Journal of Applied Mathematics and Physics, 2023, 11, 1429-1447
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Crossing Limit Cycles of Planar Piecewise Hamiltonian Systems with Linear Centers Separated by Two Parallel Straight Lines

Zhou Jin

School of Mathematics and Physics, China University of Geoscience, Wuhan, China
Email: jinzhou96@cug.edu.cn

How to cite this paper: Jin, Z. (2023) Crossing Limit Cycles of Planar Piecewise Hamiltonian Systems with Linear Centers Separated by Two Parallel Straight Lines. *Journal of Applied Mathematics and Physics*, **11**, 1429-1447.
https://doi.org/10.4236/jamp.2023.115093

Received: April 26, 2023
Accepted: May 28, 2023
Published: May 31, 2023

Abstract

In this paper, we have studied several classes of planar piecewise Hamiltonian systems with three zones separated by two parallel straight lines. Firstly, we give the maximal number of limit cycles in these classes of systems with a center in two zones and without equilibrium points in the other zone (or with a center in one zone and without equilibrium points in the other zones). In addition, we also give examples to illustrate that it can reach the maximal number.

Keywords

Limit Cycles, Planar Piecewise Hamiltonian Systems, Straight Lines, Centers, Equilibrium Points

1. Introduction

It is an important problem to study the limit cycles of differential systems, which is related to Hilbert's 16th problem [1] on the maximal number of limit cycles for polynomial differential systems. In 1977, Arnold [2] proposed the weakened Hilbert's 16th problem, which is to investigate the maximal number of simple zeros of the Abelian integral for piecewise differential systems. Recently, piecewise smooth systems are widely studied by researchers from different fields and it has a large number of applications in biology [3] [4] [5], mechanics [6] [7] and control theory [8] [9] [10].

One of the most important problems is that if a piecewise smooth differential system exists limit cycles and the maximal number of limit cycles. It is known that discontinuous piecewise linear differential systems having a straight line as separation manifold can have three limit cycles, see [11]-[16]. It is still an open problem if there are more limit cycles for this class of systems. In 1998, Freire *et al.* proved that a continuous planar piecewise linear system whose switching manifold is a straight line can have at most one limit cycle. In 2015, Llibre *et al.* [17] proved that there are at most two limit cycles when a discontinuous piecewise linear differential system has a focus, center, or weak saddle with a switching line. In particular, there are many distinguished results about the differential piecewise systems separated by two parallel straight lines. In 2020, Fonseca *et al.* [18] show that planar piecewise linear Hamiltonian systems separated by two parallel straight lines and without equilibrium points in each zone can have at most one crossing limit cycle. In [19], it proved that the piecewise differential systems continuous and separated by two parallel straight lines do not have limit cycles, and the piecewise differential systems discontinuous having two parallel straight lines with either two centers and one saddle, or two saddles and one center can have at most one limit cycle.

Motivated by the above research, the main purpose of this paper is to study how we can get the maximal limit cycles when the switching manifold is two parallel straight lines, and it has subsystems with a center in two zones and without equilibrium points in the other zone (or with a center in one zone and without equilibrium points in the other zones). We can divide plane into the following three zones:

$$\begin{aligned} U^1 &= \{(x, y) \in \mathbb{R}^2 \mid y < -1\}, \\ U^2 &= \{(x, y) \in \mathbb{R}^2 \mid -1 \le y \le 1\}, \\ U^3 &= \{(x, y) \in \mathbb{R}^2 \mid y > 1\}. \end{aligned} \tag{1}$$

And we have the following main results.

Theorem 1. The following statements hold.

1) A continuous piecewise differential system separated by two parallel straight lines, which has subsystems with a linear Hamiltonian center in two regions and a subsystem without equilibrium points in the other region has no limit cycle.

2) A discontinuous piecewise differential system separated by two parallel straight lines, which has subsystems with a linear Hamiltonian center in two regions and a subsystem without equilibrium points in the other region has at most one limit cycle.

Theorem 2. The following statements hold.

1) A class of piecewise differential system separated by two parallel straight lines, which has a subsystem with a linear Hamiltonian center in one region and subsystems without equilibrium points in the other regions has no limit cycle.

2) A discontinuous piecewise differential system separated by two parallel straight lines, which has a subsystem with a linear Hamiltonian center in one region and subsystems without equilibrium points in the other regions has at most one limit cycle.

Theorem 1 is proved in Section 3 and Theorem 2 is proved in Section 4.

2. Preliminaries

Firstly, to prove Theorem 1, we introduce the following lemmas proved in [20] and [18], which provide the normal form of the planar differential system with a linear Hamiltonian center and without equilibrium points separately.

Lemma 3. An arbitrary planar differential system with a linear Hamiltonian center can be written as

$$\dot{x} = -bx - \delta y + d,\ \dot{y} = \alpha x + by + c, \tag{2}$$

where b, c, d, α and δ are all real constants, and $\delta = b^2 + \omega^2$ with $\omega \neq 0$. The corresponding Hamiltonian function is

$$H_1(x, y) = -\frac{1}{2}x^2 - bxy - \frac{\delta}{2}y^2 - cx + dy. \tag{3}$$

Lemma 4. An arbitrary planar differential system without equilibrium points can be written as

$$\dot{x} = -\lambda hx + hy + g,\ \dot{y} = -\lambda^2 hx + \lambda hy + f, \tag{4}$$

where f, g, h and λ are all real constants, $f \neq \lambda g$ and $h \neq 0$. The corresponding Hamiltonian function is

$$H_2(x, y) = -\frac{1}{2}\lambda^2 hx^2 + \lambda hxy - \frac{h}{2}y^2 + fx - gy. \tag{5}$$

Now, we begin to prove each statement of Theorem 1.

3. Proof of Theorem 1

Considering the symmetry of the system, there are two cases with respect to the class of equilibrium points on each zone to discuss here. Firstly, we consider the following planar piecewise system:

$$\begin{cases} \dot{x} = -b_i x - \delta_i y + d_i, \\ \dot{y} = \alpha_i x + b_i y + c_i, \end{cases} (x, y) \in U^i, i = 1, 3,$$
$$\begin{cases} \dot{x} = -\lambda_1 h_1 x + h_1 y + g_1, \\ \dot{y} = -\lambda_1^2 h_1 x + \lambda_1 h_1 y + f_1, \end{cases} (x, y) \in U^2, \tag{6}$$

where $f_1, g_1, h_1, \lambda_1, b_i, c_i, d_i, \alpha_i, \delta_i, i = 1, 3$ are all real constants with $\delta_i = b_i^2 + \omega_i^2$, $\omega_i \neq 0$, $i = 1, 3$, $f_1 \neq \lambda_1 g_1$ and $h_1 \neq 0$. The corresponding Hamiltonian functions are

$$H_i(x, y) = -\frac{1}{2}x^2 - b_i xy - \frac{\delta_i}{2}y^2 - c_i x + d_i y, i = 1, 3,$$
$$H_2(x, y) = -\frac{1}{2}\lambda_1^2 h_1 x^2 + \lambda_1 h_1 xy - \frac{h_1}{2}y^2 + f_1 x - g_1 y. \tag{7}$$

We are supposed that system (6) has a limit cycle intersecting the two parallel lines $x = -1$ and $x = 1$ at exactly four points, denoted as $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$ respectively with $y_1 < y_2$ and $y_3 > y_4$. These points satisfy

$$H_1(-1, y_1) - H_1(-1, y_2) = 0,$$
$$H_2(-1, y_2) - H_2(1, y_3) = 0,$$

$$H_2(1, y_4) - H_2(-1, y_1) = 0,$$
$$H_3(1, y_3) - H_3(1, y_4) = 0, \tag{8}$$

which can be written as the expansions

$$(y_1 - y_2)(2(b_1 + d_1) - \delta_1(y_1 + y_2)) = 0,$$
$$-4f_1 - 2(\lambda_1 h_1 + g_1)y_2 - 2(\lambda_1 h_1 - g_1)y_3 - h_1(y_2^2 - y_3^2) = 0,$$
$$4f_1 + 2(\lambda_1 h_1 - g_1)y_4 + 2(\lambda_1 h_1 + g_1)y_1 - h_1(y_4^2 - y_1^2) = 0,$$
$$(2(b_3 - d_3) + \delta_3(y_3 + y_4))(y_3 - y_4) = 0. \tag{9}$$

Assume that system (6) is continuous, one has

$$b_i = \lambda_1 h_1, \delta_i = -h_1, i = 1,3, d_1 = g_1, d_3 = g_1,$$
$$c_1 = \lambda_1^2 h_1 + f_1 + \alpha_1, c_3 = -\lambda_1^2 h_1 + f_1 - \alpha_3. \tag{10}$$

From (9) and (10), we have the following expressions of y_2, y_3, y_4:

$$y_2 = -\frac{2(\lambda_1 h_1 + g_1)}{h_1} - y_1, y_3 = \frac{\lambda_1 h_1 - g_1}{h_1} \pm \frac{\sqrt{\Delta}}{h_1}, y_4 = \frac{\lambda_1 h_1 - g_1}{h_1} - \mp \frac{\sqrt{\Delta}}{h_1}, \tag{11}$$

where $\Delta = (\lambda_1 h_1 - g_1)^2 + h_1(h_1 y_1^2 + 2(\lambda_1 h_1 + g_1)y_1 + 4f_1)$. Since the points y_2, y_3, y_4 are all continuous with respect to y_1, there is a continuum of periodic orbits. It implies that if system (6) is continuous, it has no limit cycles.

