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Blowup of Solutions to the Non-Isentropic Compressible Euler Equations with Time-Dependent Damping and Vacuum

Yuping Feng[1], **Huimin Yu**[1], **Wanfang Shen**[2*]

[1]Department of Mathematics, Shandong Normal University, Jinan, China
[2]Shandong Key Laboratory of Blockchain Finance, Shandong University of Finance and Economics, Jinan, China
Email: ypfeng1128@163.com, hmyu@sdnu.edu.cn, *wfshen@sdufe.edu.cn

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Abstract

This paper mainly studies the blowup phenomenon of solutions to the compressible Euler equations with general time-dependent damping for non-isentropic fluids in two and three space dimensions. When the initial data is assumed to be radially symmetric and the initial density contains vacuum, we obtain that classical solution, especially the density, will blow up on finite time. The results also reveal that damping can really delay the singularity formation.

Keywords

Compressible Euler Equations, Blowup, General Time-Dependent Damping, Vacuum

1. Introduction

The Euler equations with general time-dependent damping for compressible non-isentropic fluids are of the form:

$$\begin{cases} \rho_t + \nabla\cdot(\rho\boldsymbol{u}) = 0, \\ (\rho\boldsymbol{u})_t + \nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p = -\alpha(t)\rho\boldsymbol{u}, \\ S_t + \boldsymbol{u}\cdot\nabla S = 0, \end{cases} \tag{1.1}$$

where $\boldsymbol{x},t$ denote space variable and time variable, $(\boldsymbol{x},t)\in R^n\times[0,\infty)$, $n=2,3$. Here, ρ, $\boldsymbol{u}$, p and S denote the density, velocity, pressure and entropy respectively, $\alpha(t)>0$ is the friction coefficient. The pressure p is given by the state equation

$$p = Ae^{S}\rho^{\gamma}, \tag{1.2}$$

where $A > 0$ and $\gamma > 1$ are the entropy constant and adiabatic index. The symbols ∇ and $\nabla\cdot$ represent gradient and divergence on space variables respectively and $\otimes$ represent convolution.

In recent years, many scholars have researched the blowup problem of the solution of compressible Euler equations. For the case that Euler equations without damping, by considering the functional $F(t) = \int_{R^3} \rho \boldsymbol{x} \cdot \boldsymbol{u} dx$, Sideris [1] showed the first general blowup result in the three space dimensions case without symmetry. The finite-time blowup result is based on the finite propagation speed property established for nonlinear hyperbolic equations. For the compressible Euler system with constant damping coefficient, Sideris *et al.* [2] proved the global existence result under the condition that the initial energy functional is sufficiently small. Then Cheung [3] [4], Yuen [5] showed that finite-time singularity will be developed for solutions of compressible Euler equations without damping or with constant damping by constructing different functional. For the case with time-dependent damping $\frac{\mu}{(1+t)^{\lambda}}$, by introducing a new functional $F(t) = \int_1^{\infty} \rho(t,r) u(t,r)\left[r^N - \alpha(t)\right] dr$ with the time-dependent parameter $\alpha(t)$, Cheung [6] obtained the result of finite time singularity formation for the initial-boundary value problem of the multidimensional compressible Euler equations with $\mu > 0$, $\lambda \geq 0$, and thus obtained a lower bound of the finite-time blowup condition for the initial kinetic energy of the fluid. However, they did not consider the case of initial density contains a vacuum state.

For the case of the initial density contains a vacuum state, Jang *et al.* [7] [8] [9] gave the local well-posedness of classical solutions of one space dimension and three space dimensions compressible Euler equations with physical vacuum. Lei [10] gave two results on the formation of a finite time singularity in the solutions of compressible Euler equations in two and three dimensions of isentropic ideal fluid flows. It has shown that for radially symmetric initial data where the initial density contains vacuum at $r = 0$, the solution of compressible Euler equations will obtain finite time singularities. Subsequently, based on [10], Liu *et al.* [11] proved the finite-time blowup results for isentropic compressible Euler equations with the time-dependent damping $\frac{\mu}{(1+t)^{\lambda}}$. However, there are few research results for non-isentropic case. We refer [12] for the local well-posedness of classical solutions for non-isentropic Euler equations with physical vacuum.

In this paper, for non-isentropic compressible Euler equations with a general time damping, *i.e.* (1.1), we consider the blowup phenomena of the classical solutions with vacuum and radially symmetric initial data. The motivations of the paper are: 1) to answer the question whether the density can concentrate at finite time if the initial density contains vacuum at one point; 2) the impact of time-depend damping on the singularity formation for the full Euler equation. Our results answer these two questions by strict mathematical derivation. Following the work of Lei [11], we select the modified Bessel function $K_0(r)$ and $\frac{1}{re^r}$ to multiply the equations in two and three space dimensions, respectively, to explore the nonlinear structure of the pressure p and overcome the difficulties brought by the point $r=0$, and finally prove the density will blowup on finite time at any small ball centered at origin. In addition, since we are considering the non-isentropic case, which bring some new difficulties to our analysis, we need to estimate entropy $S(x,t)$ along the particle path in advance. The structure of this paper is as follows: in Section 1, we give the main results of problem (1.1) - (1.4). In Sections 2 - 4, the proofs of the blowup results are given in three and two space dimensions respectively.

