Journal of Applied Mathematics and Physics, 2023, 11, 1737-1754
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Existence of Forced Waves and Their Asymptotic for Leslie-Gower Prey-Predator Model with Nonlocal Effects under Shifting Environment

Qingru Guo, Hongmei Cheng*

School of Mathematics and Statistics, Shandong Normal University, Jinan, China
Email: *hmcheng@sdnu.edu.cn

How to cite this paper: Guo, Q.R. and Cheng, H.M. (2023) Existence of Forced Waves and Their Asymptotic for Leslie-Gower Prey-Predator Model with Nonlocal Effects under Shifting Environment. *Journal of Applied Mathematics and Physics*, **11**, 1737-1754.
https://doi.org/10.4236/jamp.2023.116113

Received: May 25, 2023
Accepted: June 27, 2023
Published: June 30, 2023

Abstract

In this paper, we prove the existence of forced waves for Leslie-Gower prey-predator model with nonlocal effects under shifting environment. By constructing a pair of upper and lower solutions with the method of monotone iteration, we can obtain the existence of forced waves for any positive constant shifting speed. Finally, we show the asymptotical behavior of traveling wave fronts in two tails.

Keywords

Leslie-Gower Prey-Predator Model, Nonlocal Effects, Shifting Environment, Forced Waves

1. Introduction

Nowadays, Global warming is a hot topic and which has greatly changed the living environment of species. Climate change has caused massive changes in species distributions, abundances and diversity of species, and has led to the extinction of some vulnerable species around the globe, see [1] [2], so we have to take this factor into account when studying population dynamics.

For the phenomenon that global warming causes habitat quality to change, many researchers have produced very scientific results, see [3]-[8] etc. In the past, Berestycki developed the reaction-diffusion equation under shifting environment and studied the existence of the forced waves for the equation. A few years ago, Berestycki and Fang in [9] considered the following Fisher-KPP equation

$$u_t(x,t) = du_{xx}(x,t) + ur(x-ct) - u^2, \quad t>0, x\in\mathbb{R},$$

they established the existence and nonexistence of forced waves for this reaction-diffusion equation. It's called forced waves because the growth function *r* is related to *x* and *t*. The form $x-ct$ was forced into existence by the environmental changes, hence the name. If the rate of environmental change $c > 2\sqrt{dr(\infty)}$, there will be forced waves, and if $0 < c < 2\sqrt{dr(\infty)}$, there will be no forced waves.

In ecosystems, the dynamic relationship between predator and prey is ubiquitous and important, and it has been one of the major themes of ecology. It has been studied and still is by many researchers who have developed mathematical models to predict the interactions between prey and predator species. Of course, species interactions with each other are in a variety of ways, including cooperation, symbiosis and competition. Among them, cooperation is one of the important interactions among species and is common in both social animals and in human societies, see [10]. These species interactions are largely critical and required, see [11] [12], so many researchers built a number of different models based on these different functional responses.

Yang *et al.* in [13] concerned a Lotka-Volterra cooperation system under climate change, they obtained the existence and asymptotics of forced waves, the model as follows

$$\begin{cases} \partial_t u_1(t,x) = d_1\partial_{xx}u_1(t,x) + u_1\left[r_1(x-ct) - u_1 + a_1u_2\right], \\ \partial_t u_2(t,x) = d_2\partial_{xx}u_2(t,x) + u_2\left[r_2(x-ct) - u_2 + a_2u_1\right], \end{cases} \tag{1.1}$$

where $d_1, d_2 > 0$ respectively represent the diffusion rates of prey and predator, $r_i(x-ct)$, $i=1,2$ represent the growth function of prey and predator, *c* is the positive rate of environment change, $a_iu_1u_2$ indicates the cooperative behaviors with positive constants a_i for $i=1,2$. They showed that for any given positive speed of the shifting habitat edge, there exists a nondecreasing forced wave with the speed consistent with the habitat shifting speed. Hu *et al.* concerned with the forced traveling wave solution for the modified model of (1.1). They proved the existence of traveling wave solution for any positive constant shifting speed by constructing appropriate upper and lower solutions and using the method of monotone iteration, see [14]. The method of monotone iteration used for constructing the upper and lower solutions is an effective method for solving differential equations, and is widely used in many kinds of initial value problems and boundary value problems. This method is that if the problem has a pair of ordered lower solutions and upper solutions, then under certain conditions, the monotone iterative sequence can be constructed through the pair of lower solutions and upper solutions, so that they converge uniformly to the minimum and maximum solutions between the lower solutions and upper solutions of the equation. If the minimum solution is equal to the maximum solution, then the minimum solution (or maximum solution) is the forced traveling wave solution of the equation.

These prey-predator models are typically represented by the predator function as the increase in the number of predators after predation, while the decrease in the number of predators is due to natural death. However, a case shows that the increase of some predators is also not unlimited, and that the decrease in the number of predator populations is negatively correlated with the per capita availability of its preferred food, see [15]. Leslie in [16] first introduced a prey-predator model, combining the logistic predator equation with the carrying capacity proportional to the number of prey, in order to emphasize that predators have an upper limit on the rate of growth as prey. Since then, many researchers have studied the Leslie-Gower prey-predator model.

For the predator-prey relationships, a great deal of researchers have discussed the Leslie-Gower prey-predator model. In [17], Fang mainly concerned the spatial dynamics of a modified Leslie-Gower prey-predator model in a shifting habitat, the main concerns are the extinction and persistent conditions under the interaction between two species with different diffusion speed comparing with the shifting habitat edge constant *c*, the model as follows

$$\begin{cases} u_t(t,x) = d_1 u_{xx}(t,x) + u(t,x)\left[r_1(x-ct) - u(t,x) - \beta v(t,x)\right], & t>0, x\in\mathbb{R}, \\ v_t(t,x) = d_2 v_{xx}(t,x) + v(t,x)\left[r_2(x-ct) - \dfrac{v(t,x)}{u(t,x)+\alpha}\right], & t>0, x\in\mathbb{R}. \end{cases} \tag{1.2}$$

Changes in the environment lead to changes in the habitat boundaries of predator and prey, which are represented here by $x-ct$. Because $c>0$, the habitat range is reduced. Only when the shifting speed of environment $c > 2\sqrt{d_i r_i(\infty)}$, $i=1,2$, the population density of the two species will eventually reach an equilibrium state, that is, there is a forced wave. When the rate of species spread is less than the rate of environmental change, the species will go extinct. Recently, we considered the existence of forced waves and their asymptotic for (1.2). The existence of the forced waves indicates that the prey-predator system will eventually reach a state of equilibrium. Lee *et al.* in [18] noted that the free movement of some species can be over large areas, and nonlocal diffusion can effectively describe this phenomenon. Therefore, many researchers have considered the nonlocal diffusion species models. For example, Cheng and Yuan in [19] mainly studied the information about the existence and stability of the invasion traveling waves for the nonlocal Leslie-Gower predator-prey model. Then motivated by the aforementioned works, we concern a Leslie-Gower predator-prey model with a nonlocal predation under shifting environment as follows

$$\begin{cases} u_t(t,x) = d_1 u_{xx}(t,x) + u(t,x)\left[r_1(x-ct) - u(t,x) - \beta (J*v)(t,x)\right], & t>0, x\in\mathbb{R}, \\ v_t(t,x) = d_2 v_{xx}(t,x) + v(t,x)\left[r_2(x-ct) - \dfrac{v(t,x)}{u(t,x)+\alpha}\right], & t>0, x\in\mathbb{R}, \end{cases} \tag{1.3}$$

where all parameters are assumed to be positive,
$(J*v)(t,x) = \int_{\mathbb{R}} J(x-y)v(t,y)\mathrm{d}y$, $J(x)$ is a continuous and non-negative probability density function. u and v respectively stand for the prey populations

and the predator populations, β denotes the per capita capturing rate of the prey by a predator per unit time and $\frac{v}{u+\alpha}$ represents Leslie-Gower term.

In this paper, we will concern the forced traveling wave fronts of (1.3) connecting the trivial equilibrium and positive equilibrium. The results imply that for any given positive speed of the shifting habitat edge, there exists a forced wave with the speed in keeping with the habitat shifting speed. To prove this conclusion, we assume

(A1) $J(x)\in C(\mathbb{R},\mathbb{R}^+)$ is symmetric and compactly supported, and $\int_{\mathbb{R}} J(x)\,\mathrm{d}x = 1$;

(A2) $r_i(x):\mathbb{R}\to\mathbb{R}$ is continuous, nondecreasing with $-\infty < L_i = r_i(-\infty) < 0 < r_i(+\infty) = K_i < +\infty$, $i = 1,2$;

(A3) $\alpha > 0$, $\beta > 0$ and $K_1 > \alpha\beta K_2$;

(A4) $r_i(x)$ is continuously differentiable in $\mathbb{R}$ and both $r_i'(\pm\infty)$ exist, $i = 1,2$.

Note that the system of (1.3) admits four equilibria $E_0(0,0)$, $E_1(K_1,0)$, $E_2(0,\alpha K_2)$ and $E_*(u^*,v^*)$, where

$$u^* = \frac{K_1-\alpha\beta K_2}{1+\beta K_2}, v^* = \frac{K_1K_2+\alpha K_2}{1+\beta K_2}.$$

This paper is organized as follows. In Section 2, by constructing a pair of appropriate upper and lower solutions of the Equation (2.5) and combined with the monotone iteration approaches, we can establish the existence of forced wave. In Section 3, we consider the asymptotical behavior of forced wave in two tails.