In the other way, let us now consider the case where system (6) is discontinuous. From (9), we have the relations as follow

$$y_1 = \frac{2(b_1 + d_1)}{\delta_1} - y_2, y_3 = \frac{2(d_3 - b_3)}{\delta_3} - y_4. \tag{12}$$

Substituting the expressions of y_1 and y_3 into the second and third equations of (9) and introducing the notion L_1 and L_2, one has

$$\begin{aligned} L_1 &= 4\left(f_1\delta_3^2 + (\lambda_1 h_1 - g_1)(d_3 - b_3)\delta_3 - h_1(d_3 - b_3)^2\right) \\ &\quad + 2\delta_3^2(\lambda_1 h_1 + g_1)y_2 + 2\left(2h_1(d_3 - b_3)\delta_3 - (\lambda_1 h_1 - g_1)\delta_3^2\right)y_4 \\ &\quad + h_1\delta_3^2(y_2^2 - y_4^2) \\ &= 0, \end{aligned} \tag{13}$$

and

$$\begin{aligned} L_2 &= 4\left(f_1\delta_1^2 + (\lambda_1 h_1 + g_1)(b_1 + d_1)\delta_1 + h_1(b_1 + d_1)^2\right) \\ &\quad - 2\left((\lambda_1 h_1 + g_1)\delta_1^2 + 2h_1\delta_1(b_1 + d_1)\right)y_2 \\ &\quad + 2\delta_1^2(\lambda_1 h_1 - g_1)y_4 + h_1\delta_1^2(y_2^2 - y_4^2) \\ &= 0. \end{aligned} \tag{14}$$

Let $L_3 = \delta_1^2 L_1 - \delta_3^2 L_2 = 0$, we obtain

$$y_4 = \frac{P_0}{P_1} + \frac{P_2}{P_1}y_2, \tag{15}$$

where

$$P_0 = \delta_1\delta_3\left(\delta_3\left(\lambda_1 h_1 + g_1\right)\left(b_1 + d_1\right) - \delta_1\left(\lambda_1 h_1 - g_1\right)\left(d_3 - b_3\right)\right) + \delta_1^2 h_1\left(d_3 - b_3\right)^2 + \delta_3^2 h_1\left(b_1 + d_1\right)^2,$$
$$P_1 = \delta_1^2\delta_3\left(h_1\left(d_3 - b_3\right) - \left(\lambda_1 h_1 - g_1\right)\delta_3\right), \tag{16}$$
$$P_2 = -\delta_1\delta_3^2\left(\delta_1\left(\lambda_1 h_1 + g_1\right) + h_1\left(b_1 + d_1\right)\right).$$

If $P_1 = 0$, it has $h_1\left(d_3 - b_3\right) - \left(\lambda_1 h_1 - g_1\right)\delta_3 = 0$. Substituting it into $L_3 = 0$, one has $y_1 = y_2 = \frac{b_1 + d_1}{\delta_1}$ which leads to contradictions. Hence, we have $P_1 \neq 0$. Then, substituting (15) into (13), we have the following expression of y_2:

$$y_{2\pm} = \frac{b_1 + d_1}{\delta_1} \pm \frac{\sqrt{\Delta'}}{2P_3}, \tag{17}$$

where

$$P_3 = h_1\delta_3^2\left(\delta_1^4\delta_3^2\left(\left(-b_3 + d_1^2 + d_4\right)h_1 + \delta_3\left(g_1 - \lambda_1 h_1\right)\right)^2 - \delta_1\delta_3^4\left(\left(b_1 + d_1\right)h_1 + \delta_1\left(g_1 + \lambda_1 h_1\right)\right)^2\right),$$

$$\begin{aligned}\Delta' = \Big(& 2\delta_3^2\left(\delta_1^2\delta_3\left(h_1\left(d_3 - b_3\right) - \left(\lambda_1 h_1 - g_1\right)\right)\delta_3\right)^2\left(\lambda_1 h_1 + g_1\right) + 2\Big(\delta_1^2\delta_3\big(h_1\left(d_3 - b_3\right) \\
& -\left(\lambda_1 h_1 - g_1\right)\delta_3\big)\Big)\left(-\delta_1\delta_3^2\left(\delta_1\left(\lambda_1 h_1 + g_1\right) + h_1\left(b_1 + d_1\right)\right)\right)\left(2h_1\left(d_3 - b_3\right)\delta_3 - \left(\lambda_1 h_1 - g_1\right)\delta_3^2\right) \\
& -2h_1\delta_3^2\Big(\delta_1\delta_3\left(\delta_3\left(\lambda_1 h_1 + g_1\right)\left(b_1 + d_1\right) - \delta_1\left(\lambda_1 h_1 - g_1\right)\left(d_3 - b_3\right)\right) + \delta_1^2 h_1\left(d_3 - b_3\right)^2 \\
& +\delta_3^2 h_1\left(b_1 + d_1\right)^2\Big)\left(-\delta_1\delta_3^2\left(\delta_1\left(\lambda_1 h_1 + g_1\right) + h_1\left(b_1 + d_1\right)\right)\right)\Big)^2 - 4h_1\delta_3^2\Big(\Big(\delta_1^2\delta_3\, h_1\left(d_3 - b_3\right) \\
& -\left(\lambda_1 h_1 - g_1\right)\delta_3\Big)\Big)^2 - \left(-\delta_1\delta_3^2\left(\delta_1\left(\lambda_1 h_1 + g_1\right) + h_1\left(b_1 + d_1\right)\right)\right)^2\Big)\Big(4\Big(\delta_1^2\delta_3\big(h_1\left(d_3 - b_3\right) \\
& -\left(\lambda_1 h_1 - g_1\right)\delta_3\big)\Big)^2\left(f_1\delta_3^2 + \left(\lambda_1 h_1 - g_1\right)\left(d_3 - b_3\right)\delta_3 - h_1\left(d_3 - b_3\right)^2\right) + 2\Big(\delta_1\delta_3\big(\delta_3\left(\lambda_1 h_1 + g_1\right)\big(b_1 \\
& +d_1\big) - \delta_1\left(\lambda_1 h_1 - g_1\right)\left(d_3 - b_3\right)\big) + \delta_1^2 h_1\left(d_3 - b_3\right)^2 + \delta_3^2 h_1\left(b_1 + d_1\right)^2\Big)\Big(\delta_1^2\delta_3\big(h_1\left(d_3 - b_3\right) \\
& -\left(\lambda_1 h_1 - g_1\right)\delta_3\big)\Big)\left(2h_1\left(d_3 - b_3\right)\delta_3 - \left(\lambda_1 h_1 - g_1\right)\delta_3^2\right) - h_1\delta_3^2\Big(\delta_1\delta_3\big(\delta_3\left(\lambda_1 h_1 + g_1\right)\left(b_1 + d_1\right) \\
& -\delta_1\left(\lambda_1 h_1 - g_1\right)\left(d_3 - b_3\right)\big) + \delta_1^2 h_1\left(d_3 - b_3\right)^2 + \delta_3^2 h_1\left(b_1 + d_1\right)^2\Big)^2,\end{aligned} \tag{18}$$

whenever $P_3 \neq 0$, and if $P_3 = 0$, there is at most one solution $\left(y_1, y_2, y_3, y_4\right)$ of (8).