The compressible Euler Equations (1.1) are imposed on the following initial data

$$\rho(0,\boldsymbol{x})=\rho_0(\boldsymbol{x}), \boldsymbol{u}(0,\boldsymbol{x})=\boldsymbol{u}_0(\boldsymbol{x}), S(0,\boldsymbol{x})=S_0(\boldsymbol{x})\geq 0, \tag{1.3}$$

which satisfy

$$\rho_0(\boldsymbol{x})=\rho_0(r), \boldsymbol{u}_0(\boldsymbol{x})=\frac{\boldsymbol{x}}{r}v_0(r), S_0(\boldsymbol{x})=S_0(r)\geq 0, \tag{1.4}$$

where $r=|\boldsymbol{x}|$, $\boldsymbol{x}\in R^n, n=2,3$. We will first give the blowup result of problem (1.1) - (1.4) in three space dimensions.

Theorem 1.1 *Assume that* $\rho_0\geq 0$, $\gamma>1$ *and* $(c_0,\boldsymbol{u}_0,S_0)\in H^3(R^3)$. *Consider the solution* $(\rho,\boldsymbol{u},S)$ *of the compressible Euler Equations* (1.1) *with time-dependent damping in three space dimensions. If the initial conditions satisfy*

$$\rho_0(0)=0, \tag{1.5}$$

$$\int_{R^3}\rho_0(r)\mathrm{d}x>0 \tag{1.6}$$

and

$$-\int_{R^3}\frac{(1+r)\rho_0 v_0}{r^2\mathrm{e}^r}\mathrm{d}x\geq\sqrt{\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}}\left(\int_{R^3}\frac{\rho_0}{r\mathrm{e}^r}\mathrm{d}x\right)^{\frac{\gamma+1}{2}}, \tag{1.7}$$

then for any $r_0>0$ *the smooth solutions* (*more specifically, it is the density* $\rho(t,r)$) *will blow up for* $r\leq r_0$ *as t goes to infinity.*

Next we present the blowup result of problem (1.1) - (1.4) in two space dimensions.

Theorem 1.2 *Assume that* $\rho_0\geq 0$, $\gamma>1$ *and* $(c_0,\boldsymbol{u}_0,S_0)\in H^3(R^2)$. *Let* $K_0(r)$ *be the modified Bessel function*

$$K_0(r)=\int_0^\infty \mathrm{e}^{-r\cosh t}\mathrm{d}t. \tag{1.8}$$

Consider the solution $(\rho,\boldsymbol{u},S)$ *of the compressible Euler Equations* (*1.1*) *with time-dependent damping in two space dimensions. If the initial conditions satisfy*

$$\rho_0(0)=0, \tag{1.9}$$

$$\int_{R^2}\rho_0(r)\,\mathrm{d}x>0 \tag{1.10}$$

and

$$-\int_{R^2}\rho_0 v_0 K_0'(r)\,\mathrm{d}x \ge \sqrt{\frac{2A}{\gamma+1}}\frac{\left(\int_{R^2}\rho_0(r)K_0(r)\,\mathrm{d}x\right)^{\frac{\gamma+1}{2}}}{\left(\int_{R^2}K_0(r)\,\mathrm{d}x\right)^{\frac{\gamma-1}{2}}}, \tag{1.11}$$

then for any $r_0>0$ *the smooth solutions (more specifically, it is the density* $\rho(t,r)$*) will blow up for* $r\le r_0$ *as t goes to infinity.*

Remark 1.1 *From the proof of Theorem 1.1, the singularity formation theorem can be extend to any space dimensions* $n\ge 3$ *by using the same test function* $\frac{1}{re^r}$. *We refer [11] for isentropic case, the non-isentropic case is the same, here we omit the details.*

Remark 1.2 *Superficially, Theorem 1.1 and Theorem 1.2 are same as which in [10] for isentropic Euler equation without damping. However, for a given initial data, damping can delay the time of singularity formation, indeed. We can see this assertion from the proof in Section 3 (or Section 4):* $\rho(t,r)$ *will blow up in* B_{r_0} *as* $\int_0^t e^{-\int_0^\eta \alpha(\tau)\mathrm{d}\tau}\mathrm{d}s \to \frac{2}{(\gamma-1)C_0}H(0)^{-\frac{\gamma-1}{2}}$ *(or* $\int_0^t e^{-\int_0^\eta \alpha(\tau)\mathrm{d}\tau}\mathrm{d}s \to \frac{2}{(\gamma-1)C_1}G(0)^{-\frac{\gamma-1}{2}}$*).*

2. Some Useful Assertions

Next we will give some properties on smooth solutions of problem (1.1) - (1.4) with initial vacuum on $r=0$. Our main result is:

We first review the well-posedness of compressible Euler Equations (1.1) - (1.3) by Lemma 2.1.