2. Existence of Forced Wave

In this section, we always assume that (A1), (A2) and (A3) hold. For simplicity, denote

$$\begin{aligned} g_1(t,x,u,v) &= u\left[r_1(x-ct)-u-\beta(J*v)\right],\\ g_2(t,x,u,v) &= v\left[r_2(x-ct)-\frac{v}{u+\alpha}\right]. \end{aligned} \tag{2.1}$$

For any $0\le u_1,u_2\le K_1$, $0\le v_1,v_2\le K_2(K_1+\alpha)$ and $x\in\mathbb{R}$, we have

$$\begin{aligned} \left|g_1(t,x,u_1,v_1)-g_1(t,x,u_2,v_2)\right| &\le q_1\left[|u_1-u_2|+|v_1-v_2|\right],\\ \left|g_2(t,x,u_1,v_1)-g_2(t,x,u_2,v_2)\right| &\le q_2\left[|u_1-u_2|+|v_1-v_2|\right], \end{aligned} \tag{2.2}$$

where $q_1 = 2K_1 - L_1 + \beta\left(K_1 + K_2(K_1+\alpha)\right)$, $q_2 = \frac{2K_2(K_1+\alpha)}{\alpha} - L_2 + \frac{K_2^2(K_1+\alpha)^2}{\alpha^2}$, which imply that $g_1(t,x,u,v)$, $g_2(t,x,u,v)$ are Lipschitz continuous in $(u,v)\in[0,K_1]\times\left[0,K_2(K_1+\alpha)\right]$ for any $x\in\mathbb{R}$, $t\in\mathbb{R}^+$. Define

$$G_1(t,x,u,v) = q_1u + g_1(t,x,u,v), G_2(t,x,u,v) = q_2v + g_2(t,x,u,v). \tag{2.3}$$

Then $G_1(t,x,u,v)$ is nondecreasing in $u \in [0, K_1]$ and nonincreasing in $v \in [0, K_2(K_1+\alpha)]$, $G_2(t,x,u,v)$ is nondecreasing in $u \in [0, K_1]$, $v \in [0, K_2(K_1+\alpha)]$. Let $\mathcal{C} = \mathcal{C}(\mathbb{R})$ be all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $\mathcal{BC} = \mathcal{C} \cap \mathcal{L}^\infty(\mathbb{R})$ be all continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}$. Denote $\mathcal{Z} = \mathcal{BC} \times \mathcal{BC}$. Then it follows from [[20], Theorem 2.1] that the following conclusion holds.

Theorem 2.1. *Considering the Cauchy problem*

$$\begin{cases} u_t(t,x) = d_1 u_{xx}(t,x) - q_1 u + G_1(t,x,u,v), t > 0, x \in \mathbb{R}, \\ v_t(t,x) = d_2 v_{xx}(t,x) - q_2 v + G_2(t,x,u,v), t > 0, x \in \mathbb{R}, \\ (u(0,x), v(0,x)) = (u_0(x), v_0(x)), x \in \mathbb{R}, \end{cases} \tag{2.4}$$

where G_1, G_2 are defined in (2.3). If $u_0(x) \in \mathcal{BC}, v_0(x) \in \mathcal{BC}$ with $0 \le u_0(x) \le K_1$, $0 \le v_0(x) \le K_2(K_1+\alpha)$ in $\mathbb{R}$, then (2.4) has a unique classic solution $(u,v), u \in \mathcal{BC}, v \in \mathcal{BC}$ with $0 \le u(t,x) \le K_1$, $0 \le v(t,x) \le K_2(K_1+\alpha)$ for all $t > 0$ and $x \in \mathbb{R}$.

Let $\xi = x - ct$ and plugging it into (1.3), we have

$$\begin{cases} d_1 U''(\xi) + cU'(\xi) + U(\xi)\left[r_1(\xi) - U(\xi) - \beta(J * V(\xi))\right] = 0, \\ d_2 V''(\xi) + cV'(\xi) + V(\xi)\left[r_2(\xi) - \dfrac{V(\xi)}{U(\xi)+\alpha}\right] = 0. \end{cases} \tag{2.5}$$

Then we will consider the solution of (2.5) which satisfies the following asymptotic boundary condition

$$\lim_{\xi \to -\infty} (U(\xi), V(\xi)) = (0,0), \lim_{\xi \to +\infty} (U(\xi), V(\xi)) = (u^*, v^*). \tag{2.6}$$

Lemma 2.1 *1) Let η_i be the positive root of $d_i \lambda^2 + c\lambda + L_i = 0$ with $L_i = r_i(-\infty) < 0$, $i = 1,2$. Then there exist $\alpha_1 > \eta_1$, $\alpha_2 > \eta_2$ and small $0 < \varepsilon < K_1$, $0 < \tau < K_2(K_1+\alpha)$ such that the functions*

$$\underline{U} = \begin{cases} \varepsilon e^{\alpha_1(\xi - \xi_0)}, & \xi < \xi_0, \\ \varepsilon, & \xi \ge \xi_0, \end{cases} \quad \underline{V} = \begin{cases} \tau e^{\alpha_2(\xi - \xi_0)}, & \xi < \xi_0, \\ \tau, & \xi \ge \xi_0, \end{cases}$$

satisfy $0 \le \underline{U}(\xi) < K_1$, $0 \le \underline{V}(\xi) < K_2(K_1+\alpha)$ for all $\xi \in \mathbb{R}$, where ξ_0 satisfies $r_1(\xi_0) - \varepsilon - \beta K_2(K_1+\alpha) \ge 0$ and $r_2(\xi_0) - \dfrac{\tau}{\varepsilon+\alpha} \ge 0$.

2) Let $\xi_2 < \xi_1 \le \xi_0$ so small that $r_1(\xi_1) < 0$ and $r_2(\xi_2) < 0$. Assume that β_1 is the positive root of $d_1\lambda^2 + c\lambda + r_1(\xi_1) = 0$ and β_2 is the positive root of $d_2\lambda^2 + c\lambda + r_2(\xi_2) = 0$. Then the functions $\bar{U}(\xi) = \min\left\{K_1 e^{\beta_1(\xi-\xi_1)}, K_1\right\}$, $\bar{V}(\xi) = \min\left\{K_2(K_1+\alpha)e^{\beta_2(\xi-\xi_2)}, K_2(K_1+\alpha)\right\}$ satisfy $\underline{U}(\xi) \le \bar{U}(\xi)$, $\underline{V}(\xi) \le \bar{V}(\xi)$ for all $\xi \in \mathbb{R}$.

Furthermore, there holds

$$d_1 \underline{U}''(\xi) + c\underline{U}'(\xi) + \underline{U}(\xi)\left[r_1(\xi) - \underline{U}(\xi) - \beta(J * \bar{V}(\xi))\right] \ge 0, \tag{2.7}$$

$$d_2\underline{V}''(\xi)+c\underline{V}'(\xi)+\underline{V}(\xi)\left[r_2(\xi)-\frac{\underline{V}(\xi)}{\underline{U}(\xi)+\alpha}\right]\ge 0, \tag{2.8}$$

$$d_1\bar{U}''(\xi)+c\bar{U}'(\xi)+\bar{U}(\xi)\left[r_1(\xi)-\bar{U}(\xi)-\beta\left(J*\underline{V}(\xi)\right)\right]\le 0, \tag{2.9}$$

and

$$d_2\bar{V}''(\xi)+c\bar{V}'(\xi)+\bar{V}(\xi)\left[r_2(\xi)-\frac{\bar{V}(\xi)}{\bar{U}(\xi)+\alpha}\right]\le 0. \tag{2.10}$$

Proof. 1) Let α_1 be the positive root of the equation $d_1\lambda^2+c\lambda+L_1-\varepsilon-\beta K_2(K_1+\alpha)=0$, and α_2 be the positive root of the equation $d_2\lambda^2+c\lambda+L_2-\frac{\tau}{\alpha}=0$. Due to η_i be the positive root of $d_i\lambda^2+c\lambda+L_i=0$, we obtain $\alpha_1>\eta_1$, $\alpha_2>\eta_2$. Since $0<\varepsilon<K_1$, $0<\tau<K_2(K_1+\alpha)$, we have $0\le\underline{U}(\xi)<K_1$, $0\le\underline{V}(\xi)<K_2(K_1+\alpha)$ for all $\xi\in\mathbb{R}$.

2) Since $r_1(-\infty)<0$, $r_2(-\infty)<0$, we can obtain that there exists ξ_1,ξ_2 small enough that $r_1(\xi_1)<0$ and $r_2(\xi_2)<0$ with $\xi_2<\xi_1<0<\xi_0$. By the definition of α_i and β_i, we have $\alpha_i>\beta_i>0$ for $i=1,2$. So, when $\xi<\xi_1$, we have $\bar{U}(\xi)=K_1e^{\beta_1(\xi-\xi_1)}>K_1e^{\alpha_1(\xi-\xi_1)}>K_1e^{\alpha_1(\xi-\xi_0)}>\varepsilon e^{\alpha_1(\xi-\xi_0)}=\underline{U}(\xi)$; when $\xi>\xi_1$, $\bar{U}(\xi)=K_1>\varepsilon>\underline{U}(\xi)$, it can be seen that $\underline{U}(\xi)\le\bar{U}(\xi)$ for all $\xi\in\mathbb{R}$. Similarly, we can get that $\underline{V}(\xi)\le\bar{V}(\xi)$ for all $\xi\in\mathbb{R}$.