When $P_3 \neq 0$, given

$$y_{1\pm} = \frac{2\left(b_1 + d_1\right)}{\delta_1} - y_{2\pm} = \frac{b_1 + d_1}{\delta_1} \mp \frac{\sqrt{\Delta'}}{2P_3} = y_{2\mp}, \tag{19}$$

it exists at most one solution $\left(y_1, y_2, y_3, y_4\right)$ of (8) with $y_1 < y_2$ and $y_3 > y_4$. As a result, we have proved that there is at most one limit cycle system (6) can have in discontinuous case. Thus we have completed the proof of Theorem 1 for systems can be written as (6).

To end this part of proof, now we provide an example to illustrate that system (6) can have exactly one limit cycle when it is discontinuous. The Hamiltonian functions are given by the following expressions:

$$H_1(x, y) = y - y^2 - (x + 2y)^2,$$
$$H_2(x, y) = y^2 + x - y, \tag{20}$$
$$H_3(x, y) = -y + x^2 + y^2 - 1,$$

and the corresponding piecewise Hamiltonian system can be written as

$$\begin{cases} \dot{x} = -4x - 10y + 1, \\ \dot{y} = 2x + 4y, \end{cases} \quad (x, y) \in U^1,$$
$$\begin{cases} \dot{x} = 2y - 1, \\ \dot{y} = -1, \end{cases} \quad (x, y) \in U^2, \tag{21}$$
$$\begin{cases} \dot{x} = -2y + 1, \\ \dot{y} = 2x, \end{cases} \quad (x, y) \in U^3.$$

Since the determinant of the linear part of each subsystem is 4, 0 and 4, it implies that system (21) is without equilibrium points in U^2 and has a linear Hamiltonian center in U^1 and U^3. We verify that system (21) only has one limit cycle intersecting the two parallel lines $x = -1$ and $x = 1$ at exactly four points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$ respectively, where $(y_1, y_2, y_3, y_4) = (-1, 2, 1, 0)$ with $y_1 < y_2$ and $y_3 > y_4$. Furthermore, we can also verify that this solution satisfies Equation (9). The limit cycle we obtain is given in **Figure 1**.

We now proceed to prove the second half of Theorem 1. According to the conditions of Theorem 1, we consider the other class of planar piecewise systems as follow:

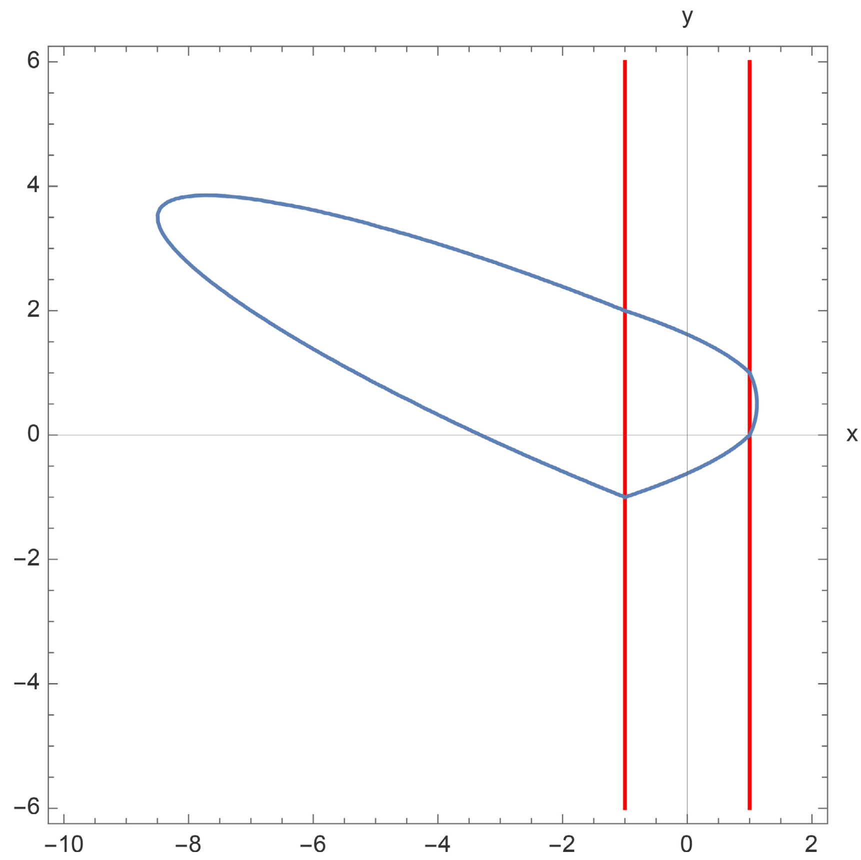

Figure 1. The limit cycle of discontinuous piecewise Hamiltonian system (21) having subsystems with a linear center in U^1 and U^3, and a subsystem without equilibrium points in U^2.

$$\begin{cases} \dot{x} = -b_i x - \delta_i y + d_i, \\ \dot{y} = \alpha_i x + b_i y + c_i, \end{cases} (x, y) \in U^i, i = 1, 2,$$
$$\begin{cases} \dot{x} = -\lambda_1 h_1 x + h_1 y + g_1, \\ \dot{y} = -\lambda_1^2 h_1 x + \lambda_1 h_1 y + f_1, \end{cases} (x, y) \in U^3, \tag{22}$$

where $f_1, g_1, h_1, \lambda_1, b_i, c_i, d_i, \alpha_i, \delta_i, i = 1, 2$ are all real constants with $\delta_i = b_i^2 + \omega_i^2$, $\omega_i \neq 0$, $i = 1, 2$, $f_1 \neq \lambda_1 g_1$ and $h_1 \neq 0$. The corresponding Hamiltonian functions are:

$$H_i(x, y) = -\frac{1}{2}x^2 - b_i xy - \frac{\delta_i}{2}y^2 - c_i x + d_i y, i = 1, 2,$$
$$H_3(x, y) = -\frac{1}{2}\lambda_1^2 h_1 x^2 + \lambda_1 h_1 xy - \frac{h_1}{2}y^2 + f_1 x - g_1 y. \tag{23}$$

If the corresponding piecewise differential system has a limit cycle intersecting the two parallel lines $x = -1$ and $x = 1$ at exactly four points respectively, denoted as $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$ with $y_1 < y_2$ and $y_3 > y_4$, then the Hamiltonian functions must satisfy the following relations:

$$\begin{aligned} &H_1(-1, y_1) - H_1(-1, y_2) = 0, \\ &H_2(-1, y_2) - H_2(1, y_3) = 0, \\ &H_2(1, y_4) - H_2(-1, y_1) = 0, \\ &H_3(1, y_3) - H_3(1, y_4) = 0, \end{aligned} \tag{24}$$

which can be written as the expansions:

$$\begin{aligned} &(y_1 - y_2)(2(b_1 + d_1) - \delta_1(y_1 + y_2)) = 0, \\ &4c_2 + 2(b_2 + d_2)y_2 + 2(b_2 - d_2)y_3 - \delta_2(y_2^2 - y_3^2) = 0, \\ &4c_2 + 2(b_2 - d_2)y_4 + 2(b_2 + d_2)y_1 + \delta_2(y_4^2 - y_1^2) = 0, \\ &(2(\lambda_1 h_1 - g_1) - h_1(y_3 + y_4))(y_3 - y_4) = 0. \end{aligned} \tag{25}$$

Assume that system (22) is a continuous piecewise system, which means the subsystems in U^1 and U^2 are coincide in the line $x = -1$, and the subsystems in U^2 and U^3 are coincide in the line $x = 1$. Therefore it has

$$\begin{aligned} &b_2 = b_1 = \lambda_1 h_1, d_2 = d_1 = g_1, \delta_2 = \delta_1 = -h_1, \\ &c_2 = c_1 + \alpha_2 - \alpha_1 = -\lambda_1^2 h_1 - \alpha_2 + f_1, \end{aligned} \tag{26}$$

From (25) and (26), under the precondition that $y_1 < y_2$, then we get the expressions of y_2, y_3, y_4 as follow:

$$y_2 = -\frac{2(\lambda_1 h_1 + g_1)}{h_1} - y_1, y_3 = \frac{\lambda_1 h_1 - g_1}{h_1} \pm \frac{\sqrt{\Delta}}{h_1}, y_4 = \frac{\lambda_1 h_1 - g_1}{h_1} \mp \frac{\sqrt{\Delta}}{h_1}, \tag{27}$$

where $\Delta = (\lambda_1 h_1 - g_1)^2 + h_1(h_1 y_1^2 + 2(\lambda_1 h_1 + g_1)y_1 - 4(\lambda_1^2 h_1 + \alpha_2 - f_1))$. Since the points y_2, y_3, y_4 are all continuous with respect to y_1, There is a continuum of periodic orbits. It implies that if system (22) is continuous, it has no limit cycles. So Theorem 1 is proved when (22) is continuous.

