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Fuzzy Henstock-Kurzweil Triple Integral

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Abstract

In this article, we use the Hausdorf distance to treat triple Simpson's rule of the Henstock triple integral of a fuzzy valued function as well as the error bound of the method. We also introduce δ-fine subdivisions for a Henstock triple integral and numerical example is presented in order to show the application and the consequence of the method.

Keywords

Fuzzy-Valued Function, Hausdorff Distance, Triple Fuzzy Integral, Triple Simpson's Rule, δ-Fine, Henstock Integral

1. Introduction

Sugeno [1] was the first to introduce the concept of fuzzy integral. Numerical methods have been developed in recent years in order to calculate a fuzzy integral. Some numerical methods are proposed by Wu [2] [3], Allahviranloo [4] and Fariborzi [5] [6] in order to compute fuzzy integrals by using quadrature methods and the definition of the set of levels. Wu and Gong [7] developed the Henstock integral of a fuzzy numeric valued function, then they applied the notion of differentiability of a fuzzy function. Bede and Gal in [8] have in turn applied the quadrature rule to calculate the integral of a function with fuzzy numerical value.

Some other integrals have been defined by Kumwimba *et al.* [9] [10] [11]. The material of this article is based on ideas developed in the article [12] to evaluate a fuzzy triple-valued function by applying the Simpson's triple rule and introduction of the fuzzy version Henstock's triple integral.

In Section 2, it will be a question of pinning down some basic definitions and properties of fuzzy sets and fuzzy numbers as well as some basic theorems useful for this work.

We introduce in Section 3, Simpson's triple rule to compute a fuzzy Henstock-Kurzweil triple integral (FHTI).

At last, in order to explain an application of the proposed method, in Section 4, one triple fuzzy integral is evaluated in order to show the efficacy of the mentioned method.

2. Preliminaries

In this section, we talk about some basic definitions of fuzzy sets theory which are being used in the following.

Definition 2.1. *Let* $\mathbb{R}$ *be a real set. Given a function* $\tilde{u}:\mathbb{R}\to[0,1]$ *satisfying the properties below*:

1) $\tilde{u}$ *is normal, i.e.* $\exists x_0\in\mathbb{R}$ *such that* $\tilde{u}(x_0)=1$,

2) $\tilde{u}$ *is a convex fuzzy set, i.e.,*

$$\tilde{u}(\lambda x+(1-\lambda)y)\geq\min\{\tilde{u}(x),\tilde{v}(y)\}\quad\forall x,y\in\mathbb{R},\lambda\in[0,1],$$

3) $\tilde{u}$ *is upper semi-continuous,*

4) *the set* $\overline{\{x\in\mathbb{R}:\tilde{u}(x)>0\}}$ *is compact, where* $\overline{B}$ *denotes the closure of B.*

This function $\tilde{u}$ *is called a fuzzy number.*

We denote by $\mathbb{R}_F$ *the set of all fuzzy real numbers. We define* $[\tilde{u}]^\alpha=\{x\in\mathbb{R}:\tilde{u}(x)\geq\alpha\}$ *and* $[\tilde{u}]^0=\overline{\{x\in\mathbb{R}:\tilde{u}(x)>0\}}$, *for* $0<\alpha\leq1$, *as the α-cut and respectively the support of a fuzzy number such as* $\tilde{u}$. *Moreover, we define* $\tilde{u}_\alpha^-=\inf[\tilde{u}]^\alpha$ *and* $\tilde{u}_\alpha^+=\sup[\tilde{u}]^\alpha$.

A triangular fuzzy number $\tilde{u}=(a,b,c)$ *where,* $a<b<c$ *and* $a,b,c\in\mathbb{R}$ *is defined by* $\tilde{u}_\alpha^-=a+(b-a)\alpha$ *and* $\tilde{u}_\alpha^+=c-(c-b)\alpha$.

For $\tilde{u},\tilde{v}\in\mathbb{R}_f$ *and* $\lambda\in\mathbb{R}$, *we can define the sum* $\tilde{u}\oplus\tilde{v}$ *and the product* $\lambda\odot\tilde{u}$ *by*

$$[\tilde{u}\oplus\tilde{v}]^\alpha=[\tilde{u}]^\alpha\oplus_{int}[\tilde{v}]^\alpha\ \text{ and }\ [\lambda\odot\tilde{u}]^\alpha=\lambda\odot_{int}[\tilde{u}]^\alpha\quad\forall\alpha\in[0,1],$$

with $[\tilde{u}]^\alpha\oplus_{int}[\tilde{v}]^\alpha$ *the usual addition of two intervals and* $\lambda[\tilde{u}]^\alpha$ *the usual product between a scalar and a subset of* $\mathbb{R}$ [13] [14].