(i) If $\xi<\xi_0$, then $\underline{U}(\xi)=\varepsilon e^{\alpha_1(\xi-\xi_0)}>0$, $\underline{V}(\xi)=\tau e^{\alpha_2(\xi-\xi_0)}<\tau$, and $\bar{V}(\xi)\le K_2(K_1+\alpha)$. By the nondecreasing of $r_i(\xi)$ and the definition of α_i for $i=1,2$, we have

$$\begin{aligned}&d_1\underline{U}''(\xi)+c\underline{U}'(\xi)+\underline{U}(\xi)\left[r_1(\xi)-\underline{U}(\xi)-\beta\left(J*\bar{V}(\xi)\right)\right]\\ &\ge\varepsilon e^{\alpha_1(\xi-\xi_0)}\left[d_1\alpha_1^2+c\alpha_1+L_1-\varepsilon-\beta K_2(K_1+\alpha)\right]=0,\end{aligned}$$

and

$$\begin{aligned}&d_2\underline{V}''(\xi)+c\underline{V}'(\xi)+\underline{V}(\xi)\left[r_2(\xi)-\frac{\underline{V}(\xi)}{\underline{U}(\xi)+\alpha}\right]\\ &\ge\tau e^{\alpha_2(\xi-\xi_0)}\left[d_2\alpha_2^2+c\alpha_2+L_2-\frac{\tau}{\alpha}\right]=0.\end{aligned}$$

(ii) For $\xi\ge\xi_0$, obviously, $\xi>\xi_2$, then $\underline{U}(\xi)=\varepsilon$, $\underline{V}(\xi)=\tau$, $\bar{V}(\xi)=K_2(K_1+\alpha)$. Due to the nondecreasing of $r_i(\xi)$ for $i=1,2$ and the chosen of ξ_0, it is easy to show that

$$\begin{aligned}&d_1\underline{U}''(\xi)+c\underline{U}'(\xi)+\underline{U}(\xi)\left[r_1(\xi)-\underline{U}(\xi)-\beta\left(J*\bar{V}(\xi)\right)\right]\\ &=\varepsilon\left[r_1(\xi)-\varepsilon-\beta K_2(K_1+\alpha)\right]\\ &\ge\varepsilon\left[r_1(\xi_0)-\varepsilon-\beta K_2(K_1+\alpha)\right]\ge 0,\end{aligned}$$

and

$$\begin{aligned}&d_2\underline{V}''(\xi)+c\underline{V}'(\xi)+\underline{V}(\xi)\left[r_2(\xi)-\frac{\underline{V}(\xi)}{\underline{U}(\xi)+\alpha}\right]\\&=\tau\left[r_2(\xi)-\frac{\tau}{\varepsilon+\alpha}\right]\\&\geq\tau\left[r_2(\xi_0)-\frac{\tau}{\varepsilon+\alpha}\right]\geq 0.\end{aligned}$$

Thus we can obtain that (2.7) and (2.8) hold for $\xi\in\mathbb{R}$.

(iii) When $\xi>\xi_1$, then $\bar{U}(\xi)=K_1$. According to the nondecreasing of $r_1(\xi)$ and $\underline{V}(\xi)>0$, we can show

$$\begin{aligned}&d_1\bar{U}''(\xi)+c\bar{U}'(\xi)+\bar{U}(\xi)\left[r_1(\xi)-\bar{U}(\xi)-\beta\left(J*\underline{V}(\xi)\right)\right]\\&=K_1\left[r_1(\xi)-K_1-\beta\left(J*\underline{V}(\xi)\right)\right]\\&\leq K_1\left[K_1-K_1\right]=0.\end{aligned}$$

If $\xi\leq\xi_1$, obviously, $\bar{U}(\xi)=K_1\mathrm{e}^{\beta_1(\xi-\xi_1)}$ and $\underline{V}(\xi)=\tau\mathrm{e}^{\alpha_2(\xi-\xi_0)}>0$, then the definition of β_1 and the nondecreasing of $r_1(\xi)$ can imply that

$$\begin{aligned}&d_1\bar{U}''(\xi)+c\bar{U}'(\xi)+\bar{U}(\xi)\left[r_1(\xi)-\bar{U}(\xi)-\beta\left(J*\underline{V}(\xi)\right)\right]\\&=K_1\mathrm{e}^{\beta_1(\xi-\xi_1)}\left[d_1\beta_1^2+c\beta_1+r_1(\xi)-K_1\mathrm{e}^{\beta_1(\xi-\xi_1)}-\beta\left(J*\underline{V}(\xi)\right)\right]\\&\leq K_1\mathrm{e}^{\beta_1(\xi-\xi_1)}\left[d_1\beta_1^2+c\beta_1+r_1(\xi)\right]\\&\leq K_1\mathrm{e}^{\beta_1(\xi-\xi_1)}\left[d_1\beta_1^2+c\beta_1+r_1(\xi_1)\right]=0.\end{aligned}$$

That is, (2.9) hold for $\xi\in\mathbb{R}$.

(iv) For $\xi>\xi_2$, $\bar{V}(\xi)=K_2(K_1+\alpha)$, $\bar{U}(\xi)\leq K_1$ and $r_2(\xi)<K_2$, then we can obtain

$$\begin{aligned}&d_2\bar{V}''(\xi)+c\bar{V}'(\xi)+\bar{V}(\xi)\left[r_2(\xi)-\frac{\bar{V}(\xi)}{\bar{U}(\xi)+\alpha}\right]\\&\leq K_2(K_1+\alpha)\left[r_2(\xi)-\frac{K_2(K_1+\alpha)}{K_1+\alpha}\right]\\&\leq K_2(K_1+\alpha)\left[K_2-K_2\right]=0.\end{aligned}$$

For $\xi\leq\xi_2$, $\bar{V}(\xi)=K_2(K_1+\alpha)\mathrm{e}^{\beta_2(\xi-\xi_2)}>0$ and $\bar{U}(\xi)>0$. Then by the definition of β_2 and the nondecreasing of $r_2(\xi)$, we can have

$$\begin{aligned}&d_2\bar{V}''(\xi)+c\bar{V}'(\xi)+\bar{V}(\xi)\left[r_2(\xi)-\frac{\bar{V}(\xi)}{\bar{U}(\xi)+\alpha}\right]\\&\leq K_2(K_1+\alpha)\mathrm{e}^{\beta_2(\xi-\xi_2)}\left[d_2\beta_2^2+c\beta_2+r_2(\xi)\right]\\&\leq K_2(K_1+\alpha)\mathrm{e}^{\beta_2(\xi-\xi_2)}\left[d_2\beta_2^2+c\beta_2+r_2(\xi_2)\right]=0.\end{aligned}$$

Thus we can deduce that (2.10) is right.□

Define

$$\Gamma=\left\{(U,V)\in\mathcal{Z}:\underline{U}\leq U\leq\bar{U},\underline{V}\leq V\leq\bar{V}\ \text{ in }\mathbb{R}\right\},$$

and

$$F_1(U,V)(\xi)=q_1U(\xi)+U(\xi)\left[r_1(\xi)-U(\xi)-\beta\left(J*V(\xi)\right)\right],$$

$$F_2(U,V)(\xi)=q_2V(\xi)+V(\xi)\left[r_2(\xi)-\frac{V(\xi)}{U(\xi)+\alpha}\right].$$

Then we can rewrite the system (2.5) as the following form

$$\begin{cases} d_1U''(\xi)+cU'(\xi)-q_1U(\xi)+F_1(U,V)(\xi)=0, \\ d_2V''(\xi)+cV'(\xi)-q_2V(\xi)+F_2(U,V)(\xi)=0. \end{cases} \tag{2.11}$$

Notice that (U,V) is a bounded solution of (2.11) if and only if (U,V) is a fixed point of the operator $Q(U,V)=\left(Q_1(U,V),Q_2(U,V)\right)$, where

$$Q_i(U,V)(\xi)=\frac{1}{d_i(\lambda_{i2}-\lambda_{i1})}\int_{-\infty}^{+\infty}J_i(\xi-s)F_i(U,V)(s)\mathrm{d}s,$$

and

$$J_i(\xi)=\begin{cases} \mathrm{e}^{\lambda_{i2}\xi}, & \xi\le 0, \\ \mathrm{e}^{\lambda_{i1}\xi}, & \xi>0, \end{cases} \quad \lambda_{ij}=\frac{-c+(-1)^j\sqrt{c^2+4d_iq_i}}{2d_i}, i,j=1,2. \tag{2.12}$$

Lemma 2.2. *For* $(U,V)\in\Gamma$, Q_1 *is nondecreasing in U and nonincreasing in V, and* Q_2 *is nondecreasing in* $(U,V)\in\Gamma$. *Moreover, the operator Q maps* Γ *into* Γ.