Now, assume that system (22) is discontinuous. From (25), we can get

$$y_1 = \frac{2(b_1+d_1)}{\delta_1} - y_2,\ y_3 = \frac{2(\lambda_1 h_1 - g_1)}{h_1} - y_4. \tag{28}$$

Substituting the expressions of y_1 and y_3 into the second and third equations of (25) and introducing the notion L_1 and L_2, then one has

$$\begin{aligned} L_1 &= 4\left(c_2 h_1^2 + \delta_2(\lambda_1 h_1 - g_1)^2 + (\lambda_1 h_1 - g_1)(b_2 - d_2)h_1\right) \\ &\quad + 2h_1^2(b_2+d_2)y_2 - 2\left(2(\lambda_1 h_1 - g_1)\delta_2 h_1 - (b_2-d_2)h_1^2\right)y_4 \\ &\quad - h_1^2\delta_2\left(y_2^2 - y_4^2\right) \\ &= 0, \end{aligned} \tag{29}$$

and

$$\begin{aligned} L_2 &= 4\left(c_2\delta_1^2 - \delta_2(b_1+d_1)^2 + \delta_1(b_1+d_1)(b_2+d_2)\right) \\ &\quad - 2\delta_1\left(\delta_1(b_2+d_2) - 2\delta_2(b_1+d_1)\right)y_2 \\ &\quad + 2\delta_1^2(b_2-d_2)y_4 - \delta_1^2\delta_2\left(y_2^2 - y_4^2\right) \\ &= 0. \end{aligned} \tag{30}$$

Let $L_3 = \delta_1^2 L_1 - h_1^2 L_2 = 0$, we obtain

$$y_4 = \frac{P_0}{P_1} + \frac{P_2}{P_1}y_2, \tag{31}$$

where

$$\begin{aligned} P_0 &= \delta_1^2(\lambda_1 h_1 - g_1)\left(\delta_2(\lambda_1 h_1 - g_1) + h_1(b_2 - d_2)\right) \\ &\quad + h_1^2(b_1+d_1)\left(\delta_2(b_1+d_1) + \delta_1(b_2+d_2)\right), \\ P_1 &= \delta_1^2\delta_2 h_1(\lambda_1 h_1 - g_1), \\ P_2 &= \delta_1 h_1^2\left(\delta_1(b_2+d_2) - \delta_2(b_1+d_1)\right). \end{aligned} \tag{32}$$

If $P_1 = 0$, it has $\lambda_1 h_1 - g_1 = 0$. Substituting it into $L_3 = 0$, one has $y_1 = y_2 = \dfrac{b_1+d_1}{\delta_1}$ which leads to contradictions. Hence, we have $P_1 \neq 0$. Then, substituting (31) into (29), we have the following expression of y_2:

$$y_{2\pm} = \frac{b_1+d_1}{\delta_1} \pm \frac{\sqrt{\Delta'}}{2P_3}, \tag{33}$$

where

$$\begin{aligned} P_3 &= h_1\delta_3^2\Big(\delta_1^4\delta_3^2\left((-b_3 + d_1^2 + d_4)h_1 + \delta_3(g_1 - \lambda_1 h_1)\right)^2 \\ &\quad - \delta_1\delta_3^4\left((b_1+d_1)h_1 + \delta_1(g_1 + \lambda_1 h_1)\right)^2\Big), \end{aligned}$$

$$\begin{aligned} \Delta' &= \left(2(b_2+d_2)h_1^2\delta_1^4\delta_2 h_1^2(-g_1+\lambda_1 h_1)^2 - 2\delta_1 h_1^4\left((b_2+d_2)\delta_1(b_1+d_1)\delta_2^2\right)\right)^2 \\ &\quad - 4\left(\delta_1^4\delta_2 h_1^2(-g_1+\lambda_1 h_1)^2 + h_1^2\delta_2(b_1+d_1)^2\delta_2^2\right)\Big(4\delta_1^4\delta_2 h_1^2(-g_1+\lambda_1 h_1)^2 \\ &\quad \times\left(c_2 h_1^2 + (b_2-d_2)h_1(-g_1+\lambda_1 h_1) + \delta_2(-g_1+\lambda_1 h_1)^2\right) \\ &\quad - 2\delta_1^2 h_1(-g_1+\lambda_1 h_1)\left(-\left((b_2-d_2)h_1^2\right) + 2\delta_2 h_1(-g_1+\lambda_1 h_1)\right) \end{aligned}$$

$$
\begin{aligned}
&+\Big(\big(b_1+d_1\big)h_1^2\big(\big(b_2+d_2\big)\delta_1+\big(b_1+d_1\big)\delta_2\big)\\
&+\delta_1^2\big(-g_1+\lambda_1h_1\big)\big(\big(b_2-d_2\big)h_1+\delta_2\big(-g_1+\lambda_1h_1\big)\big)\\
&+\delta_1h_1^2\big(\big(b_2+d_2\big)\delta_1-\big(b_1+d_1\big)\delta_2\big)y_2\big)\delta_1^4\delta_2h_1^2\big(-g_1+\lambda_1h_1\big)^2\\
&+h_1^2\delta_2\big(b_1+d_1\big)h_1^2\big(\big(b_2+d_2\big)\delta_1+\big(b_1+d_1\big)\delta_2\big)\\
&+\delta_1^2\big(-g_1+\lambda_1h_1\big)\big(\big(b_2-d_2\big)h_1\delta_2\big(-g_1+\lambda_1h_1\big)\big)h_1^2\delta_2\delta_1h_1^2\big(\big(b_2+d_2\big)\delta_1\big)^2\Big),
\end{aligned}
\tag{34}
$$

whenever $P_3 \neq 0$, and if $P_3 = 0$, there is at most one solution (y_1, y_2, y_3, y_4) of (24).

When $P_3 \neq 0$, given

$$
y_{1\pm} = \frac{2(b_1+d_1)}{\delta_1} - y_{2\pm} = \frac{b_1+d_1}{\delta_1} \mp \frac{\sqrt{\Delta'}}{2P_3} = y_{2\mp}, \tag{35}
$$

there exists at most one solution (y_1, y_2, y_3, y_4) of (24) with $y_1 < y_2$ and $y_3 > y_4$. As a result, we have proved that there is at most one limit cycle system (22) can have when it is discontinuous. Then we give an example to illustrate that there can exist one limit cycle when system (22) is discontinuous.

The Hamiltonian functions are given by the following expressions:

$$
\begin{aligned}
H_1(x,y) &= y - x^2 - y^2,\\
H_2(x,y) &= -x + y - x^2 - y^2,\\
H_3(x,y) &= -y^2 - x + y,
\end{aligned}
\tag{36}
$$

and the corresponding piecewise Hamiltonian system can be written as

$$
\begin{aligned}
&\begin{cases} \dot{x} = -2y+1, \\ \dot{y} = 2x, \end{cases} \quad (x,y)\in U^1,\\
&\begin{cases} \dot{x} = -2y+1, \\ \dot{y} = 2x+1, \end{cases} \quad (x,y)\in U^2,\\
&\begin{cases} \dot{x} = -2y+1, \\ \dot{y} = 1, \end{cases} \quad (x,y)\in U^3.
\end{aligned}
\tag{37}
$$

Since the determinant of the linear part of each subsystem is 4, 4 and 0, it means that system (37) is without equilibrium points in U^3 and has a linear Hamiltonian center in U^1 and U^2. We verify that system (37) only has one limit cycle intersecting the two parallel lines $x=-1$ and $x=1$ at exactly four points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$ respectively, where the solution $(y_1, y_2, y_3, y_4) = (-1, 2, 1, 0)$ with $y_1 < y_2$ and $y_3 > y_4$. The limit cycle is shown in **Figure 2**. This ends the proof.

