\overline{\phi_0}=(u_0+\xi,u_1+\eta)\in E_k$ and $\left\|\phi_0\right\|_{E_k},\left\|\overline{\phi_0}\right\|_{E_k}\le R$.

Let

$$\psi=\overline{u}-u-U,\phi=\psi_t+\varepsilon\psi,$$

then

$$\begin{cases}\overline{u}_{tt}+\beta(\Delta)^{2m}\overline{u}_t+\gamma M\left(\left\|D^m\overline{u}\right\|_p^p\right)\Delta^{2m}\overline{u}+\alpha\Delta^{2m}\overline{u}+\delta N\left(\left\|\overline{u}_t\right\|^2\right)\Delta^{2m}\overline{u}=f(x),\\ u_{tt}+\beta(\Delta)^{2m}u_t+\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u+\alpha\Delta^{2m}u+\delta N\left(\left\|u_t\right\|^2\right)\Delta^{2m}u=f(x),\\ U_{tt}+\gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}U+\gamma pM'\left(\left\|D^m u\right\|_p^p\right)\int_\Omega\left|D^m u\right|^{p-2}\cdot D^m u\cdot D^m U\mathrm{d}x\cdot\Delta^{2m}u\\ +\beta(\Delta)^{2m}U_t+\delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}U+2\delta N'\left(\left\|u_t\right\|^2\right)\int_\Omega u_t\cdot U_t\mathrm{d}x\cdot(\Delta)^{2m}u+\alpha\Delta^{2m}U=0.\end{cases}$$

Subtracting from three formulas can be obtained:

$$\psi_{tt}+\beta(\Delta)^{2m}\psi_t+\alpha\Delta^{2m}\psi+G=0, \tag{52}$$

where

$$
\begin{aligned}
G = {} & \gamma M\left(\left\|D^m\bar{u}\right\|_p^p\right)\Delta^{2m}\bar{u} - \gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u - \gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}U \\
& - \gamma pM'\left(\left\|D^m u\right\|_p^p\right)\int_\Omega \left|D^m u\right|^{p-2} \cdot D^m u \cdot D^m U \mathrm{d}x \cdot \Delta^{2m}u + \delta N\left(\left\|\bar{u}_t\right\|^2\right)\Delta^{2m}\bar{u} \\
& - \delta N\left(\left\|u_t\right\|^2\right)\Delta^{2m}u - \delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}U - 2\delta N'\left(\left\|u_t\right\|^2\right)\int_\Omega u_t \cdot U_t \mathrm{d}x \cdot (\Delta)^{2m}u.
\end{aligned}
$$

Let $G = g_1 + g_2$, where

$$
\begin{aligned}
g_1 = {} & \gamma M\left(\left\|D^m\bar{u}\right\|_p^p\right)\Delta^{2m}\bar{u} - \gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u - \gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}U \\
& - \gamma pM'\left(\left\|D^m u\right\|_p^p\right)\int_\Omega \left|D^m u\right|^{p-2} \cdot D^m u \cdot D^m U \mathrm{d}x \cdot \Delta^{2m}u,
\end{aligned}
$$

$$
\begin{aligned}
g_2 = {} & \delta N\left(\left\|\bar{u}_t\right\|^2\right)\Delta^{2m}\bar{u} - \delta N\left(\left\|u_t\right\|^2\right)\Delta^{2m}u - \delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}U \\
& - 2\delta N'\left(\left\|u_t\right\|^2\right)\int_\Omega u_t \cdot U_t \mathrm{d}x \cdot (\Delta)^{2m}u.
\end{aligned}
$$

