Journal of Applied Mathematics and Physics, 2023, 11, 1319-1336
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Proofs of Legendre's Conjecture and Three Related Conjectures

Wing K. Yu

Independent Researcher, Arlington, Washington State, USA
Email: wingkyu@hotmail.com

How to cite this paper: Yu, W.K. (2023) The Proofs of Legendre's Conjecture and Three Related Conjectures. *Journal of Applied Mathematics and Physics*, **11**, 1319-1336.
https://doi.org/10.4236/jamp.2023.115085

Received: April 6, 2023
Accepted: May 20, 2023
Published: May 23, 2023

Abstract

In this paper, we prove Legendre's conjecture: There is a prime number between n^2 and $(n+1)^2$ for every positive integer *n*. We also prove three related conjectures. The method that we use is to analyze binomial coefficients. It is developed by the author from the method of analyzing binomial central coefficients, that was used by Paul Erdős in his proof of Bertrand's postulate - Chebyshev's theorem.

Keywords

Legendre's Conjecture, Bertrand's Postulate - Chebyshev's Theorem, Oppermann's Conjecture, Brocard's Conjecture, Andrica's Conjecture

1. Introduction

Legendre's conjecture was proposed by Andrien-Marie Legendre (1752-1833). It states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer *n*. The conjecture was one of Landau's problems on prime numbers in 1912. Many researchers have been trying to resolve this conjecture without success, as none of the empirical, asymptotic, probabilistic, and statistical methods of proving the Legendre conjecture were considered to provide sufficient evidence. According to Wikipedia, as of 2022, the conjecture has never been proved nor disproved [1]. (See **Appendix A1** for more information)

We will use the method of analyzing binomial coefficients, $\binom{\lambda n}{n}$, to prove the Legendre's conjecture, where λ is an integer and $\lambda \geq 3$. The method is developed by the author of this paper from the method of analyzing binomial central coefficients, $\binom{2n}{n}$, that was used by Paul Erdős [2] to prove Bertrand's postulate - Chebyshev's theorem [3].

In Section 1, we will define the prime number factorization operator and clarify some terms and concepts. In Section 2, we will derive some lemmas. In Section 3, we will develop a theorem to be used in the proofs of the conjectures in the later sections. In Section 4, we will prove Legendre's conjecture, and in Section 5, we will prove Oppermann's conjecture [4], Brocard's conjecture [5], and Andrica's conjecture [6].

Definition: $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\}$ denotes the prime number factorization operator of the integer expression $\binom{\lambda n}{n}$. It is the product of the prime numbers in the decomposition of $\binom{\lambda n}{n}$ in the range of $a \ge p > b$. In this operator, p is a prime number, a and b are real numbers, and $\lambda n \ge a \ge p > b \ge 1$.

It has some properties:

It is always true that $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} \ge 1$ (1.1)

If there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \ge p > b$, then $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} = 1$, or vice versa, if $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} = 1$, then there is no prime number in $\binom{\lambda n}{n}$ within the range of $a \ge p > b$. (1.2)

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{16 \ge p > 10}\left\{\binom{20}{4}\right\} = 13^0 \cdot 11^0 = 1$. No prime number 13 or 11 is in $\binom{20}{4}$ in the range of $16 \ge p > 10$.

If there is at least one prime number in $\binom{\lambda n}{n}$ in the range of $a \ge p > b$, then $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} > 1$, or vice versa, if $\Gamma_{a \ge p > b}\left\{\binom{\lambda n}{n}\right\} > 1$, then there is at least one prime number in $\binom{\lambda n}{n}$ within the range of $a \ge p > b$. (1.3)

For example, when $\lambda = 5$ and $n = 4$, $\Gamma_{18 \ge p > 16}\left\{\binom{20}{4}\right\} = 17 > 1$. A prime number 17 is in $\binom{20}{4}$ within the range of $18 \ge p > 16$.

Let $v_p(n)$ be the *p-adic valuation* of *n*, the exponent of the highest power of *p* that divides *n*.

Similar to Paul Erdős' paper [2], we define *R*(*p*) by the inequalities $p^{R(p)} \le \lambda n < p^{R(p)+1}$, and determine the *p-adic valuation* of $\binom{\lambda n}{n}$.

$$v_p\left(\binom{\lambda n}{n}\right)=v_p\left((\lambda n)!\right)-v_p\left(((\lambda-1)n)!\right)-v_p(n!)$$
$$=\sum_{i=1}^{R(p)}\left(\left\lfloor\frac{\lambda n}{p^i}\right\rfloor-\left\lfloor\frac{(\lambda-1)n}{p^i}\right\rfloor-\left\lfloor\frac{n}{p^i}\right\rfloor\right)\le R(p)$$

because for any real numbers a and b, the expression of $\lfloor a+b\rfloor-\lfloor a\rfloor-\lfloor b\rfloor$ is 0 or 1.