4. Proof of Theorem 2

Considering the symmetry of the system, similar to the way of proving Theorem 1, there are two cases with respect to the class of equilibrium points on each zone to discuss here. Firstly, we consider the following planar piecewise system:

$$
\begin{cases} \dot{x} = -\lambda_i h_i x + h_i y + g_i, \\ \dot{y} = -\lambda_i^2 h_i x + \lambda_i h_i y + f_i, \end{cases} \quad (x,y)\in U^i, i = 1,3,
$$

$$\begin{cases} \dot{x} = -b_2 x - \delta_2 y + d_2, \\ \dot{y} = \alpha_2 x + b_2 y + c_2, \end{cases} \quad (x, y) \in U^2, \tag{38}$$

where $b_2, c_2, d_2, \alpha_2, \delta_2, f_i, g_i, h_i, \lambda_i, i = 1,3$ are all real constants with $h_i \neq 0$, $f_i \neq \lambda_i g_i$, $i = 1,3$ and $\delta_2 = b_2^2 + \omega_2^2$, $\omega_2 \neq 0$. The corresponding Hamiltonian functions are

$$\begin{aligned} H_i(x, y) &= -\frac{1}{2}\lambda_i^2 h_i x^2 + \lambda_i h_i xy - \frac{h_i}{2} y^2 + f_i x - g_i y, i = 1,3, \\ H_2(x, y) &= -\frac{1}{2} x^2 - b_2 xy - \frac{\delta_2}{2} y^2 - c_2 x + d_2 y. \end{aligned} \tag{39}$$

Assume that system (38) has a limit cycle intersecting the two parallel lines $x = -1$ and $x = 1$ at exactly four points, denoted as $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$ respectively with $y_1 < y_2$ and $y_3 > y_4$. Then we have

$$\begin{aligned} &H_1(-1, y_1) - H_1(-1, y_2) = 0, \\ &H_2(-1, y_2) - H_2(1, y_3) = 0, \\ &H_2(1, y_4) - H_2(-1, y_1) = 0, \\ &H_3(1, y_3) - H_3(1, y_4) = 0, \end{aligned} \tag{40}$$

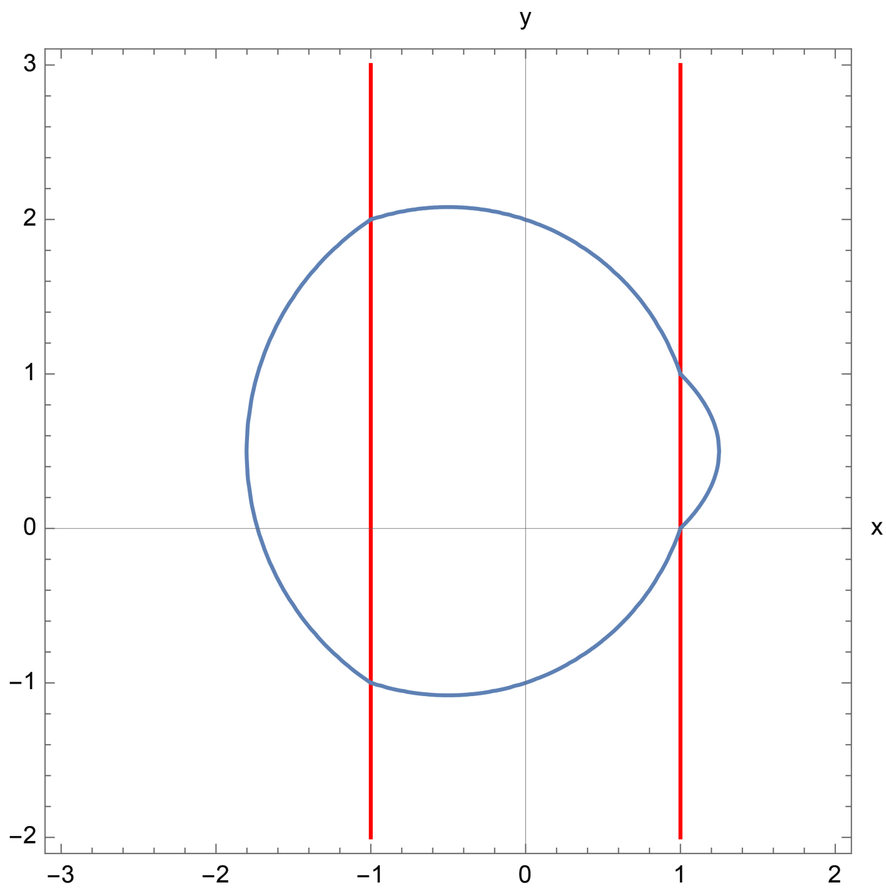

Figure 2. The limit cycle of discontinuous piecewise Hamiltonian system (37) having subsystems with a linear center in U^1 and U^2, and a subsystem without equilibrium points in U^3.

that is

$$
\begin{aligned}
&(y_1-y_2)\left(h_1(y_1+y_2)+2(\lambda_1 h_1+g_1)\right)=0,\\
&2b_2(y_2+y_3)-\delta_2\left(y_2^2-y_3^2\right)+4c_2+2d_2(y_2-y_3)=0,\\
&2b_2(y_4+y_1)+\delta_2\left(y_4^2-y_1^2\right)+4c_2-2d_2(y_4-y_1)=0,\\
&\left(h_3(y_3+y_4)-2(\lambda_3 h_3-g_3)\right)(y_3-y_4)=0.
\end{aligned} \tag{41}
$$

Assume that system (38) is continuous, we get

$$
\begin{aligned}
&b_2=\lambda_i h_i,\ \delta_2=-h_i,\ d_2=g_i,\ i=1,3,\\
&c_2=\lambda_1^2 h_1+f_1+\alpha_2,\ c_3=-\lambda_3^2 h_3+f_3-\alpha_2.
\end{aligned} \tag{42}
$$

From (41) and (42), we have the following expressions of y_2, y_3, y_4:

$$
y_2=-\frac{2(\lambda_1 h_1+g_1)}{h_1}-y_1,\ y_3=\frac{d_2-b_2}{\delta_2}\pm\frac{\sqrt{\Delta}}{\delta_2},\ y_4=\frac{d_2-b_2}{\delta_2}\mp\frac{\sqrt{\Delta}}{\delta_2}, \tag{43}
$$

where $\Delta=(b_2-d_2)^2+\delta_2\left(\delta_2 y_1^2-2(b_2+d_2)y_1-4c_2\right)$. Since the points y_2, y_3, y_4 are all continuous with respect to y_1, there is a continuum of periodic orbits. It implies that if system (38) is continuous, then it has no limit cycles.

Additionally, let us consider the case that system (38) is discontinuous. From (41), the following relations are derived:

$$
y_1=-\frac{2(\lambda_1 h_1+g_1)}{h_1}-y_2,\ y_3=\frac{2(\lambda_3 h_3-g_3)}{h_3}-y_4. \tag{44}
$$

Substituting the expressions of y_1 and y_3 into the second and third equations of (41) and introducing the notion L_1 and L_2, we can obtain that

$$
\begin{aligned}
L_1=&\ 4h_3(b_2-d_2)(\lambda_3 h_3-g_3)+4c_2 h_3^2+4\delta_2(\lambda_3 h_3-g_3)^2\\
&+2h_3^2(b_2+d_2)y_2-2h_3\left(b_2 h_3-d_2 h_3+2\delta_2(\lambda_3 h_3-g_3)\right)y_4\\
&-\delta_2 h_3^2\left(y_2^2-y_4^2\right)\\
=&\ 0,
\end{aligned} \tag{45}
$$

and

$$
\begin{aligned}
L_2=&-4h_1(b_2+d_2)(\lambda_1 h_1+g_1)+4c_2 h_1^2-4\delta_2(\lambda_1 h_1+g_1)^2\\
&+2h_1^2(b_2-d_2)y_4-2h_1\left((b_2+d_2)h_1+2(\lambda_1 h_1+g_1)\delta_2\right)y_2\\
&-\delta_2 h_1^2\left(y_2^2-y_4^2\right)\\
=&\ 0.
\end{aligned} \tag{46}
$$