To deal with

$$
\begin{aligned}
g_1 = {} & \gamma M\left(\left\|D^m\bar{u}\right\|_p^p\right)\Delta^{2m}\bar{u} - \gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}u - \gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}U \\
& - \gamma pM'\left(\left\|D^m u\right\|_p^p\right)\int_\Omega \left|D^m u\right|^{p-2} \cdot D^m u \cdot D^m U \mathrm{d}x \cdot \Delta^{2m}u \\
= {} & \gamma M'\left(\left\|D^m\bar{\zeta}\right\|_p^p\right)\left(\left\|D^m\bar{\zeta}\right\|_p^p\right)' D^m(\bar{u}-u)\Delta^{2m}\bar{u} + \gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}\psi \\
& + \gamma pM'\left(\left\|D^m u\right\|_p^p\right)\int_\Omega \left|D^m u\right|^{p-2} \cdot D^m u \mathrm{d}x D^m\psi \cdot \Delta^{2m}u \\
& - \gamma pM'\left(\left\|D^m u\right\|_p^p\right)\int_\Omega \left|D^m u\right|^{p-2} \cdot D^m u \mathrm{d}x \cdot D^m(\bar{u}-u) \cdot \Delta^{2m}u \\
= {} & R''(\varsigma)S_2\left(D^m(\bar{u}-u)\right)^2 \cdot \Delta^{2m}\bar{u} \\
& + \gamma M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m(\bar{u}-u) \cdot \left(\Delta^{2m}\bar{u} - \Delta^{2m}u\right) \\
& + \gamma M\left(\left\|D^m u\right\|_p^p\right)\Delta^{2m}\psi + \gamma M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m\psi \cdot \Delta^{2m}u.
\end{aligned}
\tag{53}
$$

where $\bar{\zeta} = s_2 D^m\bar{u} + (1-s_2)D^m u, \varsigma = s_3 D^m\bar{\zeta} + (1-s_3)D^m u, s_2, s_3 \in (0,1)$.

$$
R'(\bar{\zeta}) = \gamma M'\left(\left\|D^m\bar{\zeta}\right\|_p^p\right)\left(\left\|D^m\bar{\zeta}\right\|_p^p\right)'.
$$

Take the inner product of $g_1$ and $(-\Delta)^k\phi$, and deal with it term by using the Young's inequality,

$$
\begin{aligned}
& \left|\left(R''(\varsigma)S_2\left(D^m(\bar{u}-u)\right)^2 \cdot \Delta^{2m}\bar{u}, (-\Delta)^k\phi\right)\right| \\
& \le C_9\left|\int_\Omega \left(D^m(\bar{u}-u)\right)^2 D^{2m+k}\bar{u}D^{2m+k}\phi \mathrm{d}x\right| \\
& \le C_{10}\left\|\bar{u}-u\right\|_{H^m}^2\left\|D^{2m+k}\phi\right\| \\
& \le \frac{C_{10}^2}{2}\left\|\bar{u}-u\right\|_{H^m}^4 + \frac{1}{2}\left\|D^{2m+k}\phi\right\|^2.
\end{aligned}
\tag{54}
$$

where $C_9 = \left\|R''(\varsigma)S_2\right\|, C_{10} = C_9 \cdot \left\|D^{2m+k}\bar{u}\right\|$.

$$\begin{aligned}
&\left|\left(\gamma M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m(\bar{u}-u)\cdot\left(\Delta^{2m}\bar{u}-\Delta^{2m}u\right),(-\Delta)^k\phi\right)\right| \\
&\le C_{11}\left|\int_\Omega D^m(\bar{u}-u)\cdot\left(D^{2m+k}\bar{u}-D^{2m+k}u\right)D^{2m+k}\phi\mathrm{d}x\right| \\
&\le C_{11}\left|\int_\Omega D^m(\bar{u}-u)D^{2m+k}(\bar{u}-u)D^{2m+k}\phi\mathrm{d}x\right| \\
&\le C_{11}\left\|D^m(\bar{u}-u)\right\|\cdot\left\|D^{2m+k}(\bar{u}-u)\right\|\cdot\left\|D^{2m+k}\phi\right\| \\
&\le \frac{C_{11}}{2\lambda_1^{\frac{m+k}{2}}}\left(\left\|D^{2m+k}(\bar{u}-u)\right\|^2+\left\|D^{2m+k}(\bar{u}-u)\right\|^2\right)\cdot\left\|D^{2m+k}\phi\right\| \\
&\le \frac{C_{12}^2}{2}\left\|D^{2m+k}(\bar{u}-u)\right\|^4+\frac{1}{2}\left\|D^{2m+k}\phi\right\|^2.
\end{aligned} \tag{55}$$

where $C_{11}=\gamma M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)', C_{12}=\frac{C_{11}}{2\lambda_1^{\frac{m+k}{2}}}$.

