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Solving Different Types of Differential Equations Using Modified and New Modified Adomian Decomposition Methods

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Abstract

The Modified Adomian Decomposition Method (MADM) is presented. A number of problems are solved to show the efficiency of the method. Further, a new solution scheme for solving boundary value problems with Neumann conditions is proposed. The scheme is based on the modified Adomian decomposition method and the inverse linear operator theorem. Several differential equations with Neumann boundary conditions are solved to demonstrate the high accuracy and efficiency of the proposed scheme.

Keywords

Neumann Conditions, Modified Adomian Decomposition Method, Solution Scheme, New Modified Adomian Decomposition Method, Differential Equations

1. Introduction

The Adomian Decomposition Method (ADM) was developed by George Adomian in the mid 1980's. It is a semi-analytical method that has a wide range of applications. It is used to find solutions of differential equations [1] [2], integral equations [3] [4], algebraic equations [5], fractional differential equations [6], equations containing radical [7] and systems of equations [8]. The ADM consists of decomposing the unknown function $u(x)$ of any given equation into an infinite number of components $u_0, u_1, u_2, \cdots$ and it is expressed as,

$$u(x) = \sum_{n=0}^{\infty} u_n(x). \tag{1}$$

The nonlinear terms are dealt with by an analytic parametrization in which certain polynomials A_ns depend on the nonlinearity and order of the components are derived. The solution is then expressed as an infinite series of $u_0, u_1, u_2, \cdots$. Further, the infinite series generally converges very rapidly in real physical problems. The convergence of the series has been investigated in [9]. This paper is organized as follows. In section 2, we review the ADM and the modified Adomian decomposition method. Section 3 deals with the application of MADM. In section 4 we modify MADM and have up with the new solution scheme. The new method is called New Modified Adomian Decomposition Method (NMADM). Some examples to illustrate the new method are also included. Finally in section 5, we present the conclusion.

2. Theoretical Presentation of the Adomian Decomposition Method and the Modified Adomian Decomposition Method

In this section we present the Adomian and modified Adomian decomposition methods for solving initial value differential equations.

2.1. The Adomian Decomposition Method

Consider an Initial Value Problem (IVP) for a nonlinear Ordinary Differential Equation (ODE) in the form,

$$Lu + Ru + Nu = g, \tag{2}$$

where L is the linear operator to be inverted, which usually is the highest order differential operator, N represents the nonlinear term, R is the linear remainder operator and g is the source term. Applying the inverse operator, L^{-1} on both sides of Equation (2) gives,

$$L^{-1}Lu = L^{-1}g - L^{-1}\left[Ru + Nu\right]. \tag{3}$$

Solving for u in Equation (3) we get,

$$u = \varphi(0) + L^{-1}g - L^{-1}\left[Ru + Nu\right], \tag{4}$$

where,

$$\varphi(0) = \begin{cases} u(0) & \text{if } L = \dfrac{\mathrm{d}}{\mathrm{d}x}, \\ u(0) + xu'(0) & \text{if } L = \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}, \\ u(0) + xu'(0) + \dfrac{x^2}{2!}u''(0) & \text{if } L = \dfrac{\mathrm{d}^3}{\mathrm{d}x^3}. \\ \vdots & \\ u(0) + xu'(0) + \dfrac{x^2}{2!}u''(0) + \cdots + \dfrac{x^n}{n!}u^n(0) & \text{if } L = \dfrac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}. \end{cases}$$

The ADM decomposes the solution in the form of Equation (1), and the nonlinear term Nu is decomposed into a series,

$$Nu = \sum_{n=0}^{\infty} A_n. \tag{5}$$

The A_ns are obtained for the nonlinearity $Nu = f(u)$ by the formular in [10] which is given as,

$$A_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}\left[F\left(\sum_{k=0}^{\infty} u_k \xi^k\right)\right]_{\xi=0}, \quad n = 0,1,2,\cdots. \tag{6}$$

Upon substituting Equations (1) and (5) into Equation (4) we obtain the following equation:

$$\sum_{n=0}^{\infty} u_n = \varphi(0) + L^{-1}g - L^{-1}\left[R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right]. \tag{7}$$

The solution components $u_n(x)$ are determined by the recursive scheme,

$$u_0 = \varphi(0) + L^{-1}g$$

$$u_{n+1} = -L^{-1}\left[Ru_n + A_n\right], n \geq 0.$$

The n-term approximation of the solution is given by,

$$\phi_n(x) = \sum_{k=0}^{n-1} u_k(x). \tag{8}$$

Since its introduction, the ADM has seen several modifications with the view to improve the accuracy, reduce computational efficiency or improve on the range of application of the original method [11] [12] [13] [14].

2.2. The Modified Adomian Decomposition Method

In this subsection, we present the modified Adomian decomposition method. The method is a modification of the Adomian Decomposition given in [15] [16] [17] and requires that we introduce the terms $L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right]$ in the ADM calculations. Here p is an artificial parameter and $\forall n \in (N \cup \{0\})$, a_ns are unknown coefficients to be determined. Thus we suggest that Equation (7) be rewritten as:

$$\sum_{n=0}^{\infty} u_n = \varphi(0) + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] + L^{-1}g - L^{-1}\left[R\left(\sum_{n=0}^{\infty} u_n\right) + \sum_{n=0}^{\infty} A_n\right]. \tag{9}$$

From Equation (9), the recursive relationship for MADM is expressed as follows:

$$u_0 = \varphi(0) + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right],$$
$$u_1 = L^{-1}g - pL^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right] - L^{-1}\left[R(u_0) + A_0\right],$$
$$\vdots,$$
$$u_{n+1} = -L^{-1}\left[R(u_n) + A_n\right], \text{ for } n \geq 1.$$

To avoid calculation of $A_n, n = 0,1,2,\cdots$, we determine the coefficients a_n, for $n = 0,1,2,\cdots$. By setting $u_1 = 0$ we immediately verify that $u_n = 0$, $\forall n \geq 1$. Letting $p = 1$ we write the solution of Equation (2) as

$$u(x) = \varphi(0) + L^{-1}\left[\sum_{n=0}^{\infty} a_n x^n\right].$$

We can observe that the new algorithm reduces the number of iterations because it uses u_0 and u_1 only hence the size of calculations is minimized compared to the standard ADM. It also reduces the number of Adomian polynomials to be constructed because it involves A_0 only. Thus MADM introduces an efficient algorithm that improves the performance of ADM.

3. Application of the Modified Adomian Decomposition Method

In this section, problems are solved to illustrate the use of MADM. The solutions obtained are compared with the exact solutions. Numerical simulations confirm the validity of MADM.

Example 3.1. Let us consider the Korteweg-deVries (KdV) equation taken from [18]:

$$\begin{aligned} &u_t(x,t)+\varepsilon u(x,t)u_x(x,t)+u_{xxx}(x,t)=0, \quad t>0, \quad -\infty\le x\le\infty \\ &u(x,0)=6\operatorname{sech}^2 x, \end{aligned} \tag{10}$$

with $\varepsilon=6$. The exact solution, $u(x,t)=\dfrac{12\left(4\cosh(-8t+2x)+\cosh(-64t+4x)+3\right)}{\left(3\cosh(-28t+x)+\cosh(-36t+3x)\right)^2}$. We rewrite Equation (10) as follows:

$$\left.\begin{aligned} &u_0=6\operatorname{sech}^2 x+L_t^{-1}\left(\sum_{n=0}^{\infty}a^n t^n\right) \\ &u_1=-pL_t^{-1}\left(\sum_{n=0}^{\infty}a^n t^n\right)-6L_t^{-1}(A_0)-L_t^{-1}(u_{0xxx}) \\ &\vdots \end{aligned}\right\} \tag{11}$$

By setting $p=1$ and $u_1=0$ it can be shown that,
$a_0=48\left(6\operatorname{sech}^x+1\right)\operatorname{sech}^2 x\tanh x$; $a_1=\left(72\operatorname{sech}^2 x\tanh x\right)a_0$;
$a_2=\left(36\operatorname{sech}^2 x\tanh x\right)a_1$; $a_3=\left(24\operatorname{sech}^2 x\tanh x\right)a_2$; $a_4=\left(18\operatorname{sech}^2 x\tanh x\right)a_3$;
$a_5=\left(\frac{72}{5}\operatorname{sech}^2 x\tanh x\right)a_0$; and so on. Therefore, the approximate solution is given by,

$$u(x,t)=6\operatorname{sech}(x)^2+L^{-1}\left[\sum_{n=0}^{\infty}a_n x^n\right].$$

To verify how much the approximate solution is accurate we plot the MADM solution and the exact solution. The graph is given in **Figure 1** and it shows that MADM solution gives a good approximation.