Lemma 2.1 *Assume that* $\rho_0\ge 0$, $\gamma>1$ *and* $(\rho_0,\boldsymbol{u}_0,S_0)\in H^3(R^n)$, $n=2,3$, *then there exists a unique solution* $(\rho,\boldsymbol{u},S)$ *to the compressible Euler Equations (1.1) - (1.3) on some time interval* $[0,T)$, *which satisfies*

$$\rho\in C\left([0,T)\times R^n\right) \tag{2.1}$$

and

$$c,\boldsymbol{u},S\in C\left([0,T),H^3(R^n)\right)\cap C^1\left([0,T),H^2(R^n)\right)\cap C^2\left([0,T),H^1(R^n)\right). \tag{2.2}$$

Proof: We only give the proof of $n=3$ briefly (The case $n=2$ is similar). In the beginning, as which in [13], we rewrite the Euler equations with time dependent damping (1.1) in the form of a quasi-linear symmetric hyperbolic equations by introducing two new variables

$$c(\rho)=\sqrt{\frac{\partial p(\rho)}{\partial\rho}}=\sqrt{Ae^S\gamma}\,\rho^{\frac{\gamma-1}{2}}, \tag{2.3}$$

and

$$\tilde{c} = \sqrt{\frac{A}{\gamma - 1}} \rho^{\frac{\gamma - 1}{2}}. \tag{2.4}$$

From (1.1) (2.3) (2.4) and a series of calculations, we get

$$\begin{cases} \frac{4}{\gamma-1} e^S \tilde{c}_t + \frac{4}{\gamma-1} e^S \boldsymbol{u} \cdot \nabla \tilde{c} + 2e^S \tilde{c} \nabla \cdot \boldsymbol{u} = 0, \\ \frac{4}{(\gamma-1)\gamma} c_t + \frac{4}{(\gamma-1)\gamma} \boldsymbol{u} \cdot \nabla c + \frac{2}{\gamma} c \nabla \cdot \boldsymbol{u} = 0, \\ \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{2}{\gamma} c \nabla c + 2e^S \tilde{c} \nabla \tilde{c} = -\alpha(t) \boldsymbol{u}, \\ S_t + \boldsymbol{u} \cdot \nabla S = 0. \end{cases} \tag{2.5}$$

Rewrite (2.5) in the following matrix form for $n = 3$:

$$A_0 \frac{\partial V}{\partial t} + \sum_{i=1}^{3} A_i \frac{\partial V}{\partial x_i} = D, \tag{2.6}$$

where

$$V = (\tilde{c}, c, u_1, u_2, u_3, S)^{\mathrm{T}}, \tag{2.7}$$

$$A_0 = \begin{pmatrix} \frac{4}{\gamma-1} e^S & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{(\gamma-1)\gamma} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.8}$$

$$A_1 = \begin{pmatrix} \frac{4}{\gamma-1} e^S u_1 & 0 & 2e^S \tilde{c} & 0 & 0 & 0 \\ 0 & \frac{4}{(\gamma-1)\gamma} u_1 & \frac{2}{\gamma} c & 0 & 0 & 0 \\ 2e^S \tilde{c} & \frac{2}{\gamma} c & u_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 \end{pmatrix}, \tag{2.9}$$

$$A_2 = \begin{pmatrix} \frac{4}{\gamma-1} e^S u_2 & 0 & 0 & 2e^S \tilde{c} & 0 & 0 \\ 0 & \frac{4}{(\gamma-1)\gamma} u_2 & 0 & \frac{2}{\gamma} c & 0 & 0 \\ 0 & 0 & u_2 & 0 & 0 & 0 \\ 2e^S \tilde{c} & \frac{2}{\gamma} c & 0 & u_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_2 \end{pmatrix}, \tag{2.10}$$

$$A_3=\begin{pmatrix} \frac{4}{\gamma-1}\mathrm{e}^S u_3 & 0 & 0 & 0 & 2\mathrm{e}^S\tilde{c} & 0 \\ 0 & \frac{4}{(\gamma-1)\gamma}u_3 & 0 & 0 & \frac{2}{\gamma}c & 0 \\ 0 & 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_3 & 0 & 0 \\ 2\mathrm{e}^S\tilde{c} & \frac{2}{\gamma}c & 0 & 0 & u_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_3 \end{pmatrix}, \tag{2.11}$$

$$D=\left(0,0,-\alpha(t)u_1,-\alpha(t)u_2,-\alpha(t)u_3,0\right)^{\mathrm{T}}. \tag{2.12}$$

Denote $c_0=c(\rho_0),\tilde{c}_0=\tilde{c}(\rho_0)$, then $\tilde{c}_0=\mathrm{e}^{-\frac{S_0}{2}}c_0$. Since $\rho_0,S_0\in H^3(R^n)$ then $\mathrm{e}^{-\frac{S_0}{2}},c_0,\tilde{c}_0\in H^3(R^n)$. Then by the local existence theorem of symmetric hyperbolic equations [14], we know there exists $T>0$ such that when $t\in[0,T)$, (2.6) has a unique solution $(\tilde{c},c,\boldsymbol{u},S)$ and satisfies

$$\tilde{c},c,\boldsymbol{u},S\in C\left([0,T),H^3(R^n)\right)\cap C^1\left([0,T),H^2(R^n)\right)\cap C^2\left([0,T),H^1(R^n)\right). \tag{2.13}$$

According to Sobolev embedding theorem, we have $\rho\in C(R^n)$. The proof of Lemma 2.1 is completed.