$$\begin{aligned}
&\left|\left(\gamma M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m\psi\cdot\Delta^{2m}u,(-\Delta)^k\phi\right)\right| \\
&\le C_{13}\left|\int_\Omega D^m\psi D^{2m+k}uD^{2m+k}\phi\mathrm{d}x\right| \\
&\le C_{13}\lambda_1^{-\frac{m+k}{2}}\left\|D^{2m+k}\psi\right\|\left\|D^{2m+k}u\right\|\left\|D^{2m+k}\phi\right\| \\
&\le \frac{C_{14}^2}{2}\left\|D^{2m+k}\psi\right\|^2+\frac{1}{2}\left\|D^{2m+k}\phi\right\|^2.
\end{aligned} \tag{56}$$

where $C_{13}=\gamma M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)', C_{14}=C_{13}\lambda_1^{-\frac{m+k}{2}}\left\|D^{2m+k}u\right\|$.

$$\begin{aligned}
&\left|\gamma M\left(\left\|D^m u\right\|_p^p\right)\left(\Delta^{2m}\psi,(-\Delta)^k\phi\right)\right| \\
&=\left|\gamma M\left(\left\|D^m u\right\|_p^p\right)\left(D^{2m+k}\psi,D^{2m+k}\phi\right)\right| \\
&\le \frac{C_{15}^2}{2}\left\|D^{2m+k}\psi\right\|^2+\frac{1}{2}\left\|D^{2m+k}\phi\right\|^2,
\end{aligned} \tag{57}$$

where $C_{15}=\gamma M\left(\left\|D^m u\right\|_p^p\right)$.

So that

$$\begin{aligned}
\left|\left(g_1,(-\Delta)^k\phi\right)\right| \le{}& \frac{C_9^2}{2}\left\|\bar{u}-u\right\|_{E_k}^4+2\left\|D^{2m+k}\phi\right\|^2+\left(\frac{C_{14}^2}{2}+\frac{C_{15}^2}{2}\right)\left\|D^{2m+k}\psi\right\|^2 \\
&+\frac{C_{12}^2}{2}\left\|D^{2m+k}(\bar{u}-u)\right\|^4.
\end{aligned} \tag{58}$$

Similarly for $g_2$,

$$\begin{aligned}
g_2 ={}& \delta N\left(\left\|\bar{u}_t\right\|^2\right)\Delta^{2m}\bar{u}-\delta N\left(\left\|u_t\right\|^2\right)\Delta^{2m}u-\delta N\left(\left\|u_t\right\|^2\right)(\Delta)^{2m}U \\
&-2\delta N'\left(\left\|u_t\right\|^2\right)\int_\Omega u_t\cdot U_t\mathrm{d}x\cdot(\Delta)^{2m}u \\
={}& \delta N'\left(\left\|\sigma_t\right\|^2\right)\left(\left\|\sigma_t\right\|^2\right)'\left(\bar{u}_t-u_t\right)\Delta^{2m}\bar{u}-\delta N'\left(\left\|u_t\right\|^2\right)\left(\left\|u_t\right\|^2\right)'\left(\bar{u}_t-u_t\right)(\Delta)^{2m}u \\
&+\delta N\left(\left\|u_t\right\|^2\right)\left(\Delta^{2m}\psi\right)+\delta N'\left(\left\|u_t\right\|^2\right)\left(\left\|u_t\right\|^2\right)'\psi_t\cdot(\Delta)^{2m}u
\end{aligned}$$

$$=\delta N'\left(\|\sigma_t\|^2\right)\left(\|\sigma_t\|^2\right)'(\bar{u}_t-u_t)\Delta^{2m}\bar{u}-\delta N'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'(\bar{u}_t-u_t)\Delta^{2m}\bar{u}$$