Proof. For any $(U_1,V_1),(U_2,V_2)\in\Gamma$ with $U_1\ge U_2, V_1\ge V_2$, by the choice of q_i for $i=1,2$, we have

$$\begin{aligned} &F_1(U_1,V_2)(\xi)-F_1(U_2,V_1)(\xi) \\ &=\left[q_1+r_1(\xi)-U_1(\xi)-U_2(\xi)-\beta\left(J*V_2(\xi)\right)\right]\left[U_1(\xi)-U_2(\xi)\right] \\ &\quad+\beta U_2(\xi)\left[V_1(\xi)-V_2(\xi)\right] \\ &\ge 0, \end{aligned}$$

and

$$\begin{aligned} &F_2(U_1,V_1)(\xi)-F_2(U_2,V_2)(\xi) \\ &=q_2\left[V_1(\xi)-V_2(\xi)\right]+r_2(\xi)\left[V_1(\xi)-V_2(\xi)\right]-\frac{V_1^2(\xi)}{U_1(\xi)+\alpha}+\frac{V_2^2(\xi)}{U_2(\xi)+\alpha} \\ &=\left[q_2+r_2(\xi)-\frac{V_1(\xi)+V_2(\xi)}{U_1(\xi)+\alpha}\right]\left[V_1(\xi)-V_2(\xi)\right] \\ &\quad+\frac{V_2^2(\xi)}{\left[U_1(\xi)+\alpha\right]\left[U_2(\xi)+\alpha\right]}\left[U_1(\xi)-U_2(\xi)\right] \\ &\ge 0. \end{aligned}$$

It can be seen that F_1 is nondecreasing in *U* and nonincreasing in *V*, and F_2 is nondecreasing in $(U,V)\in\Gamma$. Hence, according to the definition of Q_i for $i=1,2$ and the nonnegativity of J_i defined in (2.12), we can obtain

$$\begin{aligned} &Q_1(U_1,V_2)(\xi)-Q_1(U_2,V_1)(\xi) \\ &=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\int_{-\infty}^{+\infty}J_1(\xi-s)\left[F_1(U_1,V_2)(s)-F_1(U_2,V_1)(s)\right]\mathrm{d}s\ge 0, \end{aligned}$$

and

$$Q_2(U_1,V_1)(\xi)-Q_2(U_2,V_2)(\xi)$$
$$=\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\int_{-\infty}^{+\infty}J_2(\xi-s)\left[F_2(U_1,V_1)(s)-F_2(U_2,V_2)(s)\right]\mathrm{d}s\geq 0.$$

So, the above inequalities show that

$$\begin{aligned}&Q_1(\underline{U},\overline{V})(\xi)\leq Q_1(U,V)(\xi)\leq Q_1(\overline{U},\underline{V})(\xi),\\&Q_2(\underline{U},\underline{V})(\xi)\leq Q_2(U,V)(\xi)\leq Q_2(\overline{U},\overline{V})(\xi)\end{aligned}\tag{2.13}$$

for all $(U,V)\in\Gamma$. Next we show that Q maps Γ into Γ. For $\xi<\xi_0$, then by (2.7), we can obtain

$$\begin{aligned}&Q_1(\underline{U},\overline{V})(\xi)\\&=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{\xi}\mathrm{e}^{\lambda_{11}(\xi-s)}F_1(\underline{U},\overline{V})(s)\mathrm{d}s+\int_{\xi}^{+\infty}\mathrm{e}^{\lambda_{12}(\xi-s)}F_1(\underline{U},\overline{V})(s)\mathrm{d}s\right]\\&\geq\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\lambda_{11}(\xi-s)}+\int_{\xi}^{\xi_0}\mathrm{e}^{\lambda_{12}(\xi-s)}+\int_{\xi_0}^{+\infty}\mathrm{e}^{\lambda_{12}(\xi-s)}\right)\\&\quad\times\left[-d_1\underline{U}''(s)-c\underline{U}'(s)+q_1\underline{U}(s)\right]\mathrm{d}s\\&=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left\{d_1(\lambda_{12}-\lambda_{11})\underline{U}(\xi)+d_1\mathrm{e}^{\lambda_{12}(\xi-\xi_0)}\left[\underline{U}'(\xi_0+0)-\underline{U}'(\xi_0-0)\right]\right.\\&\quad\left.+(d_1\lambda_{12}+c)\mathrm{e}^{\lambda_{12}(\xi-\xi_0)}\left[\underline{U}(\xi_0+0)-\underline{U}(\xi_0-0)\right]\right\}.\end{aligned}$$

For $\xi>\xi_0$, we have

$$\begin{aligned}&Q_1(\underline{U},\overline{V})(\xi)\\&\geq\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left(\int_{-\infty}^{\xi_0}\mathrm{e}^{\lambda_{11}(\xi-s)}+\int_{\xi_0}^{\xi}\mathrm{e}^{\lambda_{11}(\xi-s)}+\int_{\xi}^{+\infty}\mathrm{e}^{\lambda_{12}(\xi-s)}\right)\\&\quad\times\left[-d_1\underline{U}''(s)-c\underline{U}'(s)+q_1\underline{U}(s)\right]\mathrm{d}s\\&=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left\{d_1(\lambda_{12}-\lambda_{11})\underline{U}(\xi)+d_1\mathrm{e}^{\lambda_{11}(\xi-\xi_0)}\left[\underline{U}'(\xi_0+0)-\underline{U}'(\xi_0-0)\right]\right.\\&\quad\left.+(d_1\lambda_{11}+c)\mathrm{e}^{\lambda_{11}(\xi-\xi_0)}\left[\underline{U}(\xi_0+0)-\underline{U}(\xi_0-0)\right]\right\}.\end{aligned}$$

Similarly, when $\xi<\xi_0$, then by (2.8), we have

$$\begin{aligned}&Q_2(\underline{U},\underline{V})(\xi)\\&\geq\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\left\{d_2(\lambda_{22}-\lambda_{21})\underline{V}(\xi)+d_2\mathrm{e}^{\lambda_{22}(\xi-\xi_0)}\left[\underline{V}'(\xi_0+0)-\underline{V}'(\xi_0-0)\right]\right.\\&\quad\left.+(d_2\lambda_{22}+c)\mathrm{e}^{\lambda_{22}(\xi-\xi_0)}\left[\underline{V}(\xi_0+0)-\underline{V}(\xi_0-0)\right]\right\}.\end{aligned}$$

For $\xi>\xi_0$, we can obtain

$$\begin{aligned}&Q_2(\underline{U},\underline{V})(\xi)\\&\geq\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\left\{d_2(\lambda_{22}-\lambda_{21})\underline{V}(\xi)+d_2\mathrm{e}^{\lambda_{21}(\xi-\xi_0)}\left[\underline{V}'(\xi_0+0)-\underline{V}'(\xi_0-0)\right]\right.\\&\quad\left.+(d_2\lambda_{21}+c)\mathrm{e}^{\lambda_{21}(\xi-\xi_0)}\left[\underline{V}(\xi_0+0)-\underline{V}(\xi_0-0)\right]\right\}.\end{aligned}$$

Since $\underline{U}(\xi)$, $\underline{V}(\xi)$ are continuously differentiable in ξ_0, we have $Q_1(\underline{U},\overline{V})(\xi)\geq\underline{U}(\xi)$, $Q_2(\underline{U},\underline{V})(\xi)\geq\underline{V}(\xi)$ for $\xi\neq\xi_0$. By the continuity of

Q_1, Q_2, we can further have that $Q_1(\underline{U}, \bar{V})(\xi) \ge \underline{U}(\xi)$, $Q_2(\underline{U}, \underline{V})(\xi) \ge \underline{V}(\xi)$ for all ξ.

Using a similar argument with (2.9) and (2.10), we have

$$\begin{aligned}
&Q_1(\bar{U}, \underline{V})(\xi) \\
&\le \frac{1}{d_1(\lambda_{12}-\lambda_{11})}\Big\{d_1(\lambda_{12}-\lambda_{11})\bar{U}(\xi) + d_1 \mathrm{e}^{\lambda_{12}(\xi-\xi_1)}\big[\bar{U}'(\xi_1+0) - \bar{U}'(\xi_1-0)\big] \\
&\quad + (d_1\lambda_{12}+c)\mathrm{e}^{\lambda_{12}(\xi-\xi_1)}\big[\bar{U}(\xi_1+0) - \bar{U}(\xi_1-0)\big]\Big\}
\end{aligned}$$

for $\xi < \xi_1$,

$$\begin{aligned}
&Q_1(\bar{U}, \underline{V})(\xi) \\
&\le \frac{1}{d_1(\lambda_{12}-\lambda_{11})}\Big\{d_1(\lambda_{12}-\lambda_{11})\bar{U}(\xi) + d_1 \mathrm{e}^{\lambda_{11}(\xi-\xi_1)}\big[\bar{U}'(\xi_1+0) - \bar{U}'(\xi_1-0)\big] \\
&\quad + (d_1\lambda_{11}+c)\mathrm{e}^{\lambda_{11}(\xi-\xi_1)}\big[\bar{U}(\xi_1+0) - \bar{U}(\xi_1-0)\big]\Big\}
\end{aligned}$$

for $\xi > \xi_1$, and

$$\begin{aligned}
&Q_2(\bar{U}, \bar{V})(\xi) \\
&= \frac{1}{d_2(\lambda_{22}-\lambda_{21})}\Big[\int_{-\infty}^{\xi} \mathrm{e}^{\lambda_{21}(\xi-s)} F_2(\bar{U}, \bar{V})(s)\mathrm{d}s + \int_{\xi}^{+\infty} \mathrm{e}^{\lambda_{22}(\xi-s)} F_2(\bar{U}, \bar{V})(s)\mathrm{d}s\Big] \\
&\le \frac{1}{d_2(\lambda_{22}-\lambda_{21})}\Big(\int_{-\infty}^{\xi} \mathrm{e}^{\lambda_{21}(\xi-s)} + \int_{\xi}^{\xi_2} \mathrm{e}^{\lambda_{22}(\xi-s)} + \int_{\xi_2}^{+\infty} \mathrm{e}^{\lambda_{22}(\xi-s)}\Big) \\
&\quad \times\big[-d_2\bar{V}''(s) - c\bar{V}'(s) + q_2\bar{V}(s)\big]\mathrm{d}s \\
&= \frac{1}{d_2(\lambda_{22}-\lambda_{21})}\Big\{d_2(\lambda_{22}-\lambda_{21})\bar{V}(\xi) + d_2 \mathrm{e}^{\lambda_{22}(\xi-\xi_2)}\big[\bar{V}'(\xi_2+0) - \bar{V}'(\xi_2-0)\big] \\
&\quad + (d_2\lambda_{22}+c)\mathrm{e}^{\lambda_{22}(\xi-\xi_2)}\big[\bar{V}(\xi_2+0) - \bar{V}(\xi_2-0)\big]\Big\}
\end{aligned}$$