Lemma 2.2 *Suppose* $(\rho,\boldsymbol{u},S)(t,\boldsymbol{x})$ *is the solution of problem* (1.1) - (1.3) *with radially symmetric initial data* (1.4) *and* $\rho_0(0)=0$, *then*

$$\rho(t,0)\equiv 0, \tag{2.14}$$

$$\boldsymbol{u}(t,0)\equiv \boldsymbol{0}, \tag{2.15}$$

$$\boldsymbol{u}(t,\boldsymbol{x})=\frac{\boldsymbol{x}}{r}v(t,r) \tag{2.16}$$

and

$$S(t,\boldsymbol{x})\geq 0. \tag{2.17}$$

Proof. For smooth and radial symmetric vector $\boldsymbol{u}(t,\boldsymbol{x})$, we can naturally get (2.15). By $(1.1)_1$ and (2.15), we have

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}(t,0)=-\nabla\cdot(\rho\boldsymbol{u})(t,0)=-(\nabla\cdot\boldsymbol{u})(t,0)\rho(t,0). \tag{2.18}$$

Then integrating the above equation over $[0,t]$ and noticing $\rho_0(0)=0$, we have

$$\rho(t,0)=\rho_0(0)\mathrm{e}^{-\int_0^t\nabla\cdot u(\tau,0)\mathrm{d}\tau}\equiv 0, \tag{2.19}$$

thus (2.14) is established.

Next, we consider the following initial value problem of ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}=\boldsymbol{u}(t,\boldsymbol{x}),\ \boldsymbol{x}\big|_{t=\tau}=\xi. \tag{2.20}$$

Because $u \in C^1([0,T]; H^2(R^n)) \subset B([0,T] \times R^n)$, then by the existence and uniqueness theorem of ordinary differential equations, the above problem has a unique solution

$$x = \varphi(t) \in C^1([0,T] \times R^n). \tag{2.21}$$

Thus by $(1.1)_3$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t, \varphi(t)) = 0. \tag{2.22}$$

Integrating the above equation over $[0,\tau]$, we have

$$S(\tau, \xi) = S(0, \varphi(0)) \geq 0, \tag{2.23}$$

thus (2.17) is established.

For later use, we give the following facts, Lemma 2.3 can be find in [10], to recall some facts about the modified Bessel function $K_0(r)$ and its derivative $K_0'(r)$. Lemma 2.4 is the Sobolev embedding theorem, which can be find in [15], and it will be used later to prove the local existence of the solution.

Lemma 2.3 (Lemma 4.1 in [10]). *The modified Bessel function* $K_0(r) = \int_0^\infty \mathrm{e}^{-r\cosh t} \mathrm{d}t$ *satisfies*

$$\begin{cases} K_0(r) \leq \dfrac{3}{r}, |K_0'(r)| \leq \dfrac{1}{r^2}, & 0 < r < \dfrac{1}{2} \\ K_0(r) \leq \dfrac{C_k}{r^k}, |K_0'(r)| \leq \dfrac{C_k}{r^k}, & r > 1, \end{cases} \tag{2.24}$$

for constants C_k depending only on $k > 1$.

Lemma 2.3 gives the decay rate estimates on the modified Bessel function $K_0(r)$ and its derivative $K_0'(r)$, which will play an important role in the proof for $n = 2$.

Lemma 2.4 (*Sobolev embedding theorem*). *Suppose that* $s > \frac{n}{2} + k$ *is a real number, n is the space dimension, and k is a nonnegative integer. Then* $H^s(R^n)$ *can be embedded into* $C^k(R^n)$ *and the embedding operator is continuous.*

3. Blowup for the Smooth Solutions in Three Space Dimensions

In this section, we will give the proof of the blowup result Theorem 1.1.

Proof. Suppose that the solution $(\rho, \boldsymbol{u}, S)$ satisfies the conditions in Theorem 1.1. We apply the time derivative to $(1.1)_1$ and use $(1.1)_2$ to obtain

$$\begin{aligned} \rho_{tt} &= -\nabla \cdot (\rho \boldsymbol{u})_t = \Delta p + \nabla \cdot [\nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u})] + \alpha(t) \nabla \cdot (\rho \boldsymbol{u}) \\ &= \Delta p + \nabla \cdot [\nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u})] - \alpha(t) \rho_t. \end{aligned} \tag{3.1}$$

Then multiplying both sides of Equation (3.1) by the test function $\frac{1}{r\mathrm{e}^r}$ and integrate $\boldsymbol{x}$ in R^3, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{R^3} \frac{\rho}{r\mathrm{e}^r} \mathrm{d}x = \int_{R^3} \Delta p \frac{1}{r\mathrm{e}^r} \mathrm{d}x + \int_{R^3} \nabla \cdot [\nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u})] \frac{1}{r\mathrm{e}^r} \mathrm{d}x - \int_{R^3} \alpha(t) \rho_t \frac{1}{r\mathrm{e}^r} \mathrm{d}x. \tag{3.2}$$