Example 3.2. Consider the nonlinear telegraph equation [19]:

$$\begin{aligned} &u_{tt}-u_{xx}+2u_t-u^2=\mathrm{e}^{-2t}\cosh^2 x-2\mathrm{e}^{-t}\cosh x, \\ &u(x,0)=\cosh x, \quad u_t(x,0)=-\cosh x. \end{aligned} \tag{12}$$

The exact solution $u(x,t)=\cosh(x)\mathrm{e}^{-t}$. Using the MADM algorithm we have the following recursive of Equation (12),

$$\left.\begin{aligned} u_0 &= \cosh x - t\cosh x + L_{tt}^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) \\ u_1 &= -pL_{tt}^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) + L_{tt}^{-1}\left(\mathrm{e}^{-2t}\cosh^2 x - 2\mathrm{e}^{-t}\cosh x\right) + L_{tt}^{-1}\left(u_{0xx}\right) \\ &\quad - 2L_{tt}^{-1}\left(u_{0t}\right) + L_{tt}^{-1}\left(A_0\right) \\ &\vdots \end{aligned}\right\} \tag{13}$$

For $u_1 = 0$ and $p = 1$ we have, $a_0 = \cosh^2 x + 3\cosh x$; $a_1 = -\cosh x - 2\cosh^2 x + 2a_0$; $a_2 = a_1 + \cosh^2 x - a_0 \cosh x$; $a_3 = a_0 \cosh x - \frac{1}{3}a_1 \cosh x + \frac{2}{3}a_2$; $a_4 = \frac{1}{2}a_3 - \frac{1}{6}a_2 \cosh x + \frac{1}{3}a_1 \cosh x$; $a_5 = \frac{2}{5}a_4 - \frac{1}{10}a_3 \cosh x + \frac{1}{6}a_2 \cosh x$; $\cdots$

Figure 2 shows the comparison between the solution by MADM and the exact

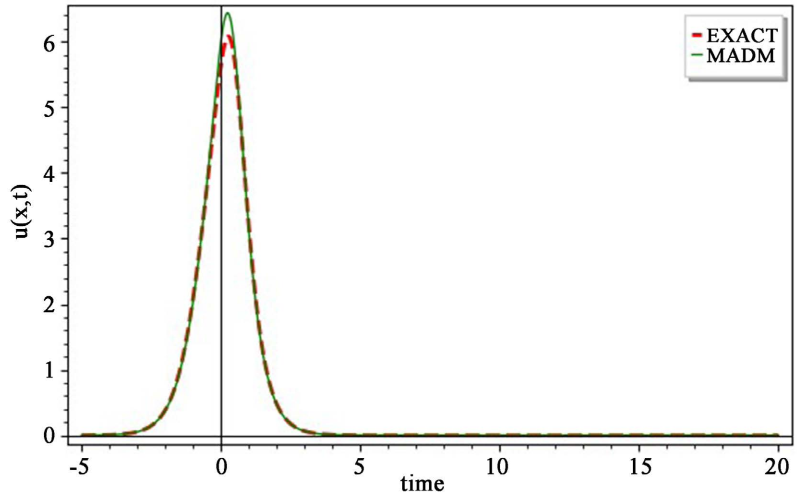

Figure 1. Comparison of the exact and MADM solutions for example 3.1 at $t = 0.01$.

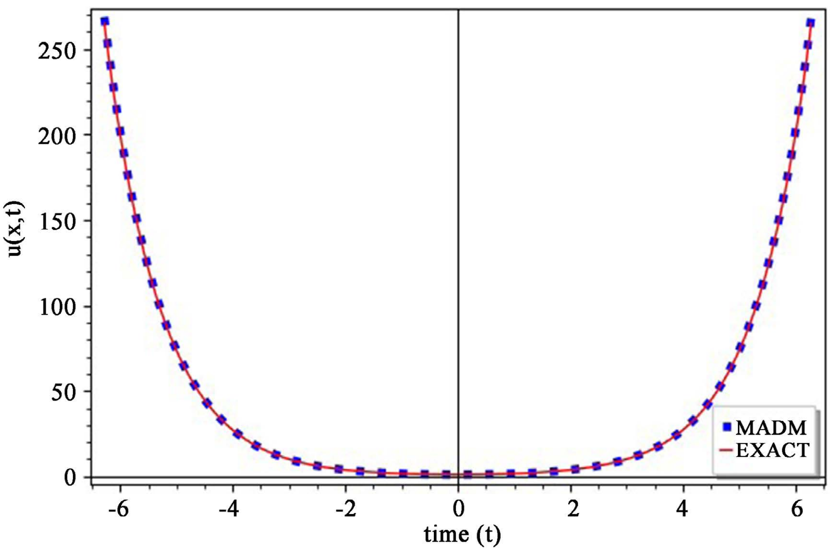

Figure 2. Comparison of the exact and MADM solutions for example 3.2 at $t = 0.01$.

solution. It is clear that the MADM solution coincides with the exact solution.

Example 3.3. Let us look at the following wave equation [20]:

$$\frac{\partial^2 w(x,t)}{\partial t^2} - 4\frac{\partial^2 w(x,t)}{\partial x^2} = 0, \quad 0 \le x \le 1, \quad 0 < t, \tag{14}$$

with the boundary conditions,

$$w(0,t) = w(1,t) = 0, \quad 0 < t$$

and initial conditions

$$w(x,0) = \sin \pi x, \quad 0 \le x \le 1, \quad \frac{\partial w(x,0)}{\partial t} = 0, \quad 0 \le x \le 1.$$

The exact solution is $w(x,t) = \sin(\pi x)\cos(2\pi t)$. By using the MADM recursive scheme we obtain the following relation for Equation (14):

$$\left.\begin{aligned} w_0 &= \sin(\pi x) + L_{tt}^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) \\ w_1 &= -pL_{tt}^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) + 4L_{tt}^{-1}\left(\frac{\partial^2 w_0(x,t)}{\partial x^2}\right) \\ &\vdots \end{aligned}\right\} \tag{15}$$

By setting $w_1 = 0$ and $p = 1$ we solve for the constants a_ns for $n = 0,1,\cdots$ and find, $a_0 = \pi^2 \sin(\pi x)$, $a_1 = a_2 = a_3 = a_4 = a_5 = \cdots = 0$.

Figure 3 depicts the graphical representation of solution obtained by MADM compared to the exact solution. It can be seen that MADM gives the same result as the exact solution.