for $\xi < \xi_2$,

$$\begin{aligned}
&Q_2(\bar{U}, \bar{V})(\xi) \\
&\le \frac{1}{d_2(\lambda_{22}-\lambda_{21})}\Big(\int_{-\infty}^{\xi_2} \mathrm{e}^{\lambda_{21}(\xi-s)} + \int_{\xi_2}^{\xi} \mathrm{e}^{\lambda_{21}(\xi-s)} + \int_{\xi}^{+\infty} \mathrm{e}^{\lambda_{22}(\xi-s)}\Big) \\
&\quad \times\big[-d_2\bar{V}''(s) - c\bar{V}'(s) + q_2\bar{V}(s)\big]\mathrm{d}s \\
&= \frac{1}{d_2(\lambda_{22}-\lambda_{21})}\Big\{d_2(\lambda_{22}-\lambda_{21})\bar{V}(\xi) + d_2 \mathrm{e}^{\lambda_{21}(\xi-\xi_2)}\big[\bar{V}'(\xi_2+0) - \bar{V}'(\xi_2-0)\big] \\
&\quad + (d_2\lambda_{21}+c)\mathrm{e}^{\lambda_{21}(\xi-\xi_2)}\big[\bar{V}(\xi_2+0) - \bar{V}(\xi_2-0)\big]\Big\}
\end{aligned}$$

for $\xi > \xi_2$. Since $\bar{U}(\xi_1+0) = \bar{U}(\xi_1-0)$, $\bar{U}'(\xi_1+0) \le \bar{U}'(\xi_1-0)$, $\bar{V}(\xi_2+0) = \bar{V}(\xi_2-0)$, $\bar{V}'(\xi_2+0) \le \bar{V}'(\xi_2-0)$, we can prove that $Q_1(\bar{U}, \underline{V})(\xi) \le \bar{U}(\xi)$ for $\xi \ne \xi_1$, $Q_2(\bar{U}, \bar{V})(\xi) \le \bar{V}(\xi)$ for $\xi \ne \xi_2$. Therefore, by the continuity of Q_1, Q_2, $Q_1(\bar{U}, \underline{V})(\xi) \le \bar{U}(\xi)$, $Q_2(\bar{U}, \bar{V})(\xi) \le \bar{V}(\xi)$ for all ξ. These, together with (2.13), we obtain that $Q = (Q_1, Q_2)$ maps Γ into Γ.

Theorem 2.2. *Assume that* (A1), (A2) *and* (A3) *hold. Then (1.3) has a forced* *wave* $(u(t,x),v(t,x))=(U(x-ct),V(x-ct))$.

Proof. Define $u_-^{(0)}=\underline{U}$, $v_-^{(0)}=\underline{V}$, $u_+^{(0)}=\bar{U}$, $v_+^{(0)}=\bar{V}$, then we have

$$u_-^{(0)}\le u_+^{(0)},\quad v_-^{(0)}\le v_+^{(0)}.$$

Define the following iterations

$$u_-^{(k)}(\xi)=Q_1\left(u_-^{(k-1)},v_+^{(k-1)}\right)(\xi)=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\int_{-\infty}^{+\infty}J_1(\xi-s)F_1\left(u_-^{(k-1)},v_+^{(k-1)}\right)(s)\mathrm{d}s,$$

$$u_+^{(k)}(\xi)=Q_1\left(u_+^{(k-1)},v_-^{(k-1)}\right)(\xi)=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\int_{-\infty}^{+\infty}J_1(\xi-s)F_1\left(u_+^{(k-1)},v_-^{(k-1)}\right)(s)\mathrm{d}s,$$

$$v_-^{(k)}(\xi)=Q_2\left(u_-^{(k-1)},v_-^{(k-1)}\right)(\xi)=\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\int_{-\infty}^{+\infty}J_2(\xi-s)F_2\left(u_-^{(k-1)},v_-^{(k-1)}\right)(s)\mathrm{d}s,$$

$$v_+^{(k)}(\xi)=Q_2\left(u_+^{(k-1)},v_+^{(k-1)}\right)(\xi)=\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\int_{-\infty}^{+\infty}J_2(\xi-s)F_2\left(u_+^{(k-1)},v_+^{(k-1)}\right)(s)\mathrm{d}s$$

for $k=1,2,\cdots$. Next, when $k=1$, $u_-^{(1)}(\xi)\ge u_-^{(0)}(\xi)=\underline{U}(\xi)$, $u_+^{(1)}(\xi)\le u_+^{(0)}(\xi)=\bar{U}(\xi)$, $v_-^{(1)}(\xi)\ge v_-^{(0)}(\xi)=\underline{V}(\xi)$, $v_+^{(1)}(\xi)\le v_+^{(0)}(\xi)=\bar{V}(\xi)$ apparently hold, according to induction, when $k=n$, $u_-^{(n)}(\xi)\ge u_-^{(n-1)}(\xi)$, $u_+^{(n)}(\xi)\le u_+^{(n-1)}(\xi)$, $v_-^{(n)}(\xi)\ge v_-^{(n-1)}(\xi)$, $v_+^{(n)}(\xi)\le v_+^{(n-1)}(\xi)$ hold.

Moreover, since F_1 is nondecreasing in U and nonincreasing in V, and F_2 is nondecreasing in $(U,V)\in\Gamma$, we have when $k=1$,

$$\begin{aligned}
&u_+^{(1)}(\xi)-u_-^{(1)}(\xi)\\
&=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\int_{-\infty}^{+\infty}J_1(\xi-s)\left[F_1\left(u_+^{(0)},v_-^{(0)}\right)(s)-F_1\left(u_-^{(0)},v_+^{(0)}\right)(s)\right]\mathrm{d}s\\
&\ge 0,\\
&v_+^{(1)}(\xi)-v_-^{(1)}(\xi)\\
&=\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\int_{-\infty}^{+\infty}J_2(\xi-s)\left[F_2\left(u_+^{(0)},v_+^{(0)}\right)(s)-F_2\left(u_-^{(0)},v_-^{(0)}\right)(s)\right]\mathrm{d}s\\
&\ge 0,
\end{aligned}$$

apparently, $u_+^{(1)}(\xi)\ge u_-^{(1)}(\xi)$ and $v_+^{(1)}(\xi)\ge v_-^{(1)}(\xi)$ hold, according to induction, when $k=n$, $u_+^{(n)}(\xi)\ge u_-^{(n)}(\xi)$ and $v_+^{(n)}(\xi)\ge v_-^{(n)}(\xi)$ hold.

From all of these, we can conclude that

$$\begin{aligned}
&\underline{U}=u_-^{(0)}\le u_-^{(1)}\le u_-^{(2)}\le\cdots\le u_-^{(k)}\le u_+^{(k)}\le\cdots\le u_+^{(2)}\le u_+^{(1)}\le u_+^{(0)}=\bar{U},\\
&\underline{V}=v_-^{(0)}\le v_-^{(1)}\le v_-^{(2)}\le\cdots\le v_-^{(k)}\le v_+^{(k)}\le\cdots\le v_+^{(2)}\le v_+^{(1)}\le v_+^{(0)}=\bar{V}
\end{aligned}\tag{2.14}$$

for all $k=1,2,\cdots$ and $\xi\in\mathbb{R}$. Thus $\lim_{k\to\infty}u_-^{(k)}=u_-^*$, $\lim_{k\to\infty}u_+^{(k)}=u_+^*$, $\lim_{k\to\infty}v_-^{(k)}=v_-^*$, $\lim_{k\to\infty}v_+^{(k)}=v_+^*$ all exist. By (2.14), it is easy to see that

$$\underline{U}(\xi)\le u_-^*(\xi)\le u_+^*(\xi)\le\bar{U}(\xi),\underline{V}(\xi)\le v_-^*(\xi)\le v_+^*(\xi)\le\bar{V}(\xi)\ \text{for}\ \xi\in\mathbb{R}.$$