$$+\delta N'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'(\bar{u}_t-u_t)\left(\Delta^{2m}\bar{u}-(\Delta)^{2m}u\right)+\delta N\left(\|u_t\|^2\right)\left(\Delta^{2m}\psi\right)$$

$$+\delta N'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'\psi_t\cdot(\Delta)^{2m}u$$

$$=Z''(\omega_t)S_4(\bar{u}_t-u_t)^2\Delta^{2m}\bar{u}+\delta N'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'(\bar{u}_t-u_t)\left(\Delta^{2m}\bar{u}-(\Delta)^{2m}u\right)$$

$$+\delta N\left(\|u_t\|^2\right)\left(\Delta^{2m}\psi\right)+\delta N'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'\psi_t\cdot(\Delta)^{2m}u$$

$$Z'(\omega_t)=\delta N'\left(\|\omega_t\|^2\right)\left(\|\omega_t\|^2\right)',\sigma_t=S_4\bar{u}+(1-S_4)u_t,\omega_t=S_5\sigma_t+(1-S_5)u_t$$

Take the inner product of $g_2$ and $(-\Delta)^k\phi$, and deal with the term by using Young's inequality,

Poincare's inequality,

$$\begin{aligned}&\left|\left(Z''(\omega_t)S_4(\bar{u}_t-u_t)^2\Delta^{2m}\bar{u},(-\Delta)^k\phi\right)\right|\\&\le C_{16}\|\bar{u}-u\|_{E_k}^2\left\|D^k\phi\right\|\le\frac{C_{16}^2}{2}\|\bar{u}-u\|_{E_k}^4+\frac{1}{2}\left\|D^k\phi\right\|^2,\end{aligned}\tag{59}$$

where $C_{16}=\left\|Z''(\omega_t)S_4\Delta^{2m}\bar{u}\right\|$.

$$\begin{aligned}&\left|\left(\delta N'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'(\bar{u}_t-u_t)\left(\Delta^{2m}\bar{u}-(\Delta)^{2m}u\right),(-\Delta)^k\phi\right)\right|\\&\le C_{17}\|\bar{u}_t-u_t\|\left\|D^{2m+k}(\bar{u}-u)\right\|\left\|D^{2m+k}\phi\right\|\\&\le\frac{C_{17}}{2\lambda_1^{\frac{2m+k}{2}}}\left(\left\|D^{2m+k}(\bar{u}_t-u_t)\right\|^2+\left\|D^{2m+k}(\bar{u}-u)\right\|^2\right)\left\|D^{2m+k}\phi\right\|\\&\le C_{18}\|\bar{u}-u\|_{E_k}^2\left\|D^{2m+k}\phi\right\|\\&\le\frac{C_{18}^2}{2}\|\bar{u}-u\|_{E_k}^4+\frac{1}{2}\left\|D^{2m+k}\phi\right\|^2,\end{aligned}\tag{60}$$

where $C_{17}=\left\|\delta N'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'\right\|,C_{18}=\frac{C_{14}}{2\lambda_1^{\frac{2m+k}{2}}}$.

$$\begin{aligned}&\left|\left(\delta N'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'\psi_t\cdot(\Delta)^{2m}u,(-\Delta)^k\phi\right)\right|\\&\le C_{16}\left|\left((\phi-\varepsilon\psi)\Delta^{2m}u,(-\Delta)^k\phi\right)\right|\\&\le C_{17}\left|\left((\phi-\varepsilon\psi),(-\Delta)^k\phi\right)\right|\\&\le C_{17}\left|\left(\phi,(-\Delta)^k\phi\right)\right|+\left|\left(\varepsilon\psi,(-\Delta)^k\phi\right)\right|\\&\le\frac{C_{17}}{2}\varepsilon^2\left\|D^k\psi\right\|^2+C_{17}\left\|D^k\phi\right\|^2,\end{aligned}\tag{61}$$

where $C_{17}=\left\|\delta N'\left(\|u_t\|^2\right)\left(\|u_t\|^2\right)'\right\|$,

$$\left|\left(\delta N\left(\|u_t\|^2\right)\left(\Delta^{2m}\psi\right),(-\Delta)^k\phi\right)\right| \le \frac{C_{18}^2}{2}\left\|D^{2m+k}\phi\right\|^2+\frac{1}{2}\left\|D^{2m+k}\psi\right\|^2, \tag{62}$$

where $C_{18}=\left\|\delta N\left(\|u_t\|^2\right)\right\|$.