Using $S>0$, $\rho(t,0)\equiv 0$, we have

$$
\begin{aligned}
\int_{R^3}\Delta p\frac{1}{r\mathrm{e}^r}\mathrm{d}x &= \int_{R^3}p\Delta\frac{1}{r\mathrm{e}^r}\mathrm{d}x-\lim_{\varepsilon\to 0}\int_{r=\varepsilon}\left(\frac{\partial p}{\partial r}\frac{\mathrm{e}^{-r}}{r}-p\frac{\mathrm{d}\left(\frac{1}{r\mathrm{e}^r}\right)}{\mathrm{d}r}\right)\mathrm{d}\xi\\
&=\int_{R^3}p\frac{1}{r\mathrm{e}^r}\mathrm{d}x+\lim_{\varepsilon\to 0}\int_{r=\varepsilon}p\left(\frac{1}{r\mathrm{e}^r}\right)'\mathrm{d}\xi\\
&=\int_{R^3}p\frac{1}{r\mathrm{e}^r}\mathrm{d}x\\
&=\int_{R^3}A\mathrm{e}^S\rho^\gamma\frac{1}{r\mathrm{e}^r}\mathrm{d}x\\
&\ge\frac{A}{(4\pi)^{\gamma-1}}\left(\int_{R^3}\frac{\rho}{r\mathrm{e}^r}\mathrm{d}x\right)^\gamma .
\end{aligned}
\tag{3.3}
$$

Then, similar to (3.3), we obtain

$$
\begin{aligned}
&\int_{R^3}\nabla\cdot\left[\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})\right]\frac{1}{r\mathrm{e}^r}\mathrm{d}x\\
&=\int_{R^3}\sum_{i,j=1}^{3}\frac{\partial^2\left(\rho u_i u_j\right)}{\partial x_i\partial x_j}\frac{1}{r\mathrm{e}^r}\mathrm{d}x\\
&=\int_{R^3}\rho v^2\left(\frac{1}{r\mathrm{e}^r}\right)''\mathrm{d}x\\
&=\int_{R^3}\rho v^2\left(\frac{1}{r}+\frac{2}{r^2}+\frac{2}{r^3}\right)\mathrm{e}^{-r}\mathrm{d}x>0.
\end{aligned}
\tag{3.4}
$$

Define $H(t)=\int_{R^3}\frac{\rho}{r\mathrm{e}^r}\mathrm{d}x$, from (3.2) we have

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(t)+\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t}H(t)\ge\frac{A}{(4\pi)^{\gamma-1}}\left(H(t)\right)^\gamma . \tag{3.5}
$$

In addition, using the equation of conservation of mass, integration by parts and (1.6), we also have

$$
\begin{aligned}
H'(0)&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^3}\frac{\rho}{r\mathrm{e}^r}\mathrm{d}x\bigg|_{t=0}\\
&=-\int_{R^3}\frac{\nabla\cdot(\rho\boldsymbol{u})}{r\mathrm{e}^r}\mathrm{d}x\bigg|_{t=0}\\
&=-\int_{R^3}\sum_{i,j=1}^{3}\rho\frac{x_i}{r}v\frac{1+r}{r^2\mathrm{e}^r}\frac{x_i}{r}\mathrm{d}x\bigg|_{t=0}\\
&=-\int_{R^3}\frac{1+r}{r^2\mathrm{e}^r}\rho_0 v_0\mathrm{d}x>0.
\end{aligned}
\tag{3.6}
$$

Now, we assert $H'(t)\ge 0$ by contradiction. The proof can also be find in [11]. If there exists $\tau>0$, such that $H'(\tau)<0$, then there is a constant τ_1 in $(0,\tau)$ satisfying

$$
\begin{cases}
H'(t)>0, & 0\le t<\tau_1,\\
H'(t)=0, & t=\tau_1,\\
H'(t)<0, & \tau_1<t\le\tau.
\end{cases}
\tag{3.7}
$$

Integrating (3.5) with respect to t over $[\tau_1,\tau]$, we have

$$\int_{\tau_1}^{\tau}\frac{\mathrm{d}^2}{\mathrm{d}t^2}H(t)\mathrm{d}t\ge\int_{\tau_1}^{\tau}\frac{A}{(4\pi)^{\gamma-1}}(H(t))^{\gamma}\mathrm{d}t-\int_{\tau_1}^{\tau}\alpha(t)\frac{\mathrm{d}}{\mathrm{d}t}H(t)\mathrm{d}t. \tag{3.8}$$

According to (3.7), we conclude $H'(\tau)\ge H'(\tau_1)=0$, which is inconsistent with the assumption $H'(\tau)<0$.