By the continuity of F_1 and F_2, we know that $F_1\left(u_-^{(k)},v_+^{(k)}\right)$, $F_1\left(u_+^{(k)},v_-^{(k)}\right)$, $F_2\left(u_-^{(k)},v_-^{(k)}\right)$, $F_2\left(u_+^{(k)},v_+^{(k)}\right)$ converge point-wise to $F_1\left(u_-^{(*)},v_+^{(*)}\right)$, $F_1\left(u_+^{(*)},v_-^{(*)}\right)$, $F_2\left(u_-^{(*)},v_-^{(*)}\right)$, $F_2\left(u_+^{(*)},v_+^{(*)}\right)$, respectively. Since

$$\left|F_1\left(u_-^{(k-1)}, v_+^{(k-1)}\right)\right| \le q_1 K_1,$$
$$\left|F_1\left(u_+^{(k-1)}, v_-^{(k-1)}\right)\right| \le q_1 K_1,$$
$$\left|F_2\left(u_-^{(k-1)}, v_-^{(k-1)}\right)\right| \le q_2 K_2\left(K_1+\alpha\right),$$
$$\left|F_2\left(u_+^{(k-1)}, v_+^{(k-1)}\right)\right| \le q_2 K_2\left(K_1+\alpha\right),$$

then by the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned} u_-^*(\xi) &= \lim_{k\to\infty} u_-^{(k)}(\xi) = \lim_{k\to\infty} Q_1\left(u_-^{(k-1)}, v_+^{(k-1)}\right)(\xi) \\ &= \lim_{k\to\infty} \frac{1}{d_1\left(\lambda_{12}-\lambda_{11}\right)} \int_{-\infty}^{+\infty} J_1(\xi-s) F_1\left(u_-^{(k-1)}, v_+^{(k-1)}\right)(s)\,\mathrm{d}s \\ &= \frac{1}{d_1\left(\lambda_{12}-\lambda_{11}\right)} \int_{-\infty}^{+\infty} J_1(\xi-s) F_1\left(u_-^*, v_+^*\right)(s)\,\mathrm{d}s = Q_1\left(u_-^*, v_+^*\right)(\xi), \end{aligned}$$

$$\begin{aligned} u_+^*(\xi) &= \lim_{k\to\infty} u_+^{(k)}(\xi) = \lim_{k\to\infty} Q_1\left(u_+^{(k-1)}, v_-^{(k-1)}\right)(\xi) \\ &= \lim_{k\to\infty} \frac{1}{d_1\left(\lambda_{12}-\lambda_{11}\right)} \int_{-\infty}^{+\infty} J_1(\xi-s) F_1\left(u_+^{(k-1)}, v_-^{(k-1)}\right)(s)\,\mathrm{d}s \\ &= \frac{1}{d_1\left(\lambda_{12}-\lambda_{11}\right)} \int_{-\infty}^{+\infty} J_1(\xi-s) F_1\left(u_+^*, v_-^*\right)(s)\,\mathrm{d}s = Q_1\left(u_+^*, v_-^*\right)(\xi), \end{aligned}$$

$$\begin{aligned} v_-^*(\xi) &= \lim_{k\to\infty} v_-^{(k)}(\xi) = \lim_{k\to\infty} Q_2\left(u_-^{(k-1)}, v_-^{(k-1)}\right)(\xi) \\ &= \lim_{k\to\infty} \frac{1}{d_2\left(\lambda_{22}-\lambda_{21}\right)} \int_{-\infty}^{+\infty} J_2(\xi-s) F_2\left(u_-^{(k-1)}, v_-^{(k-1)}\right)(s)\,\mathrm{d}s \\ &= \frac{1}{d_2\left(\lambda_{22}-\lambda_{21}\right)} \int_{-\infty}^{+\infty} J_2(\xi-s) F_2\left(u_-^*, v_-^*\right)(s)\,\mathrm{d}s = Q_2\left(u_-^*, v_-^*\right)(\xi) \end{aligned}$$

and

$$\begin{aligned} v_+^*(\xi) &= \lim_{k\to\infty} v_+^{(k)}(\xi) = \lim_{k\to\infty} Q_2\left(u_+^{(k-1)}, v_+^{(k-1)}\right)(\xi) \\ &= \lim_{k\to\infty} \frac{1}{d_2\left(\lambda_{22}-\lambda_{21}\right)} \int_{-\infty}^{+\infty} J_2(\xi-s) F_2\left(u_+^{(k-1)}, v_+^{(k-1)}\right)(s)\,\mathrm{d}s \\ &= \frac{1}{d_2\left(\lambda_{22}-\lambda_{21}\right)} \int_{-\infty}^{+\infty} J_2(\xi-s) F_2\left(u_+^*, v_+^*\right)(s)\,\mathrm{d}s = Q_2\left(u_+^*, v_+^*\right)(\xi). \end{aligned}$$

Similar to the proof of Theorem 4.2.7 in [21], we can obtain that

$$u_+^*(\xi) = u_-^*(\xi) =: U(\xi), \quad v_+^*(\xi) = v_-^*(\xi) =: V(\xi) \ \text{ for } \xi \in \mathbb{R}.$$

Thus $\left(U(\xi), V(\xi)\right)$ is a solution of (2.11) and satisfies

$$\underline{U}(\xi) \le U(\xi) \le \bar{U}(\xi), \underline{V}(\xi) \le V(\xi) \le \bar{V}(\xi).$$

Next, we prove that $\left(U(\xi), V(\xi)\right)$ satisfies the asymptotical boundary conditions (2.6). Since

$$\underline{U}(-\infty) = 0 = \bar{U}(-\infty), \underline{V}(-\infty) = 0 = \bar{V}(-\infty),$$

We can easily deduce

$$\left(U(-\infty), V(-\infty)\right) = (0,0). \tag{2.15}$$

Since $\underline{U}(\xi)\le U(\xi)\le \bar{U}(\xi)$, $\underline{V}(\xi)\le V(\xi)\le \bar{V}(\xi)$ and $\underline{U}(\xi)$, $\bar{U}(\xi)$, $\underline{V}(\xi)$, $\bar{V}(\xi)$ are bounded in $\xi\in\mathbb{R}$, we can obtain that $\overline{\lim\limits_{\xi\to+\infty}} U(\xi)$, $\underline{\lim\limits_{\xi\to+\infty}} U(\xi)$, $\overline{\lim\limits_{\xi\to+\infty}} V(\xi)$, $\underline{\lim\limits_{\xi\to+\infty}} V(\xi)$ exist and denote them by y_1, y_2, z_1, z_2, respectively. Also, $y_1, y_2\in(0,K_1]$, $z_1,z_2\in(0,K_2(K_1+\alpha)]$ and

$$\overline{\lim_{\xi\to+\infty}} F_1(U,V)(\xi)=q_1y_1+y_1(K_1-y_1-\beta z_2),$$
$$\underline{\lim_{\xi\to+\infty}} F_1(U,V)(\xi)=q_1y_2+y_2(K_1-y_2-\beta z_1),$$
$$\overline{\lim_{\xi\to+\infty}} F_2(U,V)(\xi)=q_2z_1+z_1\left(K_2-\frac{z_1}{y_1+\alpha}\right),$$
$$\underline{\lim_{\xi\to+\infty}} F_2(U,V)(\xi)=q_2z_2+z_2\left(K_2-\frac{z_2}{y_2+\alpha}\right).$$

In view of L'Hôpital's rule, we can obtain

$$\begin{aligned}
y_1 &= \overline{\lim_{\xi\to+\infty}} U(\xi)=\overline{\lim_{\xi\to+\infty}} Q_1(U,V)(\xi)\\
&=\overline{\lim_{\xi\to+\infty}}\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{\xi}\mathrm{e}^{\lambda_{11}(\xi-s)}F_1(U,V)(s)\mathrm{d}s+\int_{\xi}^{+\infty}\mathrm{e}^{\lambda_{12}(\xi-s)}F_1(U,V)(s)\mathrm{d}s\right]\\
&=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\frac{q_1y_1+y_1(K_1-y_1-\beta z_2)}{-\lambda_{11}}+\frac{q_1y_1+y_1(K_1-y_1-\beta z_2)}{\lambda_{12}}\right]\\
&=\frac{1}{q_1}\left[q_1y_1+y_1(K_1-y_1-\beta z_2)\right]=y_1+\frac{y_1(K_1-y_1-\beta z_2)}{q_1},
\end{aligned}$$

$$\begin{aligned}
y_2 &= \underline{\lim_{\xi\to+\infty}} U(\xi)=\underline{\lim_{\xi\to+\infty}} Q_1(U,V)(\xi)\\
&=\underline{\lim_{\xi\to+\infty}}\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{\xi}\mathrm{e}^{\lambda_{11}(\xi-s)}F_1(U,V)(s)\mathrm{d}s+\int_{\xi}^{+\infty}\mathrm{e}^{\lambda_{12}(\xi-s)}F_1(U,V)(s)\mathrm{d}s\right]\\
&=\frac{1}{d_1(\lambda_{12}-\lambda_{11})}\left[\frac{q_1y_2+y_2(K_1-y_2-\beta z_1)}{-\lambda_{11}}+\frac{q_1y_2+y_2(K_1-y_2-\beta z_1)}{\lambda_{12}}\right]\\
&=\frac{1}{q_1}\left[q_1y_2+y_2(K_1-y_2-\beta z_1)\right]=y_2+\frac{y_2(K_1-y_2-\beta z_1)}{q_1},
\end{aligned}$$

$$\begin{aligned}
z_1 &= \overline{\lim_{\xi\to+\infty}} V(\xi)=\overline{\lim_{\xi\to+\infty}} Q_2(U,V)(\xi)\\
&=\overline{\lim_{\xi\to+\infty}}\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\left[\int_{-\infty}^{\xi}\mathrm{e}^{\lambda_{21}(\xi-s)}F_2(U,V)(s)\mathrm{d}s+\int_{\xi}^{+\infty}\mathrm{e}^{\lambda_{22}(\xi-s)}F_2(U,V)(s)\mathrm{d}s\right]\\
&=\frac{1}{d_2(\lambda_{22}-\lambda_{21})}\left[\frac{q_2z_1+z_1\left(K_2-\frac{z_1}{y_1+\alpha}\right)}{-\lambda_{21}}+\frac{q_2z_1+z_1\left(K_2-\frac{z_1}{y_1+\alpha}\right)}{\lambda_{22}}\right]\\
&=\frac{1}{q_2}\left[q_2z_1+z_1\left(K_2-\frac{z_1}{y_1+\alpha}\right)\right]=z_1+\frac{z_1\left(K_2-\frac{z_1}{y_1+\alpha}\right)}{q_2},
\end{aligned}$$

and

$$
\begin{aligned}
z_2 &= \lim_{\xi\to+\infty} V(\xi) = \lim_{\xi\to+\infty} Q_2(U,V)(\xi) \\
&= \lim_{\xi\to+\infty} \frac{1}{d_2(\lambda_{22}-\lambda_{21})}\left[\int_{-\infty}^{\xi} e^{\lambda_{21}(\xi-s)} F_2(U,V)(s)\,ds + \int_{\xi}^{+\infty} e^{\lambda_{22}(\xi-s)} F_2(U,V)(s)\,ds\right] \\
&= \frac{1}{d_2(\lambda_{22}-\lambda_{21})}\left[\frac{q_2 z_2 + z_2\left(K_2 - \frac{z_2}{y_2+\alpha}\right)}{-\lambda_{21}} + \frac{q_2 z_2 + z_2\left(K_2 - \frac{z_2}{y_2+\alpha}\right)}{\lambda_{22}}\right] \\
&= \frac{1}{q_2}\left[q_2 z_2 + z_2\left(K_2 - \frac{z_2}{y_2+\alpha}\right)\right] = z_2 + \frac{z_2\left(K_2 - \frac{z_2}{y_2+\alpha}\right)}{q_2}.
\end{aligned}
$$