Definition 2.2. *Let be two fuzzy numbers* $\tilde{u}$ *and* $\tilde{v}$ *given. The Hausdorff distance* $D:\mathbb{R}_F\times\mathbb{R}_F\to\mathbb{R}^+\cup\{0\}$ *of* $\tilde{u}$ *and* $\tilde{v}$ *is defined by*

$$D(\tilde{u},\tilde{v})=\sup_{\alpha\in[0,1]}\max\left\{\left|\tilde{u}_\alpha^--\tilde{v}_\alpha^-\right|,\left|\tilde{u}_\alpha^+-\tilde{v}_\alpha^+\right|\right\}=\sup_{\alpha\in[0,1]}\left\{d_H\left([\tilde{u}]^\alpha,[\tilde{v}]^\alpha\right)\right\},$$

with $[\tilde{u}_\alpha]=[\tilde{u}_\alpha^-,\tilde{u}_\alpha^+],[\tilde{v}]^\alpha=[\tilde{v}_\alpha^-,\tilde{v}_\alpha^+]\subseteq\mathbb{R}$ and d_H the Hausdorff metric. We denote $\|.\|=D(.,0)$, [15].

The following theorems will also be used.

Theorem 2.3.

1) *If* $\tilde{0}=\chi_0$, *then* $\tilde{0}\in\mathbb{R}_F$ *is a neutral element with respect to* $\oplus$, *i.e.* $\tilde{u}\oplus\tilde{0}=\tilde{0}\oplus\tilde{u}=\tilde{u}\quad\forall u\in\mathbb{R}_F$.

2) *With respect to* $\tilde{0}$, *none of* $\tilde{u}\in\mathbb{R}_F,\tilde{u}\neq\tilde{0}$ *is inversible in* $\mathbb{R}_F$.

3) $\forall a,b\in\mathbb{R}$ *such that* $a,b\geq0$ (*or* $a,b\leq0$), *and any* $\tilde{u}\in\mathbb{R}_F$. *We have* $(a+b)\odot\tilde{u}=a\odot\tilde{u}\oplus b\odot\tilde{u}$.

4) $\forall \lambda \in \mathbb{R}$ *and* $\forall \tilde{u}, \tilde{v} \in \mathbb{R}_F$, *we have* $\lambda \odot (\tilde{u} \oplus \tilde{v}) = \lambda \odot \tilde{u} \oplus \lambda \odot \tilde{v}$.

5) $\forall \lambda, \mu \in \mathbb{R}$ *and* $\forall \tilde{u} \in \mathbb{R}_F$, *we have* $\lambda \odot (\mu \odot \tilde{u}) = (\lambda \odot \mu) \odot \tilde{u}$.

6) $\|.\|_F$ *has the properties of a usual norm on* $\mathbb{R}_F$, *i.e.* $\|\tilde{u}\|_F = 0$ *if* $\tilde{u} = \tilde{0}$, $\|\lambda \odot \tilde{u}\|_F = \|\lambda\| \cdot \|\tilde{u}\|_F$ *and* $\|\tilde{u} \oplus \tilde{v}\|_F \le \|\tilde{u}\|_F + \|\tilde{v}\|_F$.

7) $\|\tilde{u}\|_F \le D(\tilde{u}, \tilde{v})$ *and* $D(\tilde{u}, \tilde{v}) \le \|\tilde{u}\|_F + \|\tilde{v}\|_F$ $\forall \tilde{u}, \tilde{v} \in \mathbb{R}_F$ [7].

Theorem 2.4.

1) $(\mathbb{R}_F, D)$ *is a complete metric space*,

2) $D(\tilde{u} \oplus \tilde{v}, \tilde{v} \oplus \tilde{w}) = D(u, w)$ $\forall \tilde{u}, \tilde{v}, \tilde{w} \in \mathbb{R}_F$,

3) $D(k \odot \tilde{u}, k \odot \tilde{v}) = |k| D(\tilde{u}, \tilde{v})$ $\forall \tilde{u}, \tilde{v} \in \mathbb{R}_F$, $\forall k \in \mathbb{R}$,

4) $D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) \le D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e})$ $\forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in \mathbb{R}_F$ [15].

The concept of the Henstock integral for a fuzzy number-valued function were introduced by Wu and Gong [12]. We introduce this definition for a three-dimensional fuzzy number-valued function.

Let $\tilde{f} : [a,b] \times [c,d] \times [p,q] \to \mathbb{R}_F$ and $\Delta_m : a = x_0 < x_1 < x_2 < \cdots < x_m = b$, $\Delta_n : c = y_0 < y_1 < y_2 < \cdots < y_n = d$ and $\Delta_s : p = z_0 < z_1 < z_2 < \cdots < z_s = q$ be the partitions of the intervals $[a,b], [c,d]$ and $[p,q]$ respectively.