Example 3.4. Consider the nonlinear partial differential equation [21]:

$$u_t + uu_x = 0, \quad u(x,0) = x, \quad 0 < t, \tag{16}$$

whose exact solution is given by $w(x,t) = \frac{x}{1+t}$, $|t| < 1$. We apply the MADM

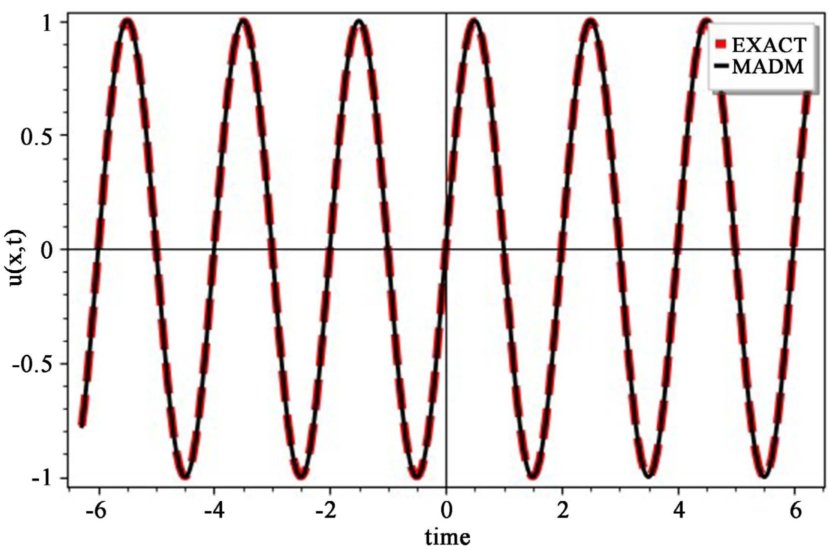

Figure 3. Comparison of the exact and MADM solutions for example 3.3 at $t = 0.01$.

recursive scheme and obtain the following relation scheme for Equation (16),

$$\left.\begin{aligned} u_0 &= x + L_t^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) \\ u_1 &= -pL_t^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) - L_t^{-1}\left(A_0\right) \\ &\vdots \end{aligned}\right\} \tag{17}$$

By setting $u_1 = 0$ and $p = 1$, it can be shown that, $a_0 = -x$, $a_1 = -a_0$, $a_2 = -\frac{1}{2}a_1$, $a_3 = -\frac{1}{6}a_2$, $a_4 = -\frac{1}{4}a_3$, $a_5 = -\frac{1}{5}a_4$ and so on.

Figure 4 shows a comparison between the exact and MADM approaches and clearly shows that the two methods give the same results.

Example 3.5. Let us consider the following Laplace equation [20]:

$$u_{xx} + u_{yy} = 0, \tag{18}$$

with boundary conditions

$$u_y\left(x,0\right) = \cos x, \quad u\left(x,0\right) = 0$$

and exact solution is given by $u\left(x,y\right) = \cos x \sinh y$. Applying the MADM recursive scheme, we obtain the relation for Equation (18) as follows:

$$\left.\begin{aligned} u_0 &= y\cos x + L_{yy}^{-1}\left(\sum_{n=0}^{\infty} a^n y^n\right) \\ u_1 &= -pL_{yy}^{-1}\left(\sum_{n=0}^{\infty} a^n y^n\right) - L_{yy}^{-1}\left(u_{0xx}\right) \\ &\vdots \end{aligned}\right\} \tag{19}$$

For $u_1 = 0$ and $p = 1$ we find the constants a_ns for $n = 0,1,\cdots$ as, $a_0 = 0$, $a_1 = \cos x$, $a_2 = a_3 = a_4 = a_5 = \cdots = 0$.

From Figure 5 we can see that the MADM solution corresponds with the

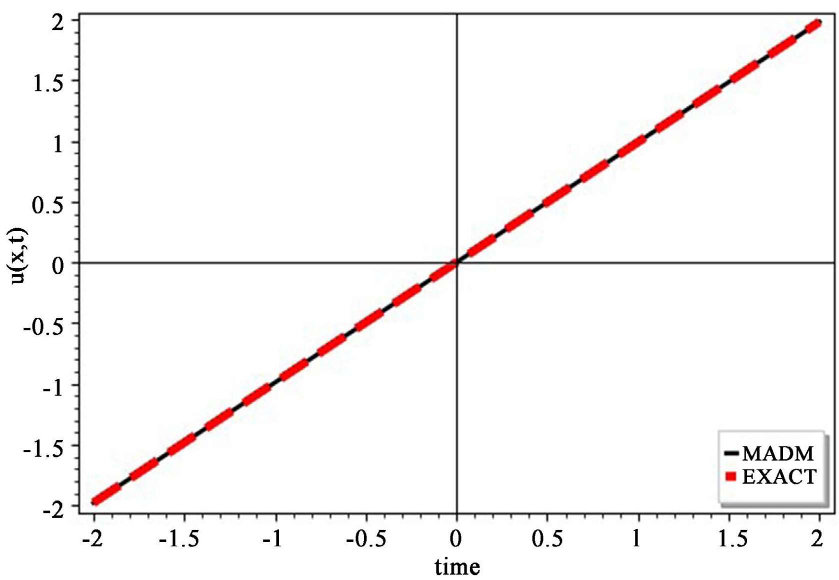

Figure 4. Comparison of the exact and MADM solutions for example 3.4 at $t = 0.01$.

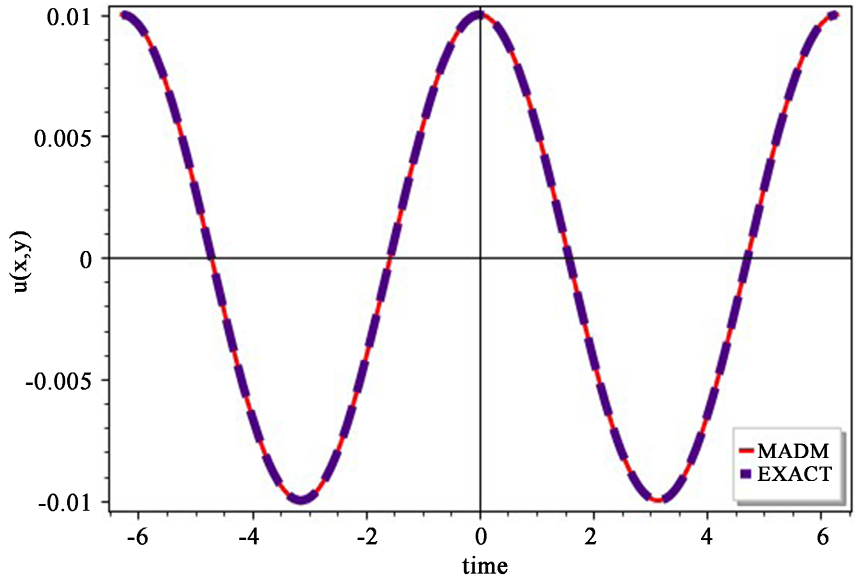

Figure 5. Comparison of the exact and MADM solutions for example 3.5 at $y = 0.01$.

exact solution.

Example 3.6. Let us consider the one dimensional unsteady heat conduction problem [20],

$$u_{xx} = 4u_t, \tag{20}$$

with initial and boundary conditions given as

$$u(0,t) = 0, \quad u(2,t) = 0, \quad u(x,0) = 2\sin\frac{\pi x}{2}.$$

Exact solution is given as $u(x,t) = 2\sin\frac{\pi x}{2} e^{\frac{-\pi^2 t}{16}}$. By using the MADM recursive scheme we obtain the following relation for Equation (20),

$$\left.\begin{array}{l} u_0 = 2\sin\frac{\pi x}{2} + L_t^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) \\ u_1 = -pL_t^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) - \frac{1}{4}L_t^{-1}(u_{0xx}) \\ \vdots \end{array}\right\} \tag{21}$$

Setting $u_1 = 0$ and $p = 1$, we can show that $a_0 = 0$, $a_1 = -\frac{1}{8}\pi^2 \sin\frac{\pi x}{2}$, $a_2 = a_3 = a_4 = a_5 = \cdots = 0$.