Then we multiply the both sides of (3.5) by $2H'(t)$, to have

$$\left[(H'(t))^2\right]'+2\alpha(t)(H'(t))^2\ge\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}\left(H(t)^{\gamma+1}\right)'. \tag{3.9}$$

Multiplying the both side of (3.9) by $\mathrm{e}^{\int_0^t 2\alpha(\tau)\mathrm{d}\tau}$, we have

$$\begin{aligned}\left[\mathrm{e}^{\int_0^t 2\alpha(\tau)\mathrm{d}\tau}(H'(t))^2\right]'&\ge\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}\mathrm{e}^{\int_0^t 2\alpha(\tau)\mathrm{d}\tau}\left(H(t)^{\gamma+1}\right)'\\&\ge\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}\left(H(t)^{\gamma+1}\right)',\end{aligned} \tag{3.10}$$

where we have used the fact that $\alpha(t)\ge0$. By integrating (3.10) over $[0,t]$, we have

$$\mathrm{e}^{\int_0^t 2\alpha(\tau)\mathrm{d}\tau}(H'(t))^2-\left(H'(t)\big|_{t=0}\right)^2\ge\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}\left(H(t)^{\gamma+1}-H(0)^{\gamma+1}\right), \tag{3.11}$$

which implies that

$$\begin{aligned}(H'(t))^2&\ge\mathrm{e}^{-\int_0^t 2\alpha(\tau)\mathrm{d}\tau}\left[\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}\left(H(t)^{\gamma+1}-H(0)^{\gamma+1}\right)+\left(H'(t)\big|_{t=0}\right)^2\right]\\&=\mathrm{e}^{-\int_0^t 2\alpha(\tau)\mathrm{d}\tau}\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}H(t)^{\gamma+1}\\&\quad+\mathrm{e}^{-\int_0^t 2\alpha(\tau)\mathrm{d}\tau}\left[-\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}\left(\int_{R^3}\frac{\rho_0}{r\mathrm{e}^r}\mathrm{d}x\right)^{\gamma+1}+\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^3}\frac{\rho}{r\mathrm{e}^r}\mathrm{d}x\bigg|_{t=0}\right)^2\right].\end{aligned} \tag{3.12}$$

Noting the assumption (1.7), we have

$$(H'(t))^2\ge\mathrm{e}^{-\int_0^t 2\alpha(\tau)\mathrm{d}\tau}\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}H(t)^{\gamma+1}. \tag{3.13}$$

Then we obtain

$$H'(t)\ge C_0\mathrm{e}^{-\int_0^t\alpha(\tau)\mathrm{d}\tau}H(t)^{\frac{\gamma+1}{2}} \tag{3.14}$$

by denoting

$$C_0=\sqrt{\frac{2A}{(4\pi)^{\gamma-1}(\gamma+1)}}. \tag{3.15}$$

By integrating (3.14) over $[0,t]$, we have

$$\int_0^t\frac{H'(\eta)}{H(\eta)^{\frac{\gamma+1}{2}}}\mathrm{d}\eta\ge\int_0^t C_0\mathrm{e}^{-\int_0^\eta\alpha(\tau)\mathrm{d}\tau}\mathrm{d}\eta \tag{3.16}$$

and

$$-\frac{2}{\gamma-1}H(t)^{-\frac{\gamma-1}{2}}+\frac{2}{\gamma-1}H(0)^{-\frac{\gamma-1}{2}}\geq C_0\int_0^t e^{-\int_0^\eta \alpha(\tau)\mathrm{d}\tau}\mathrm{d}\eta, \tag{3.17}$$

which implies that

$$H(t)\geq\left(H(0)^{-\frac{\gamma-1}{2}}-\frac{\gamma-1}{2}C_0\int_0^t e^{-\int_0^\eta \alpha(\tau)\mathrm{d}\tau}\mathrm{d}\eta\right)^{-\frac{2}{\gamma-1}}. \tag{3.18}$$

Observing (1.6), we have

$$H(0)=\int_{R^3}\frac{\rho_0}{re^r}\mathrm{d}x>0. \tag{3.19}$$

Using the mass conservation, we have

$$H(t)=\int_{R^3}\frac{\rho}{re^r}\mathrm{d}x\leq\int_{B_{r_0}}\frac{\rho}{re^r}\mathrm{d}x+\frac{1}{r_0}\int_{R^3}\rho_0\mathrm{d}x \tag{3.20}$$

for any given $r_0>0$, where B_{r_0} is the three space dimensions ball centered at the origin with radius r_0. Therefore, when $\int_0^t e^{-\int_0^\eta \alpha(\tau)\mathrm{d}\tau}\mathrm{d}s\to\frac{2}{(\gamma-1)C_0}H(0)^{-\frac{\gamma-1}{2}}$, $H(t)$ cannot be bounded. Then, $\int_{B_{r_0}}\rho(t,r)\mathrm{d}r$ cannot be bounded, which implies that $\rho(t,r)$ will blow up for $r\leq r_0$.

By Theorem 1.1., we get that though the initial contain vaccum, as long as the initial conditions satisfy the conditions (1.6) and (1.7), the density will blow up in a finite time. Due to the difference boundary terms in the two-dimensional and three-dimensional, we choose different test functions.

4. Blowup for the Smooth Solutions in Two Space Dimensions

In this section, we will give the proof of the blowup result of the Theorem 1.2.