Consider the points $\xi_i \in [x_{i-1}, x_i], i = 1, 2, \cdots, m$; $\eta_j \in [y_{j-1}, y_j], j = 1, 2, \cdots, n$; $\zeta_k \in [z_{k-1}, z_k], k = 1, 2, \cdots, s$ and $\delta : [a,b] \times [c,d] \times [p,q] \to \mathbb{R}^+$.

The divisions $P = \{([x_{i-1}, x_i]; \xi_i); i = 1, 2, \cdots, m\}$; $Q = \{([y_{j-1}, y_j]; \eta_j); j = 1, 2, \cdots, n\}$ and $R = \{([z_{k-1}, z_k]; \zeta_k); k = 1, 2, \cdots, s\}$ denoted shortly by $P = (\Delta_m, \xi), Q = (\Delta_n, \eta)$ and $R = (\Delta_s, \zeta)$ are called *δ-fines* if $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$; $[y_{j-1}, y_j] \subseteq (\eta_j - \delta(\eta_j), \eta_j + \delta(\eta_j))$ and $[z_{k-1}, z_k] \subseteq (\zeta_k - \delta(\zeta_k), \zeta_k + \delta(\zeta_k))$.

Here we give our new view of fuzzy Henstock double integral on which a third integral.

Definition 2.5. *The function is said to be Henstock triple integrable on* $I \in \mathbb{R}_f$ *if for every* $\varepsilon > 0$ *there is a function* $\delta : [a,b] \times [c,d] \times [p,q] \to \mathbb{R}^+$ *such that for any δ-fine divisions* P, Q *and* R *we obtain*

$D\left(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \bigoplus_{k=1}^{s} (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1}) \odot \tilde{f}(\xi_i, \eta_j, \zeta_k), I\right) < \varepsilon$.

Then I is called the Fuzzy Henstock Triple Integral of $\tilde{f}$ *and it's denoted by* (*FHTI*) $\int_a^b \int_c^d \int_p^q \tilde{f}(x, y, z) \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

Lemma 2.6.

1) If $\tilde{f}$ and $\tilde{h}$ are Henstock triple integrable mappings and if $D(\tilde{f}(x,y,z), \tilde{h}(x,y,z))$ is Lebesgue integrable, then

$$D\left((FHTI)\int_a^b \int_c^d \int_p^q \tilde{f}(x,y,z) \mathrm{d}x\mathrm{d}y\mathrm{d}z, (FHTI)\int_a^b \int_c^d \int_p^q \tilde{h}(x,y,z) \mathrm{d}x\mathrm{d}y\mathrm{d}z\right)$$
$$\le (L)\int_a^b \int_c^d \int_p^q D\left(\tilde{f}(x,y,z), \tilde{h}(x,y,z)\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

2) *Let* $\tilde{f} : [a,b] \times [c,d] \times [p,q] \to \mathbb{R}_F$ *be a Henstock triple integrabe bounded mapping.*

Then, $\forall (u,v,w) \in [a,b] \times [c,d] \times [p,q]$, *the function* $\phi_{(u,v,w)} : [a,b] \times [c,d] \times [p,q] \to \mathbb{R}^+$ *defined by* $\phi_{(u,v,w)}(x,y,z) = D\left(\tilde{f}(u,v,w), \tilde{f}(x,y,z)\right)$ *is Lebesgue integrable on*

$[a,b]\times[c,d]\times[p,q]$.

Proof (2) If $\tilde{f}$ is Henstock integrable and bounded on $[a,b]\times[c,d]\times[p,q]$, then it follows that $\tilde{f}_-^\alpha(x,y,z)$ and $\tilde{f}_+^\alpha(x,y,z)$ are Henstock triple integrable with $\alpha\in[0,1]$. Therefore, $\tilde{f}_-^\alpha(x,y,z)$ and $\tilde{f}_+^\alpha(x,y,z)$ are Lebesgue measurable and uniformly bounded $\forall\alpha\in[0,1]$, [7]. Moreover,

$$\begin{aligned}\phi(x,y,z)&=D\left(\tilde{f}(x_1,y_1,z_1),\tilde{f}(x_2,y_2,z_2)\right)\\&=\max_{\alpha\in[0,1]}\max\left\{\left|\tilde{f}_-^\alpha(x_1,y_1,z_1)-\tilde{f}_-^\alpha(x_2,y_2,z_2)\right|,\left|\tilde{f}_+^\alpha(x_1,y_1,z_1)-\tilde{f}_+^\alpha(x_2,y_2,z_2)\right|\right\}\\&=\max_{\alpha_n\in[0,1]}\max\left\{\left|\tilde{f}_-^{\alpha_n}(x_1,y_1,z_1)-\tilde{f}_-^{\alpha_n}(x_2,y_2,z_2)\right|,\left|\tilde{f}_+^{\alpha_n}(x_1,y_1,z_1)-\tilde{f}_+^{r_n}(x_2,y_2,z_2)\right|\right\},\end{aligned}$$