Let $L_3=h_1^2 L_1-h_3^2 L_2=0$, we obtain

$$
y_4=\frac{P_0}{P_1}+\frac{P_2}{P_1}y_2, \tag{47}
$$

where

$$
\begin{aligned}
P_0=&\ h_1 h_3\left(h_1(b_2-d_2)(\lambda_3 h_3-g_3)+h_3(b_2+d_2)(\lambda_1 h_1+g_1)\right)\\
&+\delta_2\left(h_1^2(\lambda_3 h_3-g_3)^2+h_3^2(\lambda_1 h_1+g_1)^2\right),\\
P_1=&\ h_1^2 h_3\left(\delta_2(\lambda_3 h_3-g_3)+h_3(b_2-d_2)\right),\\
P_2=&\ h_1 h_3^2\left(h_1(b_2+d_2)+\delta_2(\lambda_1 h_1+g_1)\right).
\end{aligned} \tag{48}
$$

If $P_1 = 0$, it has $\delta_2(\lambda_3 h_3 - g_3) + h_3(b_2 - d_2) = 0$. Substituting it into $L_3 = 0$, one has $y_1 = y_2 = -\frac{\lambda_1 h_1 + g_1}{h_1}$, which leads to contradictions. Hence, we have $P_1 \neq 0$. Then, substituting (47) into (45), we have the following expression of y_2:

$$y_{2\pm} = -\frac{\lambda_1 h_1 + g_1}{h_1} \pm \frac{\sqrt{\Delta'}}{2P_3}, \tag{49}$$

where

$$\begin{aligned} P_3 = \delta_2 h_3^2 \Big(& \left(h_1 h_3^2 \left(h_1 (b_2 + d_2) + \delta_2 (\lambda_1 h_1 + g_1) \right) \right)^2 \\ & - \left(h_1^2 h_3 \left(\delta_2 (\lambda_3 h_3 - g_3) + h_3 (b_2 - d_2) \right) \right)^2 \Big), \end{aligned}$$

$$\begin{aligned} \Delta' = \Big(& 2h_3^2 \left(h_1^2 h_3 \left(\delta_2 (\lambda_3 h_3 - g_3) + h_3 (b_2 - d_2) \right) \right)^2 (b_2 + d_2) \\ & - 2\left(h_1^2 h_3 \left(\delta_2 (\lambda_3 h_3 - g_3) + h_3 (b_2 - d_2) \right) \right) \Big(h_1 h_3^2 \big(h_1 (b_2 + d_2) \\ & + \delta_2 (\lambda_1 h_1 + g_1) \big) \Big) h_3 \left(h_3 (b_2 - d_2) + 2\delta_2 (\lambda_3 h_3 - g_3) \right) \\ & + 2\Big(h_1 h_3 \left(h_1 (b_2 - d_2)(\lambda_3 h_3 - g_3) + h_3 (b_2 + d_2)(\lambda_1 h_1 + g_1) \right) \\ & + \delta_2 \left(h_1^2 (\lambda_3 h_3 - g_3)^2 + h_3^2 (\lambda_1 h_1 + g_1)^2 \right) \Big) \Big(h_1 h_3^2 \big(h_1 (b_2 + d_2) \\ & + \delta_2 (\lambda_1 h_1 + g_1) \big) \Big) \delta_2 h_3^2 \Big)^2 - 4\delta_2 h_3^2 \Big(\left(h_1 h_3^2 \left(h_1 (b_2 + d_2) + \delta_2 (\lambda_1 h_1 + g_1) \right) \right)^2 \\ & - \left(h_1^2 h_3 \left(\delta_2 (\lambda_3 h_3 - g_3) + h_3 (b_2 - d_2) \right) \right)^2 \Big) \Big(\Big(h_1^2 h_3 \big(\delta_2 (\lambda_3 h_3 - g_3) \\ & + h_3 (b_2 - d_2) \big) \Big)^2 \left(4h_3 (b_2 - d_2)(\lambda_3 h_3 - g_3) + 4c_2 h_3^2 + 4\delta_2 (\lambda_3 h_3 - g_3)^2 \right) \\ & - 2\left(h_1^2 h_3 \left(\delta_2 (\lambda_3 h_3 - g_3) + h_3 (b_2 - d_2) \right) \right) h_3 \big(h_3 (b_2 - d_2) \\ & + 2\delta_2 (\lambda_3 h_3 - g_3) \big) \Big(h_1 h_3 \left(h_1 (b_2 - d_2)(\lambda_3 h_3 - g_3) + h_3 (b_2 + d_2)(\lambda_1 h_1 + g_1) \right) \\ & + \delta_2 \left(h_1^2 (\lambda_3 h_3 - g_3)^2 + h_3^2 (\lambda_1 h_1 + g_1)^2 \right) \Big) + \delta_2 h_3^2 \Big(h_1 h_3 \big(h_1 (b_2 - d_2)(\lambda_3 h_3 - g_3) \\ & + h_3 (b_2 + d_2)(\lambda_1 h_1 + g_1) \big) + \delta_2 \left(h_1^2 (\lambda_3 h_3 - g_3)^2 + h_3^2 (\lambda_1 h_1 + g_1)^2 \right) \Big)^2 \Big), \end{aligned} \tag{50}$$

whenever $P_3 \neq 0$, and if $P_3 = 0$, there is at most one solution (y_1, y_2, y_3, y_4) of (40).

When $P_3 \neq 0$, given

$$y_{1\pm} = -\frac{2(\lambda_1 h_1 + g_1)}{h_1} - y_{2\pm} = -\frac{\lambda_1 h_1 + g_1}{h_1} \mp \frac{\sqrt{\Delta'}}{2P_3} = y_{2\mp}, \tag{51}$$

there exists at most one solution (y_1, y_2, y_3, y_4) of (40) with $y_1 < y_2$ and $y_3 > y_4$. Thus, we have shown that there can be at most one limit cycle when system (38) is discontinuous.

To finish this part of proof, now we provide an example to show that system (38) can have exactly one limit cycle when it is discontinuous. The Hamiltonian functions are given by the following expressions:

$$H_1(x, y) = 2y^2 - 2x - 2y,$$

$$H_2(x,y) = x - y + x^2 + y^2 - 1,$$
$$H_3(x,y) = y^2 + x - y, \tag{52}$$

and the corresponding piecewise Hamiltonian system can be written as

$$\begin{cases}\dot{x} = 4y-2, \\ \dot{y} = 2,\end{cases} (x,y)\in U^1,$$
$$\begin{cases}\dot{x} = 2y-1, \\ \dot{y} = -2x-1,\end{cases} (x,y)\in U^2, \tag{53}$$
$$\begin{cases}\dot{x} = 2y-1, \\ \dot{y} = -1,\end{cases} (x,y)\in U^3,$$

Since the determinant of the linear part of each subsystem is 0, 4 and 0, it implies that system (53) is without equilibrium points in U^1 and U^3 and has a linear Hamiltonian center in U^2. We verify that system (53) has only one limit cycle intersecting the two parallel lines $x=-1$ and $x=1$ at exactly four points $(-1,y_1)$, $(-1,y_2)$, $(1,y_3)$ and $(1,y_4)$ respectively, where $(y_1,y_2,y_3,y_4)=(-1,2,1,0)$ with $y_1<y_2$ and $y_3>y_4$. In addition, we can check that this solution satisfies Equation (41). And the limit cycle we obtain is given in **Figure 3**.