Then

$$\begin{aligned}\left|\left(g_2,(-\Delta)^k\phi\right)\right| \le& \left(\frac{C_{16}^2}{2}+\frac{C_{18}^2}{2}\right)\|\bar{u}-u\|_{E_k}^4+\frac{2C_{17}+1}{2}\left\|D^k\phi\right\|^2+\frac{1+C_{18}^2}{2}\left\|D^{2m+k}\phi\right\|^2 \\ &+\frac{1}{2}\left\|D^{2m+k}\psi\right\|^2+\frac{C_{17}}{2}\varepsilon^2\left\|D^k\psi\right\|^2.\end{aligned} \tag{63}$$

Take the inner product of $(-\Delta)^k\phi$ and the Formula (52), and combine the above formula to get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k\phi\right\|^2+\varepsilon^2\left\|D^k\psi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2\right) \\ &+\varepsilon^3\left\|D^k\psi\right\|^2+\left(\alpha\varepsilon-\beta\varepsilon^2\right)\left\|D^{2m+k}\psi\right\|^2 \\ &\le\left(\varepsilon+\frac{1+2C_{17}}{2}\right)\left\|D^k\phi\right\|^2+\frac{1+C_{14}^2+C_{18}^2}{2}\left\|D^{2m+k}\psi\right\|^2 \\ &+\frac{C_9^2+C_{16}^2+C_{18}^2}{2}\|\bar{u}-u\|_{E_k}^4+\frac{C_{12}^2}{2}\left\|D^{2m+k}(\bar{u}-u)\right\|^4 \\ &+\frac{C_{17}}{2}\varepsilon^2\left\|D^k\psi\right\|^2+\left(2+\frac{1+C_{18}^2}{2}-\beta\right)\left\|D^{2m+k}\phi\right\|^2.\end{aligned} \tag{64}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k\phi\right\|^2+\varepsilon^2\left\|D^k\psi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2\right) \\ &\le C_{19}\left(\left\|D^k\phi\right\|^2+\varepsilon^2\left\|D^k\psi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2\right)+C_{20}\left\|D^{2m+k}(\bar{u}-u)\right\|^4,\end{aligned} \tag{65}$$

where $C_{19}=2\max\left\{\varepsilon+\frac{1+2C_{17}}{2},\frac{1+C_{14}^2+C_{18}^2}{2}-\alpha\varepsilon-\beta\varepsilon^2,\frac{C_{17}}{2}\varepsilon^2-\varepsilon^3\right\}$.

From the Gronwall's inequality

$$\begin{aligned}&\left\|D^k\phi\right\|^2+\varepsilon^2\left\|D^k\psi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2 \\ &\le C_{20}\mathrm{e}^{C_{21}t}\int_0^t\left\|D^{2m+k}(\bar{u}-u)\right\|^4\mathrm{d}\tau\le C_{22}\mathrm{e}^{C_{23}t}\|\xi-\eta\|_{E_k}^4,\end{aligned} \tag{66}$$

$$\left\|D^k\phi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2\le C_{22}\mathrm{e}^{C_{23}t}\|\xi-\eta\|_{E_k}^4, \tag{67}$$

So $S(t)\bar{\phi}_0-S(t)\phi_0-\left(U(t),U_t(t)\right)$ is bounded.

When $\|(\xi,\eta)\|_{E_k}^2\to 0$, there is

$$\frac{\left\|S(t)\bar{\phi}_0-S(t)\phi_0-\left(U(t),U_t(t)\right)\right\|^2}{\|(\xi,\eta)\|^2}\le C_{22}\mathrm{e}^{C_{23}t}\|\xi-\eta\|_{E_k}^2\to 0.$$

Lemma 5 is proved.