where the $\alpha_n(n\in\mathbb{N})$ are the rational numbers in $[0,1]$. According to Lebesgue's dominated convergence theorem, it follows that $\phi(x,y,\tilde{z})$ is Lebesgue integrable over $[a,b]\times[c,d]\times[p,q]$ and what completes the proof.

□

Keeping now three integrals we reach the following definitions.

Definition 2.7. *Let* $\tilde{f}:[a,b]\times[c,d]\times[p,q]\to\mathbb{R}_F$ *be a bounded mapping. Then the function* $\omega_{([a,b]\times[c,d]\times[p,q])}(\tilde{f},,,):\mathbb{R}^+\cup 0\to\mathbb{R}^+$ *such that*

$$\begin{aligned}&\omega_{([a,b]\times[c,d]\times[p,q])}(\tilde{f},\delta_1,\delta_2,\delta_3)\\&=\sup\left\{D\left(\tilde{f}(x_1,y_1,z_1),\tilde{f}(x_2,y_2,z_2)\right);(x_1,y_1,z_1),(x_2,y_2,z_2)\right.\\&\quad\left.\in[a,b]\times[c,d]\times[p,q],|x_1-x_2|\le\delta_1,|y_1-y_2|\le\delta_2,|z_1-z_2|\le\delta_3\right\}\end{aligned}$$

is called the modulus of oscillation of f on $[a,b]\times[c,d]\times[p,q]$.

If $\tilde{f}:[a,b]\times[c,d]\times[p,q]\to\mathbb{R}_F$ *is continuous on* $[a,b]\times[c,d]\times[p,q]$.

Then $\omega_{([a,b]\times[c,d]\times[p,q])}(f,\delta_1,\delta_2,\delta_3)$ *is called uniform modulus of continuity of f.*

We can prove the following theorem from the definition 2.7.

Theorem 2.8. *The following statements, concerning the modulus of oscillation are true.*

1) $D\left(\tilde{f}(x_1,y_1,z_1),\tilde{f}(x_2,y_2,z_2)\right)\le\omega_{([a,b]\times[c,d]\times[p,q])}\left(\tilde{f},|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\right)$
$\forall(x_1,y_1,z_1),(x_2,y_2,z_2)\in[a,b]\times[c,d]\times[p,q]$,

2) $\omega_{([a,b]\times[c,d]\times[p,q])}(\tilde{f},\delta_1,\delta_2,\delta_3)$ *is a non-decreasing mapping in* δ_1,δ_2 *and* δ_3,

3) $\omega_{([a,b]\times[c,d]\times[p,q])}(f,0,0,0)=0$,

4) $\omega_{([a,b]\times[c,d]\times[p,q])}(\tilde{f},m\delta_1,n\delta_2,s\delta_3)\le mns\,\omega_{([a,b]\times[c,d]\times[p,q])}(\tilde{f},\delta_1,\delta_2,\delta_3)$

$\forall\delta_1,\delta_2,\delta_3\ge 0$ *and* $m,n,s\in\mathbb{N}$,

5)

$$\omega_{([a,b]\times[c,d]\times[p,q])}(\tilde{f},\lambda_1\delta_1,\lambda_2\delta_2,\lambda_3\delta_3)\le(\lambda_1+1)(\lambda_2+1)(\lambda_3+1)\,\omega_{([a,b]\times[c,d]\times[p,q])}(\tilde{f},\delta_1,\delta_2,\delta_3)$$

for any $\delta_1,\delta_2,\delta_3,\lambda_1,\lambda_2,\lambda_3\ge 0$.

6) *If* $[e,f]\times[g,h]\times[i,j]\subseteq[a,b]\times[c,d]\times[p,q]$, *then*

$$\omega_{([e,f]\times[g,h]\times[i,j])}(\tilde{f},\delta_1,\delta_2,\delta_3)\le\omega_{([a,b]\times[c,d]\times[p,q])}(\tilde{f},\delta_1,\delta_2,\delta_3).$$

Proof (6) According to the hypothesis,

$$\sup\left\{D\left(\tilde f(x_1,y_1,z_1),\tilde f(x_2,y_2,z_2)\right);(x_1,y_1,z_1),(x_2,y_2,z_2)\right.$$
$$\left.\in[e,f]\times[g,h]\times[i,j],|x_1-x_2|\le\delta_1,|y_1-y_2|\le\delta_2,|z_1-z_2|\le\delta_3\right\}$$
$$\le\sup\left\{D\left(\tilde f(x_1,y_1,z_1),\tilde f(x_2,y_2,z_2)\right);(x_1,y_1,z_1),(x_2,y_2,z_2)\right.$$
$$\left.\in[a,b]\times[c,d]\times[p,q],|x_1-x_2|\le\delta_1,|y_1-y_2|\le\delta_2,|z_1-z_2|\le\delta_3\right\}$$

which is prove the relation.