Figure 6 shows the comparison between the solution by MADM and the exact solution. The MADM solution coincides with the exact solution.

Example 3.7 Consider the Sawada-Kotera (SK) equation from [22] given as:

$$u_t + 45u^2 u_{xx} + 15u_x u_{xx} + 15uu_{xx} + u_{xxxxx} = 0, \tag{22}$$

with initial condition given by

$$u_0(x,t) = 2k^2 \operatorname{sech}^2(k(x - x_0))$$

and the exact answer is given as $u(x,t) = k^2 \operatorname{sech}^2(k(x - 16k^2 t - x_0))$. Using the

MADM recursive scheme we obtain the following relation for Equation (22),

$$\left.\begin{aligned} u_0 &= 2k^2\operatorname{sech}^2\left(k\left(x-x_0\right)\right)+L_t^{-1}\left(\sum_{n=0}^{\infty}a^n t^n\right)\\ u_1 &= -pL_t^{-1}\left(\sum_{n=0}^{\infty}a^n t^n\right)-45L_t^{-1}\left(A_0\right)-15L_t^{-1}\left(B_0\right)-15L_t^{-1}\left(C_0\right)-L_t^{-1}\left(u_{0xxxxx}\right)\\ &\vdots \end{aligned}\right\} \quad (23)$$

By setting $u_1 = 0$ and $p = 1$ we find the constants a_ns for $n = 0,1,\cdots$ and obtain the approximate solution. We compare the approximate solution with the exact solution. The graph in **Figure 7** shows that MADM solution gives a good approximation of the exact solution.

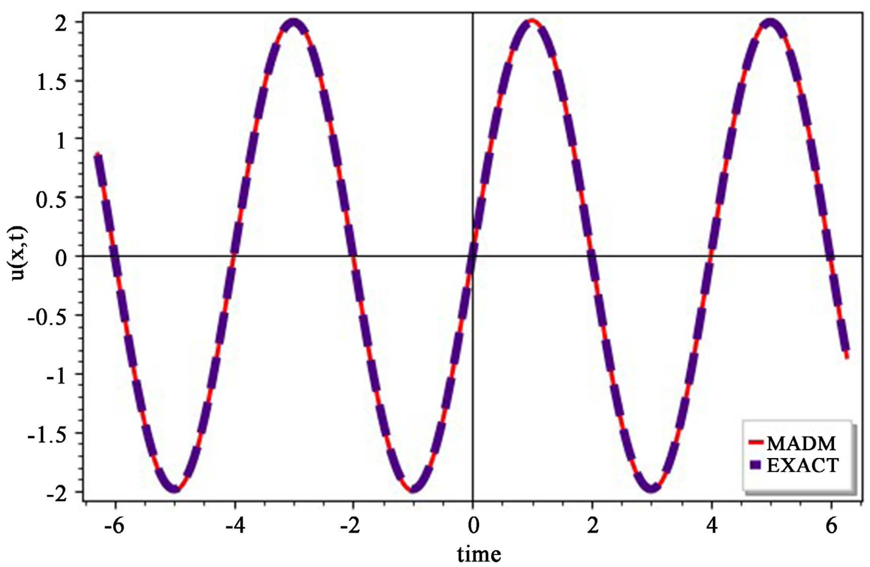

Figure 6. Comparison of the exact and MADM solutions for example 3.6 at $t = 0.01$.

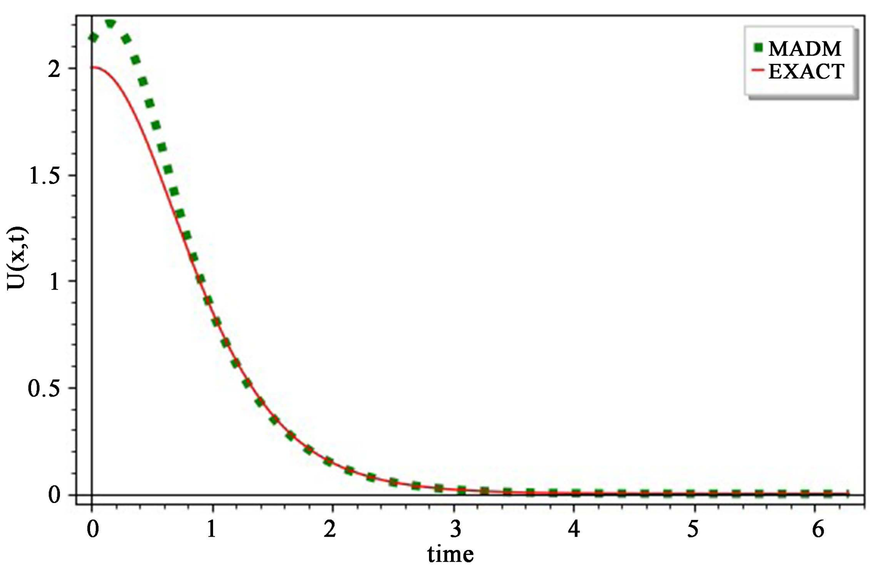

Figure 7. Comparison of the exact and MADM solutions for example 3.7 at $t = 0.001, k = 1, x_0 = 0.0$.

Example 3.8. We consider the Lax's fifth order KdV equation found in reference [22],

$$u_t + 30u^2u_x + 30u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0, \tag{24}$$

with initial condition given by

$$u(x,0) = 2k^2\left(2 - 3\tanh^2\left(k\left(x - x_0\right)\right)\right).$$

The exact solution is $u(x,t) = 2k^2\left(2 - 3\tanh^2\left(k\left(x - 56k^2t - x_0\right)\right)\right)$. By using the MADM recursive scheme we obtain the following relation for Equation (24),

$$\left.\begin{aligned} u_0 &= 2k^2\left(2 - 3\tanh^2\left(k\left(x - x_0\right)\right)\right) + L_t^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) \\ u_1 &= -pL_t^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) - 30L_t^{-1}\left(A_0\right) - 30L_t^{-1}\left(B_0\right) - 10L_t^{-1}\left(C_0\right) - L_t^{-1}\left(u_{0xxxxx}\right) \\ &\vdots \end{aligned}\right\} \tag{25}$$

By setting $u_1 = 0$ and $p = 1$ we solve for the constants a_ns for $n = 0,1,\cdots$ and find the approximate solution. Figure 8 depicts the graphical representation of solution obtained by MADM compared to the exact solution. It can be seen that MADM gives the same result as the exact solution.

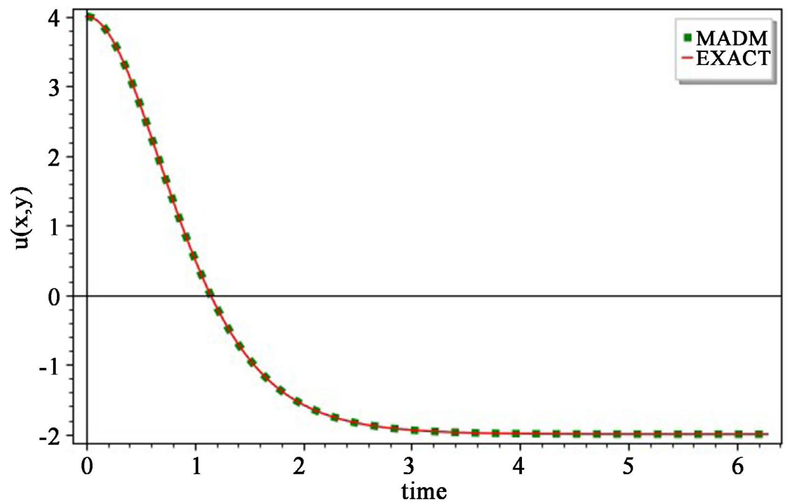

Figure 8. Comparison of the exact and MADM solutions for example 3.8 at $t = 0.0, k = 1, x_0 = 0.0$.