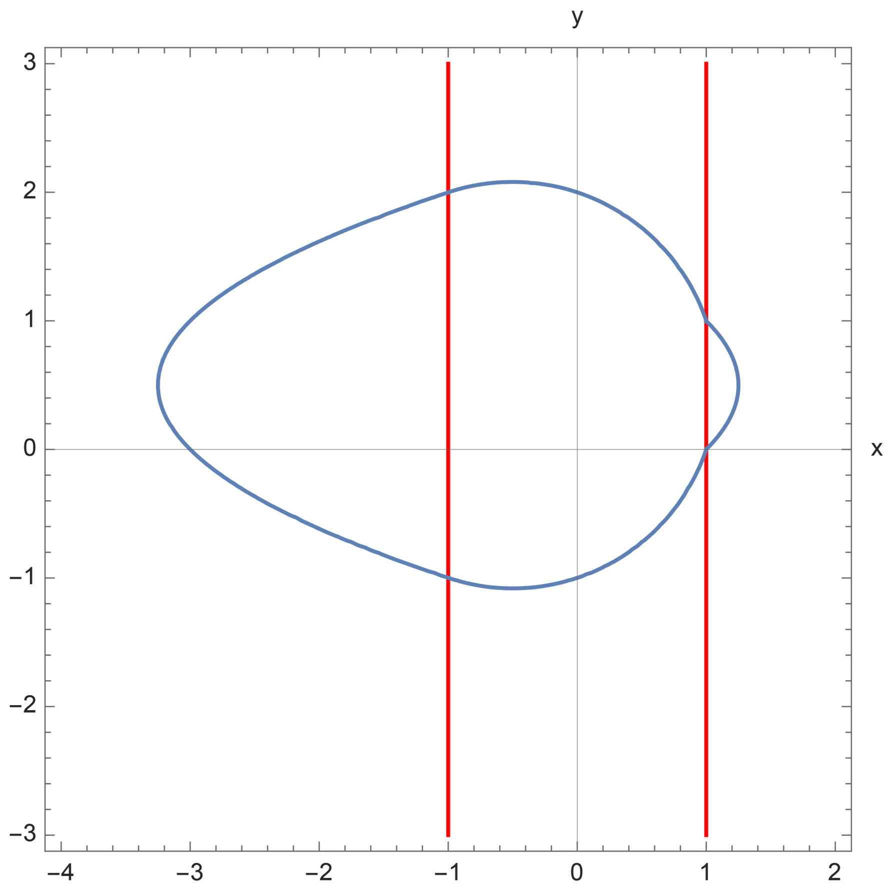

Figure 3. The limit cycle of discontinuous piecewise Hamiltonian system (53) having a subsystem with a linear center in U^2, and subsystems without equilibrium points in U^1 and U^3.

Now start to prove the latter part of Theorem 2. Under the conditions of Theorem 2, we consider the following planar piecewise system

$$\begin{cases} \dot{x} = -\lambda_i h_i x + h_i y + g_i, \\ \dot{y} = -\lambda_i^2 h_i x + \lambda_i h_i y + f_i, \end{cases} (x, y) \in U^i, i = 1, 2, \\ \begin{cases} \dot{x} = -b_3 x - \delta_3 y + d_3, \\ \dot{y} = \alpha_3 x + b_3 y + c_3, \end{cases} (x, y) \in U^3, \tag{54}$$

where $b_3, c_3, d_3, \alpha_3, \delta_3, f_i, g_i, h_i, \lambda_i, i = 1, 2$ are all real constants with $h_i \neq 0$, $f_i \neq \lambda_i g_i$, $i = 1, 2$ and $\delta_3 = b_3^2 + \omega_3^2$, $\omega_3 \neq 0$. The corresponding Hamiltonian functions are:

$$H_i(x, y) = -\frac{1}{2}\lambda_i^2 h_i x^2 + \lambda_i h_i xy - \frac{h_i}{2} y^2 + f_i x - g_i y, i = 1, 2, \\ H_3(x, y) = -\frac{1}{2} x^2 - b_3 xy - \frac{\delta_3}{2} y^2 - c_3 x + d_3 y. \tag{55}$$

If system (54) has a limit cycle intersecting the two parallel lines $x = -1$ and $x = 1$ at exactly four points respectively, denoted as $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$ with $y_1 < y_2$ and $y_3 > y_4$, then the Hamiltonian functions must satisfy the following relations:

$$\begin{aligned} &H_1(-1, y_1) - H_1(-1, y_2) = 0, \\ &H_2(-1, y_2) - H_2(1, y_3) = 0, \\ &H_2(1, y_4) - H_2(-1, y_1) = 0, \\ &H_3(1, y_3) - H_3(1, y_4) = 0, \end{aligned} \tag{56}$$

which can be written as the expansions:

$$\begin{aligned} &(y_1 - y_2)(2(\lambda_1 h_1 + g_1) + h_1(y_1 + y_2)) = 0, \\ &4 f_2 + 2(\lambda_2 h_2 + g_2) y_2 + 2(\lambda_2 h_2 - g_2) y_3 + h_2(y_2^2 - y_3^2) = 0, \\ &4 f_2 + 2(\lambda_2 h_2 - g_2) y_4 + 2(\lambda_2 h_2 + g_2) y_1 - h_2(y_4^2 - y_1^2) = 0, \\ &(2(d_3 - b_3) - \delta_3(y_3 + y_4))(y_3 - y_4) = 0. \end{aligned} \tag{57}$$

Assume that system (54) is continuous, it has

$$\begin{aligned} &b_3 = \lambda_2 h_2, d_3 = g_1 = g_2, \delta_3 = -h_1 = -h_2, \\ &\lambda_1 = \lambda_2, f_1 = f_2, c_3 = -\lambda_2^2 h_2 - \alpha_3 + f_2. \end{aligned} \tag{58}$$

From (57) and (58), under the precondition that $y_1 < y_2$, we get the expressions of y_2, y_3, y_4 as follow:

$$y_2 = -\frac{2(\lambda_1 h_1 + g_1)}{h_1} - y_1, y_3 = \frac{d_3 - b_3}{\delta_3} \pm \frac{\sqrt{\Delta}}{\delta_3}, y_4 = \frac{d_3 - b_3}{\delta_3} \mp \frac{\sqrt{\Delta}}{\delta_3}, \tag{59}$$

where $\Delta = (\lambda_2 h_2 - g_2)^2 + h_2(h_2 y_1^2 + 2(\lambda_2 h_2 + g_2) y_1 + 4 f_2)$. Since the points y_2, y_3, y_4 are all continuous with respect to y_1, there is a continuum of periodic orbits. It means that if system (54) is continuous, it has no limit cycles.

Now, assume that system (54) is discontinuous. From (57), one can obtain that

$$y_1 = -\frac{2(\lambda_1 h_1 + g_1)}{h_1} - y_2,\ y_3 = \frac{2(d_3 - b_3)}{\delta_3} - y_4. \tag{60}$$

Substituting the expressions of y_1 and y_3 into the second and third equations of (57) and introducing the notion L_1 and L_2, then one has

$$\begin{aligned} L_1 &= 4\left(\delta_3^2 f_2 + \delta_3(\lambda_2 h_2 - g_2)(d_3 - b_3) - h_2(d_3 - b_3)^2\right) \\ &\quad + 2(\lambda_2 h_2 + g_2)\delta_3^2 y_2 - 2\delta_3\left(\delta_3(\lambda_2 h_2 - g_2) - 2h_2(d_3 - b_3)\right)y_4 \\ &\quad + h_2\delta_3^2\left(y_2^2 - y_4^2\right) \\ &= 0, \end{aligned} \tag{61}$$

and

$$\begin{aligned} L_2 &= 4\left(h_1^2 f_2 - h_1(\lambda_1 h_1 + g_1)(\lambda_2 h_2 + g_2) + h_2(\lambda_1 h_1 + g_1)^2\right) \\ &\quad + 2h_1^2(\lambda_2 h_2 - g_2)y_4 - 2h_1\left(h_1(\lambda_2 h_2 + g_2) - 2h_2(\lambda_1 h_1 + g_1)\right)y_2 \\ &\quad + h_1^2 h_2\left(y_2^2 - y_4^2\right) \\ &= 0. \end{aligned} \tag{62}$$

Let $L_3 = h_1^2 L_1 - \delta_3^2 L_2 = 0$, we get

$$y_4 = \frac{P_0}{P_1} + \frac{P_2}{P_1} y_2, \tag{63}$$

where

$$\begin{aligned} P_0 &= h_1^2\left(\delta_3(\lambda_2 h_2 - g_2)(d_3 - b_3) - h_2(d_3 - b_3)^2\right) \\ &\quad + \delta_3^2\left(h_1(\lambda_1 h_1 + g_1)(\lambda_2 h_2 + g_2) - h_2(\lambda_1 h_1 + g_1)^2\right), \\ P_1 &= h_1^2\delta_3\left(\delta_3(\lambda_2 h_2 - g_2) - h_2(d_3 - b_3)\right), \\ P_2 &= \delta_3^2\left(h_1^2(\lambda_2 h_2 + g_2) - h_1 h_2(\lambda_1 h_1 + g_1)\right). \end{aligned} \tag{64}$$