We can prove similarly the other statements.

□

Definition 2.9. *A function* $\tilde f:[a,b]\times[c,d]\times[p,q]\to\mathbb{R}_F$ *is said to be* (L_1,L_2,L_3) *Lipschitz if for any* $(x_1,y_1,z_1),(x_2,y_2,z_2)\in[a,b]\times[c,d]\times[p,q]$,

$$D\left(\tilde f(x_1,y_1,z_1),\tilde f(x_2,y_2,z_2)\right)\le L_1|x_1-x_2|+L_2|y_1-y_2|+L_3|z_1-z_2|.$$

3. Triple Simpson's Rule for the Fuzzy Henstock-Kurzweil Triple Integrals

In order to introduce triple Simpson's rule for evaluating *FHTI*, firstly we prove the following theorem.

Theorem 3.1. *Let* $f:[a,b]\times[c,d]\times[p,q]\to\mathbb{R}_F$ *be a Henstock integrable, bounded mapping. Then, for any subdivision* $a=x_0<x_1<x_2<\cdots<x_m=b$, $c=y_0<y_1<y_2<\cdots<y_n=d$, $p=z_0<z_1<z_2<\cdots<z_s=q$ *and any points* $\xi_i\in[x_{i-1},x_i]$, $\eta_j\in[y_{j-1},y_j]$, $\zeta_k\in[z_{k-1},z_k]$ *we have*

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_p^q\tilde f(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x,$$
$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})\odot\tilde f(\xi_i,\eta_j,\zeta_k)\Bigg)$$
$$\le\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})$$
$$\omega_{([x_{i-1},x_i]\times[y_{j-1},y_j]\times[z_{k-1},z_k])}\left(\tilde f,(x_i-x_{i-1}),(y_j-y_{j-1}),(z_k-z_{k-1})\right).$$

Proof: Since that the Henstock integral is additive related to interval [16], hence,

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_p^q\tilde f(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x,$$
$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})(\tilde z_k-\tilde z_{k-1})\odot\tilde f(\xi_i,\eta_j,\zeta_k)\Bigg)$$
$$=D\Bigg(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(FHTI)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}\tilde f(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x,$$
$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})\odot\tilde f(\xi_i,\eta_j,\zeta_k)\Bigg).$$

Since it's clear that $(FHTI)\int_a^b\int_c^d\int_p^q k\mathrm{d}z\mathrm{d}y\mathrm{d}x=(b-a)(d-c)(q-p)\odot k$ for any fuzzy constant $k\in\mathbb{R}_F$, we obtain

$$D\left(\left(FHTI\right)\int_a^b\int_c^d\int_p^q \tilde{f}\left(x,y,z\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x,\right.$$

$$\left.\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left(z_k-z_{k-1}\right)\odot\tilde{f}\left(\xi_i,\eta_j,\zeta_k\right)\right)$$

$$=D\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\left(FHTI\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}\tilde{f}\left(x,y,z\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x,\right.$$

$$\left.\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\left(FHTI\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}\tilde{f}\left(\xi_i,\eta_j,\zeta_k\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x\right).$$

By the fourth property of the theorem 2.4, we have

$$D\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\left(FHTI\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}\tilde{f}\left(x,y,z\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x,\right.$$

$$\left.\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\left(FHTI\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}\tilde{f}\left(\xi_i,\eta_j,\zeta_k\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x\right)$$

$$\leq\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}D(\left(FHTI\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}\tilde{f}\left(x,y,z\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x,\right.$$

$$\left.\left(FHTI\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}\tilde{f}\left(\xi_i,\eta_j,\zeta_k\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x\right)$$

Since the functions $D\left(\tilde{f}\left(x,y,z\right),\tilde{f}\left(\xi_i,\eta_j,\zeta_k\right)\right)$ are Lebesgue integrable for $i=1,\cdots,m; j=1,\cdots,n$ and $k=1,\cdots,s$ from lemma 2.6 we have

$$D\left(\left(FHTI\right)\int_a^b\int_c^d\int_p^q \tilde{f}\left(x,y,z\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x,\right.$$

$$\left.\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left(z_k-z_{k-1}\right)\odot\tilde{f}\left(\xi_i,\eta_j,\zeta_k\right)\right)$$

$$\leq\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\left(L\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}D\left(\tilde{f}\left(x,y,z\right),\tilde{f}\left(\xi_i,\eta_j,\zeta_k\right)\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