Hence, $y_1(K_1 - y_1 - \beta z_2) = 0$, $y_2(K_1 - y_2 - \beta z_1) = 0$, $z_1\left(K_2 - \frac{z_1}{y_1+\alpha}\right) = 0$, $z_2\left(K_2 - \frac{z_2}{y_2+\alpha}\right) = 0$. Since $y_1, y_2 \in (0, K_1]$, $z_1, z_2 \in (0, K_2(K_1+\alpha)]$, it must be $K_1 - y_1 - \beta z_2 = 0$, $K_1 - y_2 - \beta z_1 = 0$, $K_2 - \frac{z_1}{y_1+\alpha} = 0$, $K_2 - \frac{z_2}{y_2+\alpha} = 0$. By some trivial calculations, we can show that $y_1 = y_2 = \frac{K_1 - \alpha\beta K_2}{1+\beta K_2} = u^*$,

$z_1 = z_2 = \frac{K_1K_2 + \alpha K_2}{1+\beta K_2} = v^*$, that are $U(+\infty) = u^*$, $V(+\infty) = v^*$. Combining with (2.15), the asymptotic boundary conditions (2.6) are satisfied.□

3. Asymptotic Behaviors of Forced Waves

To obtain more asymptotic information of the forced waves at $\pm\infty$, we denote the solution of (2.5) with (2.6) as

$$
(U(\xi), V(\xi)) = (u^0(\xi), v^0(\xi)) \quad \text{for} -\infty < \xi < +\infty.
$$

Differentiating (2.5) with respect to ξ, then we show that $(U'(\xi), V'(\xi)) =: (\varphi_1, \varphi_2)$ satisfies

$$
\begin{cases}
d_1\varphi_1''(\xi) + c\varphi_1'(\xi) + \left[r_1(\xi) - 2u^0 - \beta(J * v^0)\right]\varphi_1 - \beta u^0(J * \varphi_2) + r_1'(\xi)u^0 = 0, \\
d_2\varphi_2''(\xi) + c\varphi_2'(\xi) + \left[r_2(\xi) - \frac{2v^0}{u^0+\alpha}\right]\varphi_2 - \frac{{v^0}^2 \cdot \varphi_1}{(u^0+\alpha)^2} + r_2'(\xi)v^0 = 0.
\end{cases}
\tag{3.1}
$$

Theorem 3.1. *Assume that* (A1)-(A4) *hold and* $\frac{\beta v^*}{u^*+\alpha} < 1$. *Then there exist positive constants* M_i, N_i, S_i *with* $i = 1, 2$ *such that the forced wave front* $(U(\xi), V(\xi))$ *to (1.3) has the following asymptotic properties*

$$
\begin{pmatrix} U(\xi) \\ V(\xi) \end{pmatrix} = \begin{pmatrix} (M_1 + o(1))e^{\frac{-c+\sqrt{c^2-4d_1L_1}}{2d_1}\xi} \\ (M_2 + o(1))e^{\frac{-c+\sqrt{c^2-4d_2L_2}}{2d_2}\xi} \end{pmatrix}
$$

as $\xi \to -\infty$; and

$$\begin{pmatrix} U(\xi) \\ V(\xi) \end{pmatrix} = \begin{pmatrix} u^* - (N_1 + o(1))e^{\lambda_2 \xi} - (S_1 + o(1))e^{\lambda_4 \xi} \\ v^* - (N_2 + o(1))e^{\lambda_2 \xi} - (S_2 + o(1))e^{\lambda_4 \xi} \end{pmatrix}$$

as $\xi \to +\infty$ provided $d_1 = d_2$. Here λ_2, λ_4 are given in (3.7).

Proof. For $\xi \to -\infty$, the limiting equations for (3.1) is rewritten as the following form

$$\begin{cases} d_1 \psi_1''(\xi) + c\psi_1'(\xi) + L_1 \psi_1(\xi) = 0, \\ d_2 \psi_2''(\xi) + c\psi_2'(\xi) + L_2 \psi_2(\xi) = 0. \end{cases} \tag{3.2}$$

So, the first equation of (3.2) has two independent solutions

$$\psi_1^1(\xi) = e^{\frac{-c-\sqrt{c^2-4d_1L_1}}{2d_1}\xi} \text{ and } \psi_1^2(\xi) = e^{\frac{-c+\sqrt{c^2-4d_1L_1}}{2d_1}\xi}.$$

Combining (3.1) with (3.2), we have that φ_1 admits the following property as $\xi \to -\infty$,

$$\varphi_1(\xi) = m_1[1 + o(1)]\psi_1^1(\xi) + n_1[1 + o(1)]\psi_1^2(\xi).$$

Since $\lim_{\xi \to -\infty} \varphi_1(\xi) = 0$, it must be $m_1 = 0$. Hence, for $\xi \to -\infty$,

$$U'(\xi) = \varphi_1(\xi) = n_1[1 + o(1)]e^{\frac{-c+\sqrt{c^2-4d_1L_1}}{2d_1}\xi}.$$

Similarly, we can deduce that for $\xi \to -\infty$,

$$V'(\xi) = \varphi_2(\xi) = n_2[1 + o(1)]e^{\frac{-c+\sqrt{c^2-4d_2L_2}}{2d_2}\xi}.$$

Using the integration on $U'(\xi)$ and $V'(\xi)$ from $-\infty$ to ξ, there are two constants $M_1 > 0$ and $M_2 > 0$ such that

$$\begin{pmatrix} U(\xi) \\ V(\xi) \end{pmatrix} = \begin{pmatrix} (M_1 + o(1))e^{\frac{-c+\sqrt{c^2-4d_1L_1}}{2d_1}\xi} \\ (M_2 + o(1))e^{\frac{-c+\sqrt{c^2-4d_2L_2}}{2d_2}\xi} \end{pmatrix}$$

as $\xi \to -\infty$. For $\xi \to +\infty$, the limiting equations for (3.1) are

$$\begin{cases} d_1 \omega_1'(\xi) + c\omega_1'(\xi) + M\omega_1(\xi) - N\omega_2(\xi) = 0, \\ d_2 \omega_2''(\xi) + c\omega_2'(\xi) - S\omega_1(\xi) + T\omega_2(\xi) = 0, \end{cases} \tag{3.3}$$

where

$$M = K_1 - 2u^* - \beta v^* = -u^* < 0,\ N = \beta u^* > 0,$$

$$T = K_2 - \frac{2v^*}{u^* + \alpha} = -\frac{v^*}{u^* + \alpha} < 0,\ S = \frac{v^{*2}}{(u^* + \alpha)^2} > 0.$$

Note that in (3.3), we have used the fact that $r_i'(+\infty) = 0$ with $i = 1, 2$. Indeed, by applying L'Hôpital's rule, we have

$$\lim_{\xi \to +\infty} r_i(\xi) = \lim_{\xi \to +\infty} \frac{e^{\xi} r_i(\xi)}{e^{\xi}} = \lim_{\xi \to +\infty} \frac{e^{\xi}[r_i(\xi) + r_i'(\xi)]}{e^{\xi}} = \lim_{\xi \to +\infty} [r_i(\xi) + r_i'(\xi)],$$

which implies $r_i'(+\infty)=0$. Choosing $\omega_1'=\tilde{\omega}_1$, $\omega_2'=\tilde{\omega}_2$, then we can rewrite (3.3) as the following first-order differential equations

$$\begin{cases}\omega_1'(\xi)=\tilde{\omega}_1,\\ \tilde{\omega}_1'(\xi)=-\dfrac{c}{d_1}\tilde{\omega}_1-\dfrac{M}{d_1}\omega_1+\dfrac{N}{d_1}\omega_2,\\ \omega_2'(\xi)=\tilde{\omega}_2,\\ \tilde{\omega}_2'(\xi)=-\dfrac{c}{d_2}\tilde{\omega}_2+\dfrac{S}{d_2}\omega_1-\dfrac{T}{d_2}\omega_2.\end{cases}\tag{3.4}$$

Then we can get the characteristic equation of (3.4) as

$$\lambda^2\left(\lambda+\frac{c}{d_1}\right)\left(\lambda+\frac{c}{d_2}\right)+\frac{T}{d_2}\lambda\left(\lambda+\frac{c}{d_1}\right)+\frac{M}{d_1}\lambda\left(\lambda+\frac{c}{d_2}\right)+\frac{MT-NS}{d_1d_2}=0.\tag{3.5}$$

If $d_1=d_2=d$, then (3.5) can be simplified as

$$\lambda^2\left(\lambda+\frac{c}{d}\right)^2+\frac{1}{d}(M+T)\lambda\left(\lambda+\frac{c}{d}\right)+\frac{1}{d^2}(MT-NS)=0.$$

Define $s=:\lambda\left(\lambda+\frac{c}{d}\right)$. Then s satisfies $s^2+\frac{1}{d}(M+T)s+\frac{1}{d^2}(MT-NS)=0$. Since

$$(M+T)^2-4(MT-NS)=(M-T)^2+4NS>0,$$
$$MT=\frac{u^*v^*}{u^*+\alpha}>\frac{\beta u^*v^{*2}}{(u^*+\alpha)^2}=NS,\ M+T<0,$$

We have $s_j=\dfrac{-(M+T)+(-1)^j\sqrt{(M+T)^2-4(MT-NS)}}{2d}>0$, $j=1,2$.