If $P_1 = 0$, it has $\delta_3(\lambda_2 h_2 - g_2) - h_2(d_3 - b_3) = 0$. Substituting it into $L_3 = 0$, one has $y_1 = y_2 = -\frac{\lambda_1 h_1 + g_1}{h_1}$, which leads to contradictions. Hence, we have $P_1 \neq 0$. Then, substituting (63) into (61), we obtain the following expression of y_2:

$$y_{2\pm} = -\frac{\lambda_1 h_1 + g_1}{h_1} \pm \frac{\sqrt{\Delta'}}{2P_3}, \tag{65}$$

where

$$\begin{aligned} P_3 = h_1\delta_3^2 &\left(\left(h_1^2\delta_3\left(\delta_3(\lambda_2 h_2 - g_2) - h_2(d_3 - b_3)\right)\right)^2\right. \\ &\left. - \left(\delta_3^2\left(h_1^2(\lambda_2 h_2 + g_2) - h_1 h_2(\lambda_1 h_1 + g_1)\right)\right)^2\right), \end{aligned}$$

$$\begin{aligned} \Delta' = \Big(&\left(2h_1^2\delta_3\left(\delta_3(\lambda_2 h_2 - g_2) - h_2(d_3 - b_3)\right)\right)^2(\lambda_2 h_2 + g_2)\delta_3^2 \\ &- 2\delta_3 h_1^2\delta_3\left(\delta_3(\lambda_2 h_2 - g_2) - h_2(d_3 - b_3)\right)\delta_3^2\left(h_1^2(\lambda_2 h_2 + g_2)\right. \\ &\left. - h_1 h_2(\lambda_1 h_1 + g_1)\right)\left(\delta_3(\lambda_2 h_2 - g_2) - 2h_2(d_3 - b_3)\right)\Big)^2 \end{aligned}$$

$$
\begin{aligned}
&-4h_2\delta_3^2\left(h_1^2\delta_3\left(\delta_3\left(\lambda_2h_2-g_2\right)-h_2\left(d_3-b_3\right)\right)^2\right.\\
&-\delta_3^2\left(h_1^2\left(\lambda_2h_2+g_2\right)-h_1h_2\left(\lambda_1h_1+g_1\right)\right)^2\Big)\Big(4h_1^2\delta_3\left(\delta_3\left(\lambda_2h_2-g_2\right)\right.\\
&\left.-h_2\left(d_3-b_3\right)\right)^2\left(\delta_3^2f_2+\delta_3\left(\lambda_2h_2-g_2\right)\left(d_3-b_3\right)-h_2\left(d_3-b_3\right)^2\right)\\
&-2\delta_3h_1^2\left(\delta_3\left(\lambda_2h_2-g_2\right)\left(d_3-b_3\right)-h_2\left(d_3-b_3\right)^2\right)\\
&+\delta_3^2\left(h_1\left(\lambda_1h_1+g_1\right)\left(\lambda_2h_2+g_2\right)-h_2\left(\lambda_1h_1+g_1\right)^2\right)h_1^2\delta_3\left(\delta_3\left(\lambda_2h_2-g_2\right)\right.\\
&\left.-h_2\left(d_3-b_3\right)\right)\left(\delta_3\left(\lambda_2h_2-g_2\right)-2h_2\left(d_3-b_3\right)\right)\\
&-h_2\delta_3^2h_1^2\left(\delta_3\left(\lambda_2h_2-g_2\right)\left(d_3-b_3\right)-h_2\left(d_3-b_3\right)^2\right) \qquad (66)\\
&\left.+\delta_3^2\left(h_1\left(\lambda_1h_1+g_1\right)\left(\lambda_2h_2+g_2\right)-h_2\left(\lambda_1h_1+g_1\right)^2\right)^2\right),
\end{aligned}
$$

whenever $P_3 \neq 0$, and if $P_3 = 0$, there is at most one solution (y_1, y_2, y_3, y_4) of (56).

When $P_3 \neq 0$, given

$$
y_{1\pm} = \frac{2(b_1+d_1)}{\delta_1} - y_{2\pm} = \frac{b_1+d_1}{\delta_1} \mp \frac{\sqrt{\Delta'}}{2P_3} = y_{2\mp}, \tag{67}
$$

there exists at most one solution (y_1, y_2, y_3, y_4) of (56) with $y_1 < y_2$ and $y_3 > y_4$. Therefore, we have proved that there can be at most one limit cycle when system (54) is discontinuous. In addition, an example of (54) with exactly one limit cycle is given below.

The Hamiltonian functions are given by the following expressions:

$$
\begin{aligned}
&H_1(x,y) = y^2 - x - y,\\
&H_2(x,y) = y^2 + x - y,\\
&H_3(x,y) = x - y + x^2 + y^2,
\end{aligned} \tag{68}
$$

and the corresponding piecewise Hamiltonian system can be written as

$$
\begin{aligned}
&\begin{cases} \dot{x} = 2y - 1, \\ \dot{y} = 1, \end{cases} \quad (x,y) \in U^1,\\
&\begin{cases} \dot{x} = 2y - 1, \\ \dot{y} = -1, \end{cases} \quad (x,y) \in U^2,\\
&\begin{cases} \dot{x} = 2y - 1, \\ \dot{y} = -2x - 1, \end{cases} \quad (x,y) \in U^3,
\end{aligned} \tag{69}
$$

Since the determinant of the linear part of each subsystem is 0, 0 and 4, it means that (69) is without equilibrium points in U^3 and has a linear Hamiltonian center in U^1 and U^2. We verify that system (69) only has one limit cycle intersecting the two parallel lines $x = -1$ and $x = 1$ at exactly four points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$ respectively, where the solution $(y_1, y_2, y_3, y_4) = (-1, 2, 1, 0)$ with $y_1 < y_2$ and $y_3 > y_4$. And the limit cycle is shown in **Figure 4**. This ends the proof.

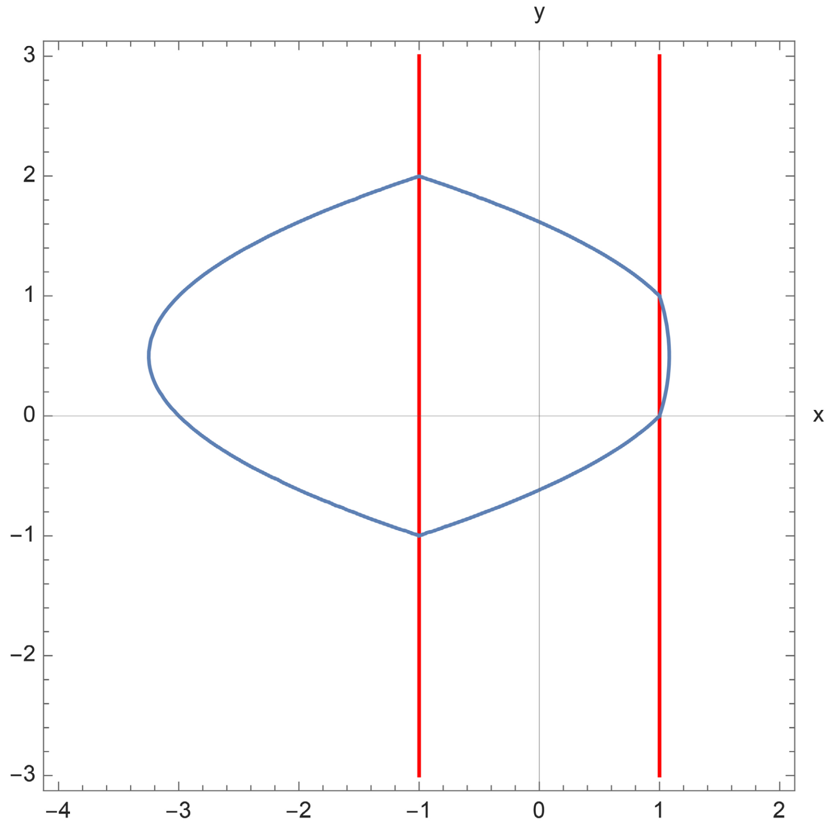

Figure 4. The limit cycle of discontinuous piecewise Hamiltonian system (69) having a subsystem with a linear center in U^3, and subsystems without equilibrium points in U^1 and U^2.

5. Conclusion

We have studied a class of planar piecewise Hamiltonian systems with a center in two zones and without equilibrium points in the other zone separated by two parallel straight lines and the maximal number of limit cycles it can have. According to the type of equilibrium point for each subsystem, we discussed this problem in two cases. When these systems are continuous, it proved that there are no limit cycles for each case. In the other hand, when the systems are discontinuous, we show that they can have at most one limit cycle in each case. Subsequently, we discussed another class of planar piecewise Hamiltonian systems with a center in one zone and without equilibrium points in the other zones separated by two parallel straight lines in a similar way. In addition, we give the examples of the systems with exactly one limit cycle and provide their figures.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 11772306).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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