From the first property of the theorem 2.8 applied to each of the above integrals we have

$$D\left(\left(FHTI\right)\int_a^b\int_c^d\int_p^q \tilde{f}\left(x,y,z\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x,\right.$$

$$\left.\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left(z_k-z_{k-1}\right)\odot\tilde{f}\left(\xi_i,\eta_j,\zeta_k\right)\right)$$

$$\leq\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\left(L\right)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}\omega_{\left(\left[x_{i-1},x_i\right]\times\left[y_{j-1},y_j\right]\times\left[z_{k-1},z_k\right]\right)}$$

$$\left(\tilde{f},x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left(z_k-z_{k-1}\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left(z_k-z_{k-1}\right)$$

$$\omega_{\left(\left[x_{i-1},x_i\right]\times\left[y_{j-1},y_j\right]\times\left[z_{k-1},z_k\right]\right)}\left(\tilde{f},\left(x_i-x_{i-1}\right)\left(y_j-y_{j-1}\right)\left(z_k-z_{k-1}\right)\right),$$

which completes the proof.

□

Corollary 3.2. *Let* $\tilde{f}:[a,b]\times[c,d]\times[p,q]\to\mathbb{R}_F$ *be a Henstock triple integrable, bounded mapping. Then,*

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_p^q \tilde{f}(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})\odot\tilde{f}(\xi_i,\eta_j,\zeta_k)\Bigg)$$

$$\le(\alpha-a)(\beta-c)(\gamma-p)\omega_{[a,\alpha]\times[c,\beta]\times[p,\gamma]}\left(\tilde{f},(\alpha-a),(\beta-c),(\gamma-p)\right)$$

$$+(\alpha-a)(\beta-c)(q-\gamma)\omega_{[a,\alpha]\times[c,\beta]\times[\gamma,q]}\left(\tilde{f},(\alpha-a),(\beta-c),(q-\gamma)\right)$$

$$+(\alpha-a)(d-\beta)(\gamma-p)\omega_{[a,\alpha]\times[\beta,d]\times[p,\gamma]}\left(\tilde{f},(\alpha-a),(d-\beta),(\gamma-p)\right)$$

$$+(\alpha-a)(d-\beta)(q-\gamma)\omega_{[a,\alpha]\times[\beta,d]\times[\gamma,q]}\left(\tilde{f},(\alpha-a),(d-\beta),(q-\gamma)\right)$$

$$+(b-\alpha)(\beta-c)(\gamma-p)\omega_{[\alpha,b]\times[c,\beta]\times[p,\gamma]}\left(\tilde{f},(b-\alpha),(\beta-c)(\gamma-p)\right)$$

$$+(b-\alpha)(\beta-c)(q-\gamma)\omega_{[\alpha,b]\times[c,\beta]\times[\gamma,q]}\left(\tilde{f},(b-\alpha),(\beta-c)(q-\gamma)\right)$$

$$+(b-\alpha)(d-\beta)(\gamma-p)\omega_{[\alpha,b]\times[\beta,d]\times[p,\gamma]}\left(\tilde{f},(b-\alpha),(d-\beta),(\gamma-p)\right)$$

$$+(b-\alpha)(d-\beta)(q-\gamma)\omega_{[\alpha,b]\times[\beta,d]\times[\gamma,q]}\left(\tilde{f},(b-\alpha),(d-\beta),(q-\gamma)\right),$$

for any $\alpha\in[a,b],\beta\in[c,d]$ *and* $\gamma\in[p,q]$, $(u,v,w)\in[a,\alpha]\times[c,\beta]\times[p,\gamma]$ *and* $(u',v',w')\in[\alpha,b]\times[\beta,d]\times[\gamma,q]$ *where* $\xi_1=u$, $\xi_2=u'$; $\eta_1=v$, $\eta_2=v'$; $\zeta_1=w$, $\zeta_2=w'$.

Proof It's clear that for $m=2,n=2$ and $s=2$ in the theorem 3.1 the inequality stated above is obtained.