Example 3.9. Consider the following one-dimensional nonhomogeneous wave equation in [23],

$$u_{tt} - u_{xx} = t\sin x \tag{26}$$

with initial conditions given by

$$u(x,0) = \sin x, \quad u_t(x,0) = \sin 3x.$$

and exact solution is $u(x,t) = \sin x\cos t + \frac{1}{3}\sin 3x\sin 3t + (t - \sin t)\sin x$ By using the MADM recursive scheme we obtain the following relation for Equation (26),

$$\left.\begin{aligned} u_0 &= \sin x + t\sin 3x + L_{tt}^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) \\ u_1 &= -pL_{tt}^{-1}\left(\sum_{n=0}^{\infty} a^n t^n\right) + L_{tt}^{-1}\left(t\sin x\right) + L_{tt}^{-1}\left(u_{0xx}\right) \\ &\vdots \end{aligned}\right\} \quad (27)$$

By setting $u_1 = 0$ and $p = 1$ we solve for the constants a_ns for $n = 0,1,\cdots$ and find the approximate solution. From Figure 9 we can see that the MADM solution matches with the exact solution.

4. The New Modified Adomian Decomposition Method

In this section we modify MADM by incorporating the inverse linear operator theorem in [24] into the relation scheme of MADM. The new method is called New Modified Adomian Decomposition Method (NMADM). The NMADM gives a simplified way of solving complicated linear and nonlinear boundary value problems with Nuemann boundary conditions.

4.1. Theoretical Presentation of the NMADM

We first present the inverse operator theorem, without proof, that is used in the new method. The proof of the theorem is found in reference [24].

The Inverse Linear Operator Theorem [24]

Theorem 1. *If* $u'(a) = \alpha$ *and* $u'(b) = \beta$ *are Neumann boundary conditions of a second-order ordinary differential equation then,*

$$L_{xx}^{-1}u''(x) = u(x) - (x - \Omega)u'(a) - \frac{\Omega}{2}u'(b) - \frac{1}{\Omega}\int_0^{\Omega} u(x)\,\mathrm{d}x\mathrm{d}x,\ a \le x \le b$$

where,

$$L_{xx}^{-1}[\cdot] = \int_{\Omega}^{x}\mathrm{d}x'\int_a^{x'}[\cdot]\,\mathrm{d}x'' + \frac{1}{\Omega}\int_0^{\Omega}\mathrm{d}x'\left(x'\int_b^{x'}[\cdot]\,\mathrm{d}x''\right),$$

where Ω *is an arbitrary finite element.*

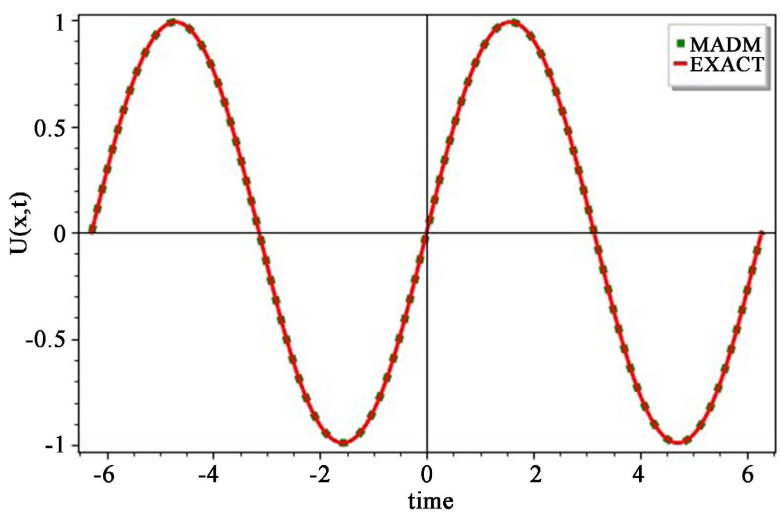

Figure 9. Comparison of the exact and MADM solutions for example 3.9 at $t = 0.01$.

To illustrate the NMADM let us consider the following equation,

$$u''(x)+m(x)u'(x)+n(x)h(u(x))=g(x), \quad \text{for } x\in[a,b] \tag{28}$$

with Nuemann boundary conditions

$$u'(a)=\beta_1, \quad u'(b)=\beta_2. \tag{29}$$

We rewrite Equation (28) as

$$u''(x)=g(x)-m(x)u'(x)-n(x)h(u(x)), \tag{30}$$

where $m(x)u'(x)$ is a linear term, $n(x)h(u(x))$ is a nonlinear term and $g(x)$ is a source term. We combine the inverse linear operator theorem and MADM recursive relation to come up with the new solution equation for Equation (28) and condition Equation (29) as,

$$\begin{aligned} u(x)=(x-\Omega)u'(a)+\frac{\Omega}{2}u'(b)+\frac{1}{\Omega}\int_0^\Omega u_n(x)\mathrm{d}x+L_{xx}^{-1}\left[\sum_{n=0}^{\infty}a_nx^n\right] \\ -pL_{xx}^{-1}\left[\sum_{n=0}^{\infty}a_nx^n\right]+L_{xx}^{-1}g(x)-m(x)L_{xx}^{-1}\left[\sum_{n=0}^{\infty}u_n'(x)\right]-L_{xx}^{-1}\left[\sum_{n=0}^{\infty}A_n\right]. \end{aligned} \tag{31}$$

From Equation (31), the recursive scheme for NMADM is given as follows:

$$\left.\begin{aligned} u_0&=(x-\Omega)u'(a)+\frac{\Omega}{2}u'(b)+L_{xx}^{-1}\left[\sum_{n=0}^{\infty}a_nx^n\right] \\ u_1&=\frac{1}{\Omega}\int_0^\Omega u_0(x)\mathrm{d}x-pL_{xx}^{-1}\left[\sum_{n=0}^{\infty}a_nx^n\right]+L_{xx}^{-1}g(x)-m(x)L_{xx}^{-1}\left[u_0'(x)\right]-L_{xx}^{-1}\left[A_0\right] \\ u_{n+1}&=-m(x)L_{xx}^{-1}\left[u_n'(x)\right]-L_{xx}^{-1}\left[A_n\right], \quad n\geq 1. \end{aligned}\right\} \tag{32}$$

It should be noted that in the evaluation of u_0 and u_1, $\Omega\to 0$ and consequently $\frac{\Omega}{2}u'(b)=0$ and after evaluation, $\frac{1}{\Omega}\int_0^\Omega u_0(x)\mathrm{d}x=0$. We compute the coefficients a_n, $n\geq 0$, by putting $u_1=0$ and setting $p=1$. This yields the solution of Equation (28) and the boundary condition Equation (29) in the form:

$$u(x)=x\beta_1+L_{xx}^{-1}\left[\sum_{n=0}^{\infty}a_nx^n\right]. \tag{33}$$

4.2. Application of New Modified Adomian Decomposition Method

We now test the efficiency of the proposed method on different boundary value problems with Neumann conditions. Numerical simulations are done to compare the solutions from NMADM and the exact solutions.