**Theorem 5.** By lemma 5, for some n, define

$$q_n(t)=\sup_{\Psi_0\in D_{0k}}\sup_{\substack{\eta_j\in E_k\\ \|\eta_j\|_{E_k}\le 1}}\left(\frac{1}{t}\int_0^t trF'\left(S(\tau)\Psi_0\right)\cdot Q_n(\tau)\mathrm{d}\tau\right)<0,$$

then a family of global attracters $A_k$ of the initial boundary value problem (1) - (3) has Hausdorff dimension and Fractal dimension is finite, and

$$d_H(A_k) < \frac{2}{7}n,\ d_F(A_k) < \frac{4}{7}n.$$

**Proof.** Let $P_\varepsilon : \{u, u_t\} \to \{u, u_t + \varepsilon u\}$ be an isomorphic mapping, then

$$\Psi = P_\varepsilon \bar{\varphi} = (U, V)^{\mathrm{T}},$$

where

$$\bar{\varphi} = (U, U_t)^{\mathrm{T}}, V = U_t + \varepsilon U.$$

The Frechet differentiability of $S(t): E_k \to E_k$ is known from lemma 5 to estimate the Hausdorff dimension and Fractal dimension of problem (48) - (50). The variational equation of Equation (49) under initial conditions is considered

$$\Psi_t + \Lambda_\varepsilon \Psi = 0, \tag{68}$$

$$\Lambda_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ (\delta N(\|D^m u_t\|^2) + \gamma M(\|D^m u\|_p^p) + \alpha - \beta\varepsilon)(\Delta)^{2m} & \\ +\gamma M'(\|D^m u\|_p^p)(\|D^m u\|_p^p)'\Delta^{2m} u D^m) & (\beta(\Delta)^{2m} + \delta N'(\|D^m u_t\|^2)(\|D^m u_t\|^2)' \\ -\varepsilon k_0 (\Delta)^{2m} u D^m & \cdot(\Delta)^{2m} u D^m - \varepsilon) I \\ \varepsilon^2) I & \end{pmatrix},$$

where $k_0 = \delta N'\left(\|D^m u_t\|^2\right)\left(\|D^m u_t\|^2\right)'$.

For fixed $(u_0, v_0) \in E_k$, let $\gamma_1, \gamma_2, \cdots, \gamma_n$ be $n$ elements of $E_k$, and let $U_1(t), U_2(t), \cdots, U_n(t)$ be $n$ solutions of the linear Equation (68) with corresponding initial values of $U_1(0) = \gamma_1, U_2(0) = \gamma_2, \cdots, U_n(0) = \gamma_n$ respectively, we can obtain

$$\begin{aligned} &\|U_1(t) \Lambda U_2(t) \Lambda \cdots \Lambda U_n(t)\|_{\Lambda E_k}^2 \\ &= \|\gamma_1 \Lambda \gamma_1 \Lambda \cdots \Lambda \gamma_n\|_{\Lambda E_k} \exp\left(\int_0^t tr F'(\Psi(\tau)) \cdot Q_n(\tau) \mathrm{d}\tau\right), \end{aligned}$$

where $\Lambda$ represents the outer product, *tr* represents the trace of the operator, and $Q_N$ is the orthogonal projection from the space $E_k$ to $span\{U_1(t), U_2(t), \cdots, U_n(t)\}$.

For a given time $\tau$, let $\omega_j(\tau) = (\xi_j(\tau), \eta_j(\tau))^{\mathrm{T}}$, $j = 1, 2, \cdots, n$ be an standard orthonormal basis for $span\{U_1(t), U_2(t), \cdots, U_n(t)\}$.