□

Theorem 3.3. *Let* $\tilde{f}:[a,b]\times[c,d]\times[p,q]\to\mathbb{R}_F$ *be a Lipschitz mapping with the constants* L_1,L_2 *and* L_3. *Then, for any subdivision*

$$\Delta_m: a=x_0<x_1<x_2<\cdots<x_m=b,\quad \Delta_n: c=y_0<y_1<y_2<\cdots<y_n=d$$

and

$$\Delta_s: p=z_0<z_1<z_2<\cdots<z_s=q.\quad \forall\xi_i\in[x_{i-1},x_i], i=1,2,\cdots,m;$$

$\eta_j\in[y_{j-1},y_j], j=1,2,\cdots,n$ and $\zeta_k\in[z_{k-1},z_k], k=1,2,\cdots,s$; we have

$$D\Bigg((FHTI)\int_a^b\int_c^d\int_p^q \tilde{f}(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x,$$

$$\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})\odot\tilde{f}(\xi_i,\eta_j,\zeta_k)\Bigg)$$

$$\le\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\Big(L_1(y_j-y_{j-1})(z_k-z_{k-1})(x_i-x_{i-1})^2$$

$$+L_2(x_i-x_{i-1})(z_k-z_{k-1})(y_j-y_{j-1})^2$$

$$+L_3(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})^2\Big).$$

Proof Similar to the proof of theorem 3.1 we have

$$D\left((FHTI)\int_a^b\int_c^d\int_p^q \tilde{f}(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x,\ \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})\odot\tilde{f}(\xi_i,\eta_j,\zeta_k)\right)$$
$$\leq\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(L)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}D\left(\tilde{f}(x,y,z),\tilde{f}(\xi_i,\eta_j,\zeta_k)\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

We obtain by the definition of a Lipschitz mapping

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}(L)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}D\left(\tilde{f}(x,y,z),\tilde{f}(\xi_i,\eta_j,\zeta_k)\right)\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$
$$\leq\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\Big(L_1\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}|x-\xi_i|\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$+L_2\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}|y-\eta_j|\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$+L_3\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}|z-\zeta_k|\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\Big)$$

It follows by direct computation that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\Big(L_1\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}|x-\xi_i|\,\mathrm{d}z\mathrm{d}y\mathrm{d}x+L_2\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}|y-\eta_j|\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$$
$$+L_3\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\int_{z_{k-1}}^{z_k}|z-\zeta_k|\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\Big)$$
$$=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\Big(L_1(y_j-y_{j-1})(z_k-z_{k-1})\left[(x_i-\xi_i)^2-(x_{i-1}-\xi_i)^2\right]$$
$$+L_2(x_i-x_{i-1})(z_k-z_{k-1})\left[(y_j-\eta_j)^2-(y_{j-1}-\eta_j)^2\right]$$
$$+L_3(x_i-x_{i-1})(y_j-y_{j-1})\left[(z_k-\zeta_k)^2-(z_{k-1}-\zeta_k)^2\right]\Big)$$
$$\leq\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\Big(L_1(y_j-y_{j-1})(z_k-z_{k-1})(x_i-x_{i-1})^2$$
$$+L_2(x_i-x_{i-1})(z_k-z_{k-1})(y_j-y_{j-1})^2$$
$$+L_3(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})^2\Big).$$

□

Remark 3.4. *If* $x_i-x_{i-1}=h, y_j-y_{j-1}=k$ *and* $z_k-z_{k-1}=l$, *then*,

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\Big(L_1(y_j-y_{j-1})(z_k-z_{k-1})(x_i-x_{i-1})^2$$
$$+L_2(x_i-x_{i-1})(z_k-z_{k-1})(y_j-y_{j-1})^2+L_3(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})^2\Big)$$
$$=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{s}\left(L_1h^2kl+L_2hk^2l+hkl^2\right),$$

where $mh=b-a, nk=d-c$ *and* $sl=q-p$. *Therefore, we obtain*

$$D\left((FHTI)\int_a^b\int_c^d\int_p^q \tilde{f}(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x,\ \bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n}\bigoplus_{k=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})(z_k-z_{k-1})\odot\tilde{f}(\xi_i,\eta_j,\zeta_k)\right)\tag{1}$$
$$\leq U(h,k,l)=\frac{(b-a)(d-c)(q-p)}{2}(L_1h+L_2k+L_3l).$$

4. Numerical Example

Let $\tilde{f}:[0,1]\times[1,2]\times[1,2]\to\mathbb{R}_F$, $\tilde{f}(x,y,z)=(\tilde{x}\otimes\tilde{x})\oplus(\tilde{3}\otimes\tilde{y})\oplus(\tilde{1}\otimes\tilde{z})$ where $\tilde{x}=(x-1,x,x+1)$; $\tilde{1}=(0,1,2)$; $\tilde{3}=(2,3,4)$; $\tilde{y}=(y-1,y,y+1)$; $\tilde{z}=(z-1,z,z+1)$, and where (a_1,a_2,a_3) is a triangular fuzzy number such that

$$\mu(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1} & a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2} & a_2\le x\le a_3\\ 0 & \text{otherwise}\end{cases}$$

We must compute the integral

$$(FHTI)\int_0^1\int_1^2\int_1^2\tilde{f}(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

numerically.