Example 4.1 Let us consider the nonlinear Boundary Value Problem (BVP) which is taken from reference [24]

$$y''-(y')^2=0, \quad 0\leq x\leq 1, \tag{34}$$

with conditions,

$$y'(0)=-1, \quad y'(1)=-\frac{1}{2}. \tag{35}$$

Exact solution is given by $y(x)=-\log(x+1)$. The NMADM solution scheme for equations (34) and (35) is given by

$$\left.\begin{aligned} y_0 &= (x-\Omega)(-1)+\frac{\Omega}{2}\left(\frac{-1}{2}\right)+L_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right) \\ y_1 &= \frac{1}{\Omega}\int_0^{\Omega} y_0(x,t)\,\mathrm{d}x - pL_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right)-L_{xx}^{-1}(A_0) \\ &\vdots \end{aligned}\right\} \quad (36)$$

We can easily show that $a_0=1, a_1=-2, a_2=3, a_3=-4,\cdots$ so that the solution of Equations (34)-(35) is given by:

$$y(x)=-x+\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{4}x^4-\frac{1}{5}x^5+\cdots.$$

The numerical results shown in **Figure 10** implies the effectiveness of NMADM.

Example 4.2. In this example we look at the nonlinear Burger equation which is in reference [24]

$$y''+yy'+y=\frac{1}{2}\sin(2x),\quad 0\le x\le\frac{\pi}{2}, \quad (37)$$

with conditions,

$$y'(0)=1,\quad y'\left(\frac{\pi}{2}\right)=0. \quad (38)$$

The exact solution is given by $y(x)=\sin(x)$. The NMADM solution scheme for Equations (37) and (38) is given by

$$\left.\begin{aligned} y_0 &= (x-\Omega)(1)+\frac{\Omega}{2}(0)+L_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right) \\ y_1 &= \frac{1}{\Omega}\int_0^{\Omega} y_0(x,t)\,\mathrm{d}x - pL_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right)+L_{xx}^{-1}\left(\frac{\sin(2x)}{2}\right)-L_{xx}^{-1}(y_0)-L_{xx}^{-1}(A_0) \\ &\vdots \end{aligned}\right\} \quad (39)$$

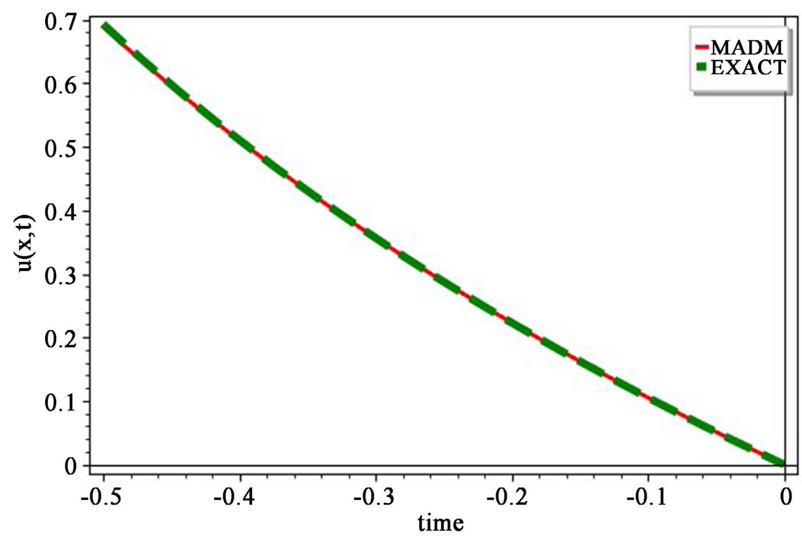

Figure 10. Comparison of the exact and NMADM solutions for example 4.1.

It can be shown that if $u_1 = 0$ and $p = 1$, then $a_0 = 0, a_1 = -1, a_2 = 0, a_3 = \frac{1}{6}, \cdots$ so that the solution of Equations ((37), (38)) is written as

$$y(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots.$$

Figure 11 shows that the result by NMADM is accurate.

Example 4.3. Consider the following linear partial BVP for the heat equation in [25],

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}, \quad 0 \le x \le 1, \quad t \ge 0, \tag{40}$$

with specified conditions,

$$y_x(0,t) = \mathrm{e}^t, \quad y_x(1,t) = \mathrm{e}^t \cosh(1), \tag{41}$$

and the exact solution is given by $y(x,t) = \mathrm{e}^t \sinh(x)$. We apply NMADM and write the solution scheme for Equations (40) and (41) as,

$$\left.\begin{aligned} y_0 &= (x-\Omega)\mathrm{e}^t + \frac{\Omega}{2}\left(\mathrm{e}^t \cosh(1)\right) + L_{xx}^{-1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_1 &= \frac{1}{\Omega}\int_0^{\Omega} y_0(x,t)\,\mathrm{d}x - pL_{xx}^{-1}\left(\sum_{n=0}^{\infty} a_n x^n\right) + L_{xx}^{-1}(L_t y_0) \\ &\vdots \end{aligned}\right\} \tag{42}$$

It can be shown that $a_0 = 0, a_1 = \mathrm{e}^t, a_2 = 0, a_3 = \frac{\mathrm{e}^t}{6}, a_4 = 0, a_5 = \frac{\mathrm{e}^t}{120}, \cdots$ so that the solution of Equations ((40), (41)) is given by

$$y(x,t) = \mathrm{e}^t\left(x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \cdots\right).$$

From Figure 12, it is easily seen that the NMADM and the exact solutions are

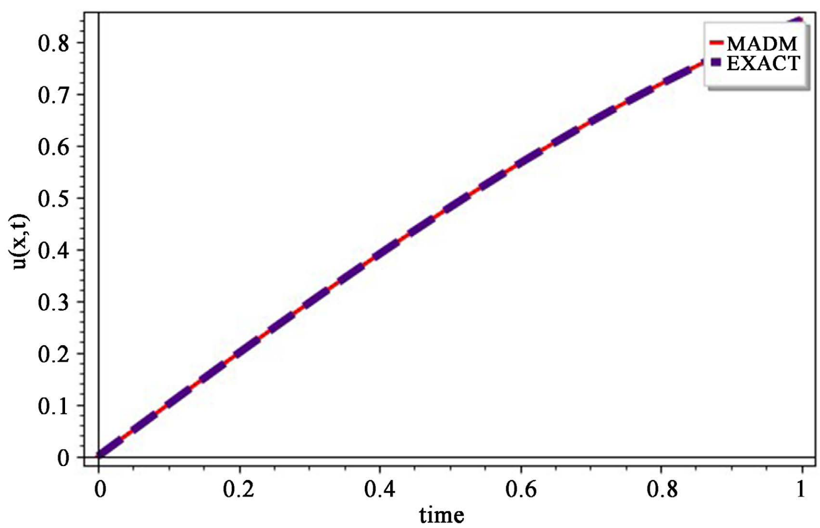

Figure 11. Comparison of the exact and NMADM solutions for example 4.2.

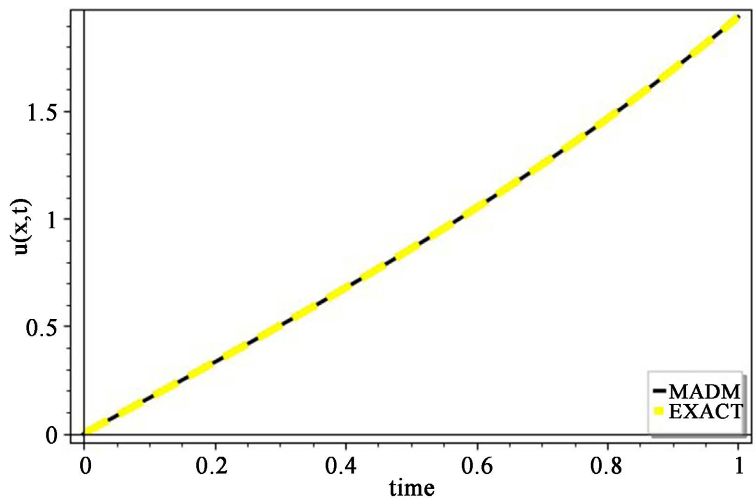

Figure 12. Comparison of the exact and NMADM solutions for example 4.3 at $t = 0.5$.

the same.