Thus the general solution corresponding to (3.4) can be expressed as

$$(\omega_1,\tilde{\omega}_1,\omega_2,\tilde{\omega}_2)^{\mathrm{T}}=l_1x_1\mathrm{e}^{\lambda_1\xi}+l_2x_2\mathrm{e}^{\lambda_2\xi}+l_3x_3\mathrm{e}^{\lambda_3\xi}+l_4x_4\mathrm{e}^{\lambda_4\xi},\tag{3.6}$$

where

$$\begin{aligned}\lambda_1&=\frac{-c+\sqrt{c^2+4d^2s_1}}{2d}>0,\quad \lambda_2=\frac{-c-\sqrt{c^2+4d^2s_1}}{2d}<0,\\ \lambda_3&=\frac{-c+\sqrt{c^2+4d^2s_2}}{2d}>0,\quad \lambda_4=\frac{-c-\sqrt{c^2+4d^2s_2}}{2d}<0,\end{aligned}\tag{3.7}$$

and x_i are eigenvectors corresponding to λ_i, l_i are arbitrary constants with $i=1,2,3,4$. Since $(\omega_1,\tilde{\omega}_1,\omega_2,\tilde{\omega}_2)^{\mathrm{T}}\to(0,0,0,0)$ as $\xi\to+\infty$, we deduce that $l_1=0$ and $l_3=0$ from (3.6). Thus, $(\omega_1,\tilde{\omega}_1,\omega_2,\tilde{\omega}_2)^{\mathrm{T}}=l_2x_2\mathrm{e}^{\lambda_2\xi}+l_4x_4\mathrm{e}^{\lambda_4\xi}$. Thus, (φ_1,φ_2) satisfies the following property

$$\begin{pmatrix}\varphi_1(\xi)\\ \varphi_2(\xi)\end{pmatrix}=\begin{pmatrix}c_1(a_1+o(1))\mathrm{e}^{\lambda_2\xi}+c_2(a_2+o(1))\mathrm{e}^{\lambda_4\xi}\\ c_1(b_1+o(1))\mathrm{e}^{\lambda_2\xi}+c_2(b_2+o(1))\mathrm{e}^{\lambda_4\xi}\end{pmatrix},$$

as $\xi\to+\infty$, where a_i,b_i,c_i are constants and c_i cannot be zero simultaneously, $i=1,2$. Meanwhile, $a_i\neq0$, $b_i\neq0$. For the solution $x_i\mathrm{e}^{\lambda_i\xi}$ to (3.4), if one of the first and third components of the eigenvectors x_i is zero, then the linear system (3.4) leads to the other components are also zero. By integrating from ξ to $+\infty$, it follows that

$$\begin{pmatrix} U(\xi) \\ V(\xi) \end{pmatrix} = \begin{pmatrix} u^* - (N_1 + o(1))e^{\lambda_2 \xi} - (S_1 + o(1))e^{\lambda_4 \xi} \\ v^* - (N_2 + o(1))e^{\lambda_2 \xi} - (S_2 + o(1))e^{\lambda_4 \xi} \end{pmatrix}$$

as $\xi \to +\infty$ and $d_1 = d_2$.

Acknowledgments

The authors would like to thank the anonymous referee for their careful reading and valuable comments.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Thomas, C.D., Cameron, A. and Green, R.E. (2004) Extinction Risk from Climate Change. *Nature*, **427**, 145-148. https://doi.org/10.1038/nature02121

[2] Wu, C., Wang, Y. and Zou, X. (2019) Spatial-Temporal Dynamics of a Lotka-Volterra Competition Model with Nonlocal Dispersal under Shifting Environment. *Journal of Differential Equations*, **267**, 4890-4921. https://doi.org/10.1016/j.jde.2019.05.019

[3] Alfaro, M., Berestycki, H. and Raoul, G. (2017) The Effect of Climate Shift on a Species Submitted to Dispersion, Evolution, Growth and Nonlocal Competition. *SIAM Journal on Mathematical Analysis*, **49**, 562-596. https://doi.org/10.1137/16M1075934

[4] Cantrell, R.S. and Cosner, C. (2003) Spatial Ecology via Reaction-Diffusion Equations. John Wiley and Sons, Hoboken. https://doi.org/10.1002/0470871296

[5] Fang, J., Lou, Y. and Wu, J. (2016) Can Pathogen Spread Keep Pace with Its Host Invasion? *SIAM Journal on Applied Mathematics*, **76**, 1633-1657. https://doi.org/10.1137/15M1029564

[6] Potapov, A. and Lewis, M. (2004) Climate and Competition: The Effect of Moving Range Boundaries on Habitat Invasibility. *Bulletin of Mathematical Biology*, **66**, 975-1008. https://doi.org/10.1016/j.bulm.2003.10.010

[7] Vo, H.-H. (2015) Persistence versus Extinction under a Climate Change in Mixed Environments. *Journal of Differential Equations*, **259**, 4947-4988. https://doi.org/10.1016/j.jde.2015.06.014

[8] Walther, G.R., Post, E., Convey, P., Menzel, A., Parmesan, C., Beebee, T.J.C., Fromentin, J.-M., Hoegh-Guldberg, O. and Bairlein, F. (2002) Ecological Responses to Recent Climate Change. *Nature*, **416**, 389-395. https://doi.org/10.1038/416389a

[9] Berestycki, H. and Fang, J. (2018) Forced Waves of the Fisher-KPP Equation in a Shifting Environment. *Journal of Differential Equations*, **264**, 2157-2183. https://doi.org/10.1016/j.jde.2017.10.016

[10] Nakata, Y. and Muroya, Y. (2010) Permanence for Nonautonomous Lotka-Volterra Cooperative Systems with Delays. *Nonlinear Analysis*: *Real World Applications*, **11**, 528-534. https://doi.org/10.1016/j.nonrwa.2009.01.002

[11] Cosner, C. (2014) Challenges in Modeling Biological Invasions and Population Distributions in a Changing Climate. *Ecological Complexity*, **20**, 258-263. https://doi.org/10.1016/j.ecocom.2014.05.007

[12] Lin, X.B., Weng, P. and Wu, C.F. (2011) Traveling Wave Solutions for a Predator-Prey System with Sigmoidal Response Function. *Journal of Dynamics and Differential Equations*, **23**, 903-921. https://doi.org/10.1007/s10884-011-9220-7

[13] Yang, Y., Wu, C. and Li, Z. (2019) Forced Waves and Their Asymptotics in a Lotka-Volterra Cooperative Model under Climate Change. *Applied Mathematics and Computation*, **353**, 254-264. https://doi.org/10.1016/j.amc.2019.01.058

[14] Hu, H., Deng, L. and Huang, J. (2021) Traveling Wave of a Nonlocal Dispersal Lotka-Volterra Cooperation Model under Shifting Habitat. *Journal of Mathematical Analysis and Applications*, **500**, Article ID: 125100. https://doi.org/10.1016/j.jmaa.2021.125100

[15] Wollkind, D.J. and Logan, J.A. (1978) Temperature-Dependent Predator-Prey Mite Ecosystem on Apple Tree Foliage. *Journal of Mathematical Biology*, **6**, 265-283. https://doi.org/10.1007/BF02547801

[16] Leslie, P.H. (1948) Some Further Notes on the Use of Matrices in Population Mathematics. *Biometrika*, **35**, 213-245. https://doi.org/10.1093/biomet/35.3-4.213

[17] Fang, Q., Cheng, H. and Yuan, R. (2023) Spatial Dynamics of Some Modified Leslie-Gower Prey-Predator Model with Shifting Habitat. *Journal of Mathematical Analysis and Applications*, **518**, Article ID: 126713. https://doi.org/10.1016/j.jmaa.2022.126713

[18] Lee, C.T., Hoopes, M.F., Diehl, J., Gilliland, W., Huxel, G., Leaver, E.V., McCain, K., Umbanhowar, J. and Mogilner, A. (2001) Non-Local Concepts and Models in Biology. *Journal of Theoretical Biology*, **210**, 201-219. https://doi.org/10.1006/jtbi.2000.2287

[19] Cheng, H. and Yuan, R. (2017) Existence and Stability of Traveling Waves for Leslie-Gower Predator-Prey System with Nonlocal Diffusion. *Discrete and Continuous Dynamical Systems*, **37**, 5422-5454. https://doi.org/10.3934/dcds.2017236

[20] Pao, C.V. (1992) Nonlinear Parabolic and Elliptic Equations. Plenum Press, New York. https://doi.org/10.1007/978-1-4615-3034-3

[21] Ye, Q. and Li, Z. (1990) Introduction to Reaction-Diffusion Equations. Science Press, Beijing.