Firstly we calculate

$$\tilde{x}\otimes\tilde{x}=(x^2-1,x^2,x^2+2x+1)$$

$$\tilde{3}\otimes\tilde{y}=(2y-2,3y,4y+4)$$

$$\tilde{1}\otimes\tilde{z}=(0,z,2z+2),$$

so

$$\tilde{f}(x,y,z)=(2y+x^2-3,3y+z+x^2,4y+2z+2x+x^2+7).$$

We obtain

$$\left[\tilde{f}(x,y,z)\right]_-^\alpha=\alpha(y+z+3)+2y+x^2-3$$

$$\left[\tilde{f}(x,y,z)\right]_+^\alpha=-\alpha(y+z+2x+7)+4y+2z+2x+x^2+7$$

Remark that

$$\begin{aligned}&D\left(\tilde{f}(x_1,y_1,z_1),\tilde{f}(x_2,y_2,z_2)\right)\\&=\sup_{\alpha\in[0,1]}\max\left\{\left|\tilde{f}_-^\alpha(x_1,y_1,z_1)-\tilde{f}_-^\alpha(x_2,y_2,z_2)\right|,\left|\tilde{f}_+^\alpha(x_1,y_1,z_1)-\tilde{f}_+^\alpha(x_2,y_2,z_2)\right|\right\}\\&=\sup_{\alpha\in[0,1]}\max\left\{\alpha\left|(y_1-y_2)+(z_1-z_2)\right|+\left|2(y_1-y_2)+(x_1^2-x_2^2)\right|,\right.\\&\quad\alpha\left|2(x_1-x_2)+(y_1-y_2)+(z_1-z_2)\right|\\&\quad\left.+\left|2(x_1-x_2)+4(y_1-y_2)+2(z_1-z_2)+(x_1^2-x_2^2)\right|\right\}\\&\le\sup_{\alpha\in[0,1]}\max\left\{|y_1-y_2|\left(|x_1+x_2|,2\alpha+2+|x_1+x_2|\right)\right\}\\&\quad+\sup_{\alpha\in[0,1]}\max\left\{|y_1-y_2|(\alpha+2,\alpha+4)\right\}+\sup_{\alpha\in[0,1]}\max\left\{|z_1-z_2|(\alpha,\alpha+2)\right\}\\&\le 6|x_1-x_2|+5|y_1-y_2|+3|z_1-z_2|.\end{aligned}$$

i.e. $\tilde{f}$ is a Lipschitz mapping with $L_1=6,L_2=5$ and $L_3=3$. We have for $\alpha=1$:

$$\underline{I}^1=\overline{I}^1=\int_0^1\int_1^2\int_1^2(x^2+3y+z)\mathrm{d}z\mathrm{d}y\mathrm{d}x=6.33333$$

Table 1 shows the results for different α and $m=60,n=50$ and $s=30$.

Table 1. The results of example.

α	$\underline{I}_{\alpha}^{m,n,s}$	$\overline{I}_{\alpha}^{m,n,s}$
1	6.33333	6.33333
0.9	5.73333	7.43333
0.8	5.13333	8.53333
0.7	4.53333	9.63333
0.6	3.93333	10.73330
0.5	3.33333	11.83330
0.4	2.73333	12.93330
0.3	2.13333	14.03330
0.2	1.53333	15.13330
0.1	0.93333	16.23330
0.0	0.33333	17.33333

In this table, the notations $\underline{I}_{\alpha}^{m,n,s}$ and $\overline{I}_{\alpha}^{m,n,s}$ are the approximate values of α-cut for $(FHTI)\int_0^1\int_1^2\int_1^2 \tilde{f}(x,y,z)\text{d}z\text{d}y\text{d}x$ obtained by the triple Simpson's rule with $h=\frac{b-a}{m}$, $k=\frac{d-c}{n}$ and $l=\frac{q-p}{s}$ [17].

We have $U(h,k,l)=0.15$ from 3.1 in this case.

5. Conclusion

We generalize the evaluating of fuzzy Henstock double integral using double Simpson's rule [12] by introduce and evaluate Henstock's fuzzy triple integral by applying Simpson's triple rule. Therefore, a theorem has been demonstrated to show the upper limit of the distance between the exact and approximate values. In the following, the Monte Carlo method [3] can be used for Henstock's fuzzy triple integral and thus compare the results of the methods with each other. We finished our paper by a numerical example of a fuzzy function in wich triple Simpson's rule is used.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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