Example 4.4. Let us consider the nonlinear Burger equation in [24]

$$y_t + yy_x - y_{xx} = 0, \quad 0 \le x \le \frac{\pi}{2}, \quad t \ge 0, \tag{43}$$

with conditions,

$$y_x(0,t) = \frac{1}{t} - \frac{\pi^2}{2t^2}, \quad y_x(2,t) = \frac{1}{t} - \frac{\pi^2}{2t^2}\operatorname{sech}^2\left(\frac{\pi}{t}\right). \tag{44}$$

The exact solution is given as $y(x,t) = \frac{x}{t} - \frac{x}{2}\tanh\left(\frac{\pi x}{2t}\right)$. Using NMADM, the solution scheme for Equation (43) and conditions Equation (44) is given by

$$\left.\begin{aligned} y_0 &= (x-\Omega)\left(\frac{1}{t} - \frac{\pi^2}{2t^2}\right) + \frac{\Omega}{2}\left(\frac{1}{t} - \frac{\pi^2}{2t^2}\operatorname{sech}^2\left(\frac{\pi}{t}\right)\right) + L_{xx}^{-1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_1 &= \frac{1}{\Omega}\int_0^{\Omega} y_0(x,t)\,\mathrm{d}x - pL_{xx}^{-1}\left(\sum_{n=0}^{\infty} a_n x^n\right) + L_{xx}^{-1}(L_t y_0) + L_{xx}^{-1}(A_0) \\ &\vdots \end{aligned}\right\} \tag{45}$$

where A_0 is the Adomian polynomial of the nonlinear term yy_x. A_0 is given by $y_0 y_{0x}$. By setting $u_1 = 0$ and $p-1$, we solve for the coefficients a_n for $n = 0,1,2,3,\cdots$ and obtain $a_0 = 0$, $a_1 = \frac{\pi^4}{4t^4}$, $a_2 = 0$, $a_3 = \frac{\pi^4}{6t^5} - \frac{\pi^6}{12t^6}$, $a_4 = 0$, $a_5 = \frac{\pi^4}{20t^6} - \frac{\pi^2}{20t^7} + \frac{17\pi^8}{960t^8}$, $\cdots$. Thus the solution of Equations ((43), (44)) is,

$$y(x,t) = \frac{x}{t} - \frac{\pi}{t}\left[\left(\frac{\pi x}{2t}\right) - \frac{1}{3}\left(\frac{\pi x}{2t}\right)^3 + \frac{2}{15}\left(\frac{\pi x}{2t}\right)^5 - \frac{17}{315}\left(\frac{\pi x}{2t}\right)^7 + \cdots\right] + \xi_7,$$

where ξ_7 is a constant. It is noted from Figure 13 that NMADM is effective.

Example 4.5. We consider the following nonlinear oscillator equation which is found in reference [24]

$$y'' + \omega^2 y = \lambda y^m, \quad 0 \le x \le 1, \quad t \ge 0, \tag{46}$$

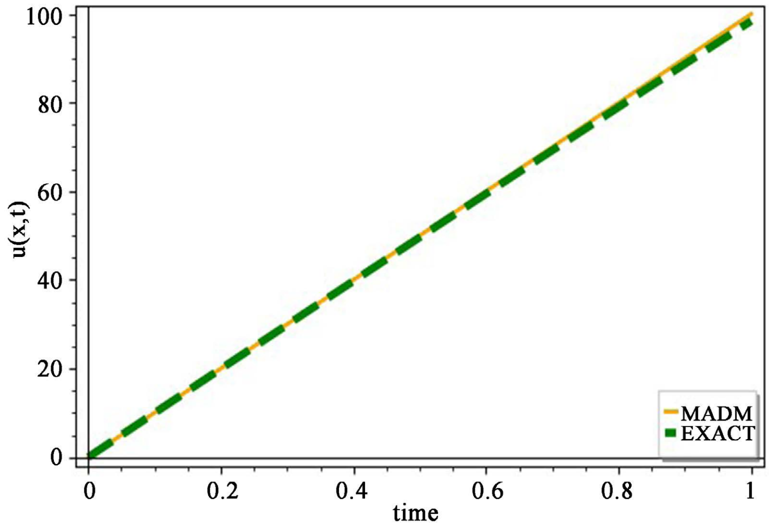

Figure 13. Comparison of the exact and NMADM solutions for example 4.4 at $t = 0.5$.

with conditions,

$$y'(0)=1, \quad y'(1)=cn\left(1\,|\,\frac{1}{4}\right)dn\left(1\,|\,\frac{1}{4}\right), \tag{47}$$

where m is a positive integer. The problem has the exact solution $y=sn\left(x\,|\,\frac{1}{4}\right)$ where $m=3$ (Duffing oscillator), $\lambda=\frac{1}{2}$ and $\omega^2=\frac{5}{4}$ and sn, cn, dn are Jacobi elliptic functions. Using NMADM solution scheme the recursive relation for Equation (46) and conditions Equation (47) can be easily given as,

$$\left.\begin{aligned} y_0 &= (x-\Omega)(1)+\frac{\Omega}{2}\left(cn\left(1\,|\,\frac{1}{4}\right)dn\left(1\,|\,\frac{1}{4}\right)\right)+L_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right) \\ y_1 &= \frac{1}{\Omega}\int_0^{\Omega} y_0(x,t)\,\mathrm{d}x - pL_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right)-\frac{5}{4}L_{xx}^{-1}(y_0)+\frac{1}{2}L_{xx}^{-1}(A_0) \\ &\vdots \end{aligned}\right\} \tag{48}$$

where A_0 is the Adomian polynomial of the nonlinear term y^3 and it is given by y_0^3. It can also be shown that $a_0=0, a_1=-\frac{5}{4}, a_2=0, a_3=\frac{73}{96}, a_4=0, a_5=-\frac{553}{1536}, \cdots$ so that the solution of Equations ((46), (47)) is given by,

$$y(x)=x-\frac{5}{24}x^3+\frac{73}{1920}x^5-\frac{79}{9216}x^7+\cdots.$$

In **Figure 14** the comparison between NMADM and the exact solution is shown. It is clear that NMADM gives an accurate solution.

Example 4.6. Consider the following linear ordinary boundary problem [24],

$$u''(x)+u(x)+x=0, \quad 0\le x\le 1 \tag{49}$$

with boundary conditions,

$$u'(0)=-1+\csc(1), \quad u'(1)=-1+\cot(1) \tag{50}$$

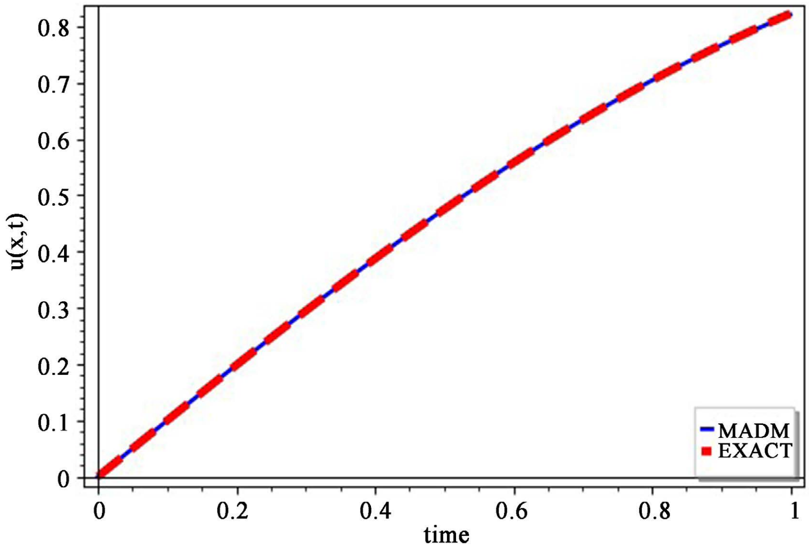

Figure 14. Comparison of the exact and NMADM solutions for example 4.5.