Define an inner product on $E_k$

$$((\xi, \eta), (\bar{\xi}, \bar{\eta})) = \left(\left(D^{2m+k}\xi, D^{2m+k}\bar{\xi}\right) + \left(D^k \eta, D^k \bar{\eta}\right)\right).$$

And

$$\begin{aligned} tr F'(\Psi(\tau)) \cdot Q_n(\tau) &= -\sum_{j=1}^n \left(\Lambda_\varepsilon(\Psi(\tau)) \cdot Q_n(\tau) \omega_j(\tau), \omega_j(\tau)\right)_{E_k} \\ &= -\sum_{j=1}^n \left(\Lambda_\varepsilon(\Psi(\tau)) \omega_j(\tau), \omega_j(\tau)\right)_{E_k}, \end{aligned}$$

where

$$\begin{aligned}
&\left(\Lambda_\varepsilon \omega_j, \omega_j\right) \\
&= \Bigg( \Bigg( \varepsilon \xi_j - \eta_j, \Big[ \left( \delta N \left( \left\| D^m u_t \right\|^2 \right) + \gamma M \left( \left\| D^m u \right\|_p^p \right) + \alpha \right) (\Delta)^{2m} \\
&\quad + \gamma M' \left( \left\| D^m u \right\|_p^p \right) \left( \left\| D^m u \right\|_p^p \right)' \Delta^{2m} u D^m - \varepsilon \beta \Big] \xi_j \\
&\quad - \varepsilon \delta N' \left( \left\| D^m u_t \right\|^2 \right) \left( \left\| D^m u_t \right\|^2 \right)' \cdot (\Delta)^{2m} u D^m \xi_j + \varepsilon^2 \xi_j + \beta (\Delta)^{2m} \eta_j \\
&\quad + \delta N' \left( \left\| D^m u_t \right\|^2 \right) \left( \left\| D^m u_t \right\|^2 \right)' \cdot (\Delta)^{2m} u D^m \eta_j - \varepsilon \eta_j \Bigg), \left( \xi_j, \eta_j \right) \Bigg) \\
&= \varepsilon \left\| D^{2m+k} \xi_j \right\|^2 - \left( \alpha - \beta \varepsilon - 1 + C + A \right) \left( D^{2m+k} \eta_j, D^{2m+k} \xi_j \right) \\
&\quad + \left( B \Delta^{2m} u - \varepsilon k_0 \Delta^{2m} u \right) \left( D^{m+k} \xi_j, D^k \eta_j \right) + \varepsilon^2 \left( D^k \xi_j, D^k \eta_j \right) \\
&\quad + k_0 \Delta^{2m} u \left( D^{m+k} \eta_j, D^k \eta_j \right) + \beta \left( \Delta^{2m} \eta_j, \Delta^k \eta_j \right) - \varepsilon \left( D^k \eta_j, D^k \eta_j \right) \\
&= \varepsilon \left\| D^{2m+k} \xi_j \right\|^2 - \frac{\alpha - \beta \varepsilon - 1 + C + A}{2} \left\| D^{2m+k} \eta_j \right\|^2 - \frac{\alpha - \beta \varepsilon - 1 + C + A}{2} \left\| D^{2m+k} \xi_j \right\|^2 \\
&\quad - \frac{B_1}{2} \left\| D^{m+k} \xi_j \right\|^2 - \frac{B_1}{2} \left\| D^k \eta_j \right\|^2 - \frac{\varepsilon^2}{2} \left\| D^k \xi_j \right\|^2 - \frac{\varepsilon^2}{2} \left\| D^k \eta_j \right\|^2 - \frac{B_2}{2} \left\| D^{m+k} \eta_j \right\|^2 \\
&\quad - \frac{B_2}{2} \left\| D^k \eta_j \right\|^2 + \beta \left\| D^{2m+k} \eta_j \right\|^2 - \varepsilon \left\| D^k \eta_j \right\|^2 \\
&\geq \left( \varepsilon - k_1 \right) \left\| D^{2m+k} \xi_j \right\|^2 + k_2 \left\| D^{2m+k} \eta_j \right\|^2,
\end{aligned}$$

where

$$k_1 = \min \left\{ \frac{\alpha - \beta \varepsilon - 1 + C + A}{2}, \frac{B_1 \lambda^{-m}}{2} \right\},$$