Proof. Suppose that the solution $(\rho,\boldsymbol{u},S)$ satisfies the conditions in Theorem 1.2. We apply the time derivative to $(1.1)_1$ and by $(1.1)_2$, to obtain

$$\begin{aligned}\rho_{tt}&=-\nabla\cdot(\rho\boldsymbol{u})_t=\Delta p+\nabla\cdot[\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})]+\alpha(t)\nabla\cdot(\rho\boldsymbol{u})\\&=\Delta p+\nabla\cdot[\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})]-\alpha(t)\rho_t.\end{aligned} \tag{4.1}$$

Multiplying both sides of Equation (4.1) by the function $K_0(r)$ and integrate $\boldsymbol{x}$ in R^2, we obtain

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{R^2}\rho K_0(r)\mathrm{d}x&=\int_{R^2}\Delta pK_0(r)\mathrm{d}x+\int_{R^2}\nabla\cdot[\nabla\cdot(\rho\boldsymbol{u}\otimes\boldsymbol{u})]K_0(r)\mathrm{d}x\\&\quad-\int_{R^2}\alpha(t)\rho_tK_0(r)\mathrm{d}x.\end{aligned} \tag{4.2}$$

Using the state equation, $\rho(t,\mathbf{0})\equiv0$, $S>0$, we obtain

$$\begin{aligned}\int_{R^2}\Delta pK_0(r)\mathrm{d}x&=\int_{R^2}\nabla\cdot(\nabla pK_0(r))-\nabla p\cdot\nabla K_0(r)\mathrm{d}x\\&=-\lim_{\varepsilon\to0}\int_{r=\varepsilon}\frac{\partial p}{\partial r}K_0(r)\mathrm{d}\xi-\int_{R^2}\nabla\cdot(p\nabla K_0(r))-p\Delta K_0(r)\mathrm{d}x\\&=\lim_{\varepsilon\to0}\int_{r=\varepsilon}pK_0'(r)\mathrm{d}\xi+\int_{R^2}p\Delta K_0(r)\mathrm{d}x\\&=\int_{R^2}p\Delta K_0(r)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \int_{R^2} pK_0(r)\,\mathrm{d}x \\
&= \int_{R^2} Ae^S \rho^\gamma K_0(r)\,\mathrm{d}x \\
&\geq \frac{A}{\left(\int_{R^2} K_0(r)\,\mathrm{d}x\right)^{\gamma-1}} \left(\int_{R^2} \rho K_0(r)\,\mathrm{d}x\right)^\gamma .
\end{aligned} \tag{4.3}
$$

Similar to the proof of Theorem 1.1, we obtain

$$
\begin{aligned}
\int_{R^2} \nabla\cdot\left[\nabla\cdot(\rho \boldsymbol{u}\otimes \boldsymbol{u})\right]K_0(r)\,\mathrm{d}x &= \int_{R^2} \sum_{i,j=1}^{2} \frac{\partial^2(\rho u_i u_j)}{\partial x_i \partial x_j} K_0(r)\,\mathrm{d}x \\
&= \int_{R^2} \rho v^2 K_0''(r)\,\mathrm{d}x \\
&= \int_{R^2} \rho v^2 \left(K_0(r) - \frac{1}{r}K_0'(r)\right)\mathrm{d}x > 0
\end{aligned} \tag{4.4}
$$

and

$$
\frac{\mathrm{d}^2}{\mathrm{d}t^2} I(t) + \alpha(t)\frac{\mathrm{d}}{\mathrm{d}t}G(t) \geq \frac{A}{\left(\int_{R^2} K_0(r)\,\mathrm{d}x\right)^{\gamma-1}} \left(I(t)\right)^\gamma , \tag{4.5}
$$

where $I(t) = \int_{R^2} \rho K_0(r)\,\mathrm{d}x$. Since

$$
\begin{aligned}
I'(0) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{R^2} \rho K_0(r)\,\mathrm{d}x\Big|_{t=0} \\
&= -\int_{R^2} \nabla\cdot(\rho \boldsymbol{u}) K_0(r)\,\mathrm{d}x\Big|_{t=0} \\
&= \int_{R^2} \sum_{i,j=1}^{2} \rho \frac{x_i}{r} v K_0'(r) \frac{x_i}{r}\,\mathrm{d}x\Big|_{t=0} \\
&= \int_{R^2} \rho_0 v_0 K_0'(r)\,\mathrm{d}x > 0,
\end{aligned} \tag{4.6}
$$

and (4.5), we have $I'(t) \geq 0$. Then multiplying the both side of (4.6) by $2\frac{\mathrm{d}}{\mathrm{d}t}I(t)$, we have

$$
\left[\left(I'(t)\right)^2\right]' + 2\alpha(t)\left(I'(t)\right)^2 \geq \frac{2A}{(\gamma+1)\left(\int_{R^2} K_0(r)\,\mathrm{d}x\right)^{\gamma-1}} \left(I(t)^{\gamma+1}\right)' . \tag{4.7}
$$