The NMADM solution scheme for Equation (49) and conditions Equation (50) is given by

$$\left.\begin{array}{l} u_0=(x-\Omega)(-1+\csc(1))+\dfrac{\Omega}{2}(-1+\cot(1))+L_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right) \\ u_1=\dfrac{1}{\Omega}\int_0^{\Omega}u_0(x,t)\mathrm{d}x-pL_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right)-L_{xx}^{-1}(x)-L_{xx}^{-1}(u_0) \\ \vdots \end{array}\right\} \tag{51}$$

For $u_1=0$ and $p=1$, it can be shown that: $a_0=0$, $a_1=-\csc(1)$, $a_2=0$, $a_3=\frac{1}{6}\csc(1)$, $a_4=0$, $a_5=-\frac{1}{120}\csc(1)$, $\cdots$ so that the solution of Equations ((49), (50)) is calculated as:

$$u(x)=-x+x\csc(1)-\frac{x^3}{120}\csc(1)+\frac{x^7}{5040}\csc(1)+\cdots=-x+\csc(1)\sin(x),$$

as obtained in reference [24].

Example 4.7. Consider the following linear second-order two-point BVP [25],

$$y''+xy=(3-x-x^2+x^3)\sin(x)+4x\cos(x),\quad 0\le x\le 1 \tag{52}$$

with boundary conditions

$$y'(0)=-1,\quad y'(1)=2\sin(1). \tag{53}$$

Exact solution is $y(x)=(x^2-1)\sin x$. The NMADM solution scheme for Equation (52) and conditions Equation (53) is given by

$$\left.\begin{array}{l} y_0=(x-\Omega)(-1)+\dfrac{\Omega}{2}(2\sin(1))+L_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right) \\ y_1=\dfrac{1}{\Omega}\int_0^{\Omega}y_0(x,t)\mathrm{d}x-pL_{xx}^{-1}\left(\sum_{n=0}^{\infty}a_n x^n\right)+L_{xx}^{-1}\left((3-x-x^2+x^3)\sin x+4x\cos x\right) \\ \quad -L_{xx}^{-1}(xy_0) \\ \vdots \end{array}\right\} \tag{54}$$

For $y_1 = 0$ and $p = 1$, it can be shown that, $a_0 = 0$, $a_1 = 7$, $a_2 = 0$, $a_3 = -\frac{2}{7}$, $a_4 = 0$, $a_5 = \frac{43}{120}$, $\cdots$. Thus the solution to equations (52) and (53) is given by

$$y(x) = -x + \frac{7}{3!}x^3 - \frac{21}{5!}x^5 + \frac{43}{7!}x^7 + \cdots.$$

The numerical results shown in **Figure 15** implies the effectiveness of the new method discussed in this section.

Example 4.8. Consider the following nonlinear second order two-point BVP [25]:

$$y'' + \mathrm{e}^{-2y} = 0, \quad 0 \le x \le 1 \tag{55}$$

with boundary conditions

$$y'(0) = 1, \quad y'(1) = \frac{1}{2}. \tag{56}$$

The exact solution is $y(x) = \ln(1+x)$. The NMADM solution scheme for Equation (55) and conditions Equation (56) is given by

$$\left.\begin{aligned} y_0 &= (x-\Omega)(1) + \frac{\Omega}{2}\left(\frac{1}{2}\right) + L_{xx}^{-1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_1 &= \frac{1}{\Omega}\int_0^{\Omega} y_0(x,t)\,\mathrm{d}x - pL_{xx}^{-1}\left(\sum_{n=0}^{\infty} a_n x^n\right) - L_{xx}^{-1}(A_0) \\ &\vdots \end{aligned}\right\} \tag{57}$$

where $A_0 = \mathrm{e}^{-2y_0}$ is the Adomian polynomial of the nonlinear term e^{-2y}. For $y_1 = 0$ and $p = 1$, it can be shown that

$a_0 = -1, a_1 = 2, a_2 = -3, a_3 = 4, a_4 = -5, a_5 = 6, \cdots$

Therefore the solution of Equations (55) and (56) is given as

$$y(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 + \cdots.$$

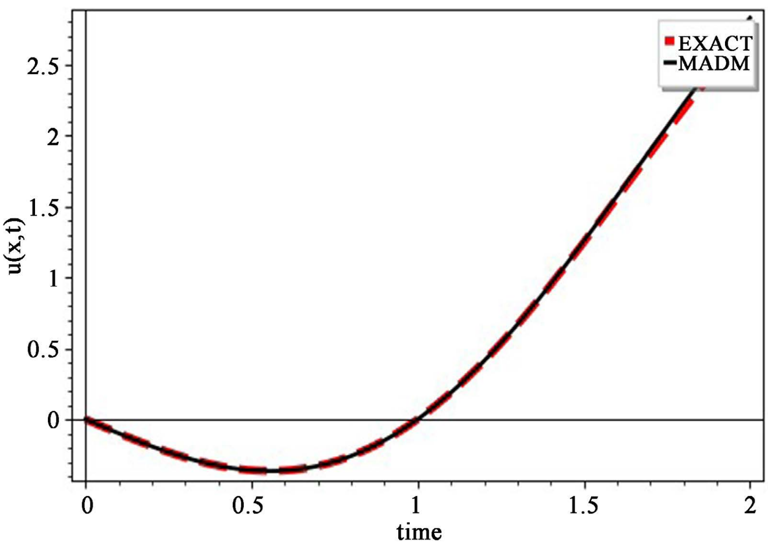

Figure 15. Comparison of the exact and NMADM solutions for example 4.7.

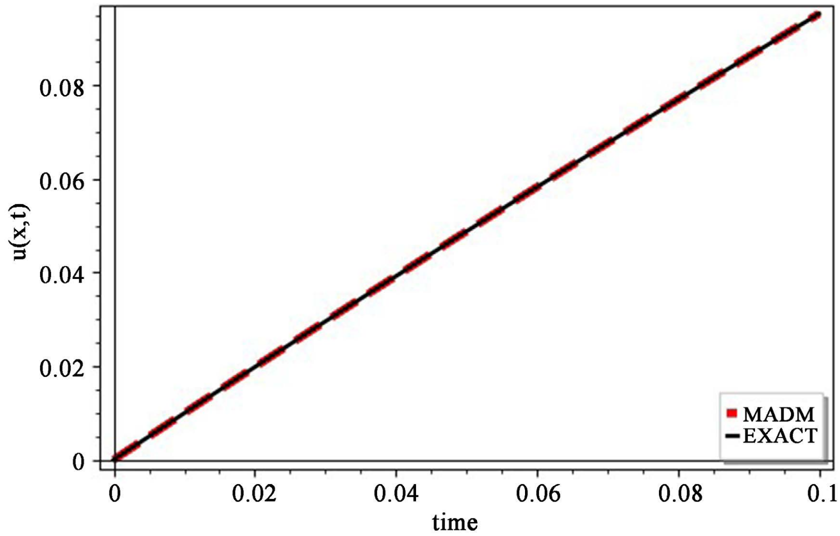

Figure 16. Comparison of the exact and NMADM solutions for example 4.8.

In **Figure 16**, the NMADM solution is plotted against the exact solution. It is seen that NMADM is effective.

5. Conclusion

The modifications of Adomian decomposition method for solving different types of differential equations is presented and provided with the solution schemes. The proposed schemes need one Adomian polynomial (A_0), for nonlinear terms hence the steps are really short. The comparisons between the new schemes and the exact solutions show that the solutions are the same. The methods give alternative ways of solutions of differential equations.

Data Availability

Due to the nature of research, supporting data is not available.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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