$$k_2 = \beta - \frac{\alpha - \beta \varepsilon - 1 + C + A}{2} - \left( \frac{B_1}{2} + \frac{\varepsilon^2}{2} + \frac{B_2}{2} + \varepsilon \right) \lambda^{-2m} - \frac{B_2}{2} \lambda^{-m}$$

$$\left( \Lambda_\varepsilon \left( \Psi(\tau) \right) \omega_j(\tau), \omega_j(\tau) \right)_{E_k} \geq \left( \varepsilon - k_1 \right) \left\| D^{2m+k} \xi_j \right\|^2 + k_2 \left\| D^{2m+k} \eta_j \right\|^2. \tag{69}$$

Owing to the

$$\omega_j(\tau) = \left( \xi_j(\tau), \eta_j(\tau) \right)^{\mathrm{T}}, j = 1, 2, \cdots, n,$$

is an orthonormal basis for $span\left\{ U_1(t), U_2(t), \cdots, U_n(t) \right\}$, so

$$\sum_{j=1}^{n} \left( F'\left( \psi(\tau) \right) \omega_j(\tau), \omega_j(\tau) \right)_{E_k} \leq -n\left( \varepsilon - k_1 \right) + r \sum_{j=1}^{n} \left\| D^{2m+k} \xi_j \right\|^2, \tag{70}$$

For any *t*, there is

$$\sum_{j=1}^{n} \left\| D^k \xi_j \right\|^2 \leq \sum_{j=1}^{n} \lambda_j^{s-1},$$

So

$$TrF'\left( \Psi(\tau) \right) \cdot Q_n(\tau) \leq -nC_{24} + r \sum_{j=1}^{n} \lambda_j^{s-1}. \tag{71}$$

Let

$$q_n(t) = \sup_{\Psi_0 \in D_{0k}} \sup_{\substack{\eta_j \in E_k \\ \|\eta_j\|_{E_k} \le 1}} \left( \frac{1}{t} \int_0^t trF'(S(\tau)\Psi_0) \cdot Q_n(\tau) \mathrm{d}\tau \right), q_n = \lim_{t\to\infty} \sup q_n(t),$$

then

$$q_n \le -nC_{24} + r\sum_{j=1}^{n} \lambda_j^{s-1}.$$

Therefore, the Lyapunov exponent $\tilde{\mu}_1, \tilde{\mu}_2, \cdots, \tilde{\mu}_j$ of $D_{0k}$ is uniformly bounded, and

$$\tilde{\mu}_1 + \tilde{\mu}_2 + \cdots + \tilde{\mu}_j \le -nC_{24} + r\sum_{j=1}^{n} \lambda_j^{s-1}. \tag{72}$$

There is a $s \in [0,1)$, such that

$$(q_j)_+ \le -nC_{24} + r\sum_{j=1}^{n} \lambda_j^{s-1} \le r\sum_{j=1}^{n} \lambda_j^{s-1} \le \frac{nC_{24}}{8}. \tag{73}$$

where $\lambda_j$ is the eigenvalue of $(-\Delta)^m$, and $\lambda_1 < \lambda_2 < \cdots < \lambda_{2m}$.

$$q_n \le -\frac{nC_{24}}{2}\left(1 - \frac{2r}{nC_{24}}\sum_{j=1}^{n} \lambda_j^{s-1}\right) \le -\frac{7}{16} nC_{24}. \tag{74}$$

Then

$$\max_{1\le j\le n} \frac{(q_j)_+}{|q_n|} \le \frac{2}{7}. \tag{75}$$

It can be concluded that n-dimensional volume elements decay exponentially in $E_k$ and $d_H(A_k) < \frac{2}{7}n, d_F(A_k) < \frac{4}{7}n$, so the Hausdorff dimension and Fractal dimension of the whole attractor family are limited. Theorem 5 is proved.

## 4. Conclusion

We prove the existence and uniqueness of the global solution and the existence of a family of global attractors, and estimate the upper bound of the Hausdorff dimension for the family global attractors, and obtain the global stability of the problem.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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