Like the calculations in Section 3, we get

$$
\left[\mathrm{e}^{\int_0^t 2\alpha(\tau)\mathrm{d}\tau}\left(I'(t)^2\right)\right]' \geq \frac{2A}{(\gamma+1)\left(\int_{R^2} K_0(r)\,\mathrm{d}x\right)^{\gamma-1}} \left(I(t)^{\gamma+1}\right)' , \tag{4.8}
$$

and

$$
\begin{aligned}
\left(I'(t)\right)^2 &\geq \mathrm{e}^{-\int_0^t 2\alpha(\tau)\mathrm{d}\tau}\left[\frac{2A}{(\gamma+1)\left(\int_{R^2} K_0(r)\,\mathrm{d}x\right)^{\gamma-1}}\left(I(t)^{\gamma+1} - I(0)^{\gamma+1}\right) + \left(I'(t)\big|_{t=0}\right)^2\right] \\
&= \mathrm{e}^{-\int_0^t 2\alpha(\tau)\mathrm{d}\tau} \frac{2A}{(\gamma+1)\left(\int_{R^2} K_0(r)\,\mathrm{d}x\right)^{\gamma-1}} I(t)^{\gamma+1} \\
&- \mathrm{e}^{-\int_0^t 2\alpha(\tau)\mathrm{d}\tau}\left[\frac{2A}{(\gamma+1)\left(\int_{R^2} K_0(r)\,\mathrm{d}x\right)^{\gamma-1}}\left(\int_{R^2} \rho_0 K_0(r)\,\mathrm{d}x\right)^{\gamma+1} - \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^2} \rho K_0(r)\,\mathrm{d}x\Big|_{t=0}\right)^2\right].
\end{aligned} \tag{4.9}
$$

Again, noting the assumption (1.11) and (4.6), we have

$$\left(I'(t)\right)^2 \ge \mathrm{e}^{-\int_0^t 2\alpha(\tau)\mathrm{d}\tau} \frac{2A}{(\gamma+1)\left(\int_{R^2} K_0(r)\mathrm{d}x\right)^{\gamma-1}} I(t)^{\gamma+1}. \tag{4.10}$$

and

$$I'(t) \ge C_1 \mathrm{e}^{-\int_0^t \alpha(\tau)\mathrm{d}\tau} I(t)^{\frac{\gamma+1}{2}} \tag{4.11}$$

by denoting

$$C_1 = \sqrt{\frac{2A}{(\gamma+1)\left(\int_{R^2} K_0(r)\mathrm{d}x\right)^{\gamma-1}}}. \tag{4.12}$$

Therefore, by integrating (4.11) over $[0,t]$, we have

$$\int_0^t \frac{I'(\eta)}{I(\eta)^{\frac{\gamma+1}{2}}} \mathrm{d}\eta \ge \int_0^t C_1 \mathrm{e}^{-\int_0^\eta \alpha(\tau)\mathrm{d}\tau} \mathrm{d}\eta \tag{4.13}$$

and

$$-\frac{2}{\gamma-1} I(t)^{-\frac{\gamma-1}{2}} + \frac{2}{\gamma-1} I(0)^{-\frac{\gamma-1}{2}} \ge C_1 \int_0^t \mathrm{e}^{-\int_0^\eta \alpha(\tau)\mathrm{d}\tau} \mathrm{d}\eta, \tag{4.14}$$

which implies that

$$I(t) \ge \left[I(0)^{-\frac{\gamma-1}{2}} - \frac{\gamma-1}{2} C_1 \int_0^t \mathrm{e}^{-\int_0^\eta \alpha(\tau)\mathrm{d}\tau} \mathrm{d}\eta \right]^{-\frac{2}{\gamma-1}}. \tag{4.15}$$

Observing (1.10), we have

$$\int_{R^2} \rho_0 K_0(r)\mathrm{d}x > 0. \tag{4.16}$$

On the other hand, using the mass conservation, we have

$$I(t) = \int_{R^2} \rho K_0(r)\mathrm{d}x \le \int_{B_{r_0}} \rho K_0(r)\mathrm{d}x + \frac{\max_{r \ge r_0} K_0(r)}{r_0} \int_{R^2} \rho_0 \mathrm{d}x \tag{4.17}$$

for any given $r_0 > 0$, where B_{r_0} is the two space dimensions ball centered at the origin with radius r_0. Therefore, when $\int_0^t \mathrm{e}^{-\int_0^\eta \alpha(\tau)\mathrm{d}\tau} \mathrm{d}\eta \to \frac{2}{(\gamma-1)C_1} G(0)^{-\frac{\gamma-1}{2}}$, $\rho(t,r)$ will blow up for $r \le r_0$.

5. Conclusion

In this paper, we research the blowup phenomenon of solutions to the compressible Euler equations with general time-dependent damping for non-isentropic fluids in two and three space dimensions. When the initial data is assumed to be radially symmetric and the initial density contains vacuum, by selecting the different test functions, we explore the nonlinear structure of the pressure p and overcome the difficulties brought by the point $r = 0$, and obtain that classical solution will blow up on finite time. The results also show that damping can indeed delay the formation of singularity.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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