Thus, if p divides $\binom{\lambda n}{n}$, then $v_p\left(\binom{\lambda n}{n}\right)\le R(p)\le\log_p(\lambda n)$, or

$$p^{v_p\left(\binom{\lambda n}{n}\right)}\le p^{R(p)}\le\lambda n \tag{1.4}$$

If $\lambda n\ge p>\lfloor\sqrt{\lambda n}\rfloor$, then $0\le v_p\left(\binom{\lambda n}{n}\right)\le R(p)\le 1$. (1.5)

Let $\pi(n)$ be the number of distinct prime numbers less than or equal to n. Among the first six consecutive natural numbers are three prime numbers 2, 3 and 5. Then, for each additional six consecutive natural numbers, at most one can add two prime numbers, $p\equiv 1(\text{MOD } 6)$ and $p\equiv 5(\text{MOD } 6)$. Thus,

$$\pi(n)\le\left\lfloor\frac{n}{3}\right\rfloor+2\le\frac{n}{3}+2. \tag{1.6}$$

From the prime number decomposition, when $n>\lfloor\sqrt{\lambda n}\rfloor$,

$$\binom{\lambda n}{n}=\Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}\cdot\Gamma_{n\ge p>\lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$$
$$\cdot\Gamma_{\lfloor\sqrt{\lambda n}\rfloor\ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$$

when $n\le\lfloor\sqrt{\lambda n}\rfloor$, $\binom{\lambda n}{n}\le\Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}\cdot\Gamma_{\lfloor\sqrt{\lambda n}\rfloor\ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$

Thus, $\binom{\lambda n}{n}\le\Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}\cdot\Gamma_{n\ge p>\lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$

$\cdot\Gamma_{\lfloor\sqrt{\lambda n}\rfloor\ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}$.

$\Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}=\Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$ since all prime numbers in $n!$ do not appear in the range of $\lambda n\ge p>n$.

Referring to (1.5), $\Gamma_{n\ge p>\lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}\le\prod_{n\ge p}p$. It has been proven [7] that for $n\ge 3$, $\prod_{n\ge p}p<2^{2n-3}$. Thus, for $n\ge 3$,

$$\Gamma_{n\ge p>\lfloor\sqrt{\lambda n}\rfloor}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}\le\prod_{n\ge p}p<2^{2n-3}.$$

Referring to (1.4) and (1.6), $\Gamma_{\lfloor\sqrt{\lambda n}\rfloor\ge p}\left\{\frac{(\lambda n)!}{n!\cdot((\lambda-1)n)!}\right\}\le(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$.

Thus, for $\lambda\ge 3$ and $n\ge 3$, $\binom{\lambda n}{n}<\Gamma_{\lambda n\ge p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\cdot 2^{2n-3}\cdot(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$

(1.7)

2. Lemmas

Lemma 1: If a real number $x\ge 3$, then $\frac{2(2x-1)}{x-1}>\left(\frac{x}{x-1}\right)^x$ (2.1)

Proof:

Let $f_1(x)=\frac{2(2x-1)}{x-1}$; then,

$$f_1'(x)=\frac{2(x-1)(2x-1)'-2(2x-1)(x-1)'}{(x-1)^2}=\frac{-2}{(x-1)^2}<0.$$

Thus, $f_1(x)$ is a strictly decreasing function for $x>1$.

Since $f_1(3)=5$, and $\lim_{x\to\infty}f_1(x)=4$, for $x\ge 3$, we have

$$5\ge f_1(x)=\frac{2(2x-1)}{x-1}\ge 4.$$

Let $f_2(x)=\left(\frac{x}{x-1}\right)^x$, then

$$f_2'(x)=\left(\left(\frac{x}{x-1}\right)^x\right)'=\left(e^{x\cdot\ln\frac{x}{x-1}}\right)'=e^{x\cdot\ln\frac{x}{x-1}}\cdot\left(x\cdot\ln\frac{x}{x-1}\right)'$$

$$f_2'(x)=\left(\frac{x}{x-1}\right)^x\cdot\left(\ln\frac{x}{x-1}+x\cdot\left(\ln\frac{x}{x-1}\right)'\right)$$

$$=\left(\frac{x}{x-1}\right)^x\cdot\left(\ln\frac{x}{x-1}+x\cdot\frac{x-1}{x}\cdot\frac{x-1-x}{(x-1)^2}\right)$$

$$f_2'(x)=\left(\frac{x}{x-1}\right)^x\cdot\left(\ln\frac{x}{x-1}-\frac{1}{x-1}\right) \quad (2.1.1)$$

In (2.1.1), for $x\ge 3$, $\frac{1}{x-1}=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}+\frac{1}{x^6}+\cdots$

Using the formula: $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\frac{x^6}{6}+\cdots$, we have

$$\ln\frac{x}{x-1}=\ln\frac{1}{1+\frac{-1}{x}}=-\ln\left(1+\frac{-1}{x}\right)=\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{3x^3}+\frac{1}{4x^4}+\frac{1}{5x^5}+\frac{1}{6x^6}+\cdots$$

Thus, for $x\ge 3$, $\ln\frac{x}{x-1}-\frac{1}{x-1}<0$.

Since $\left(\frac{x}{x-1}\right)^x$ is a positive number for $x\ge 3$,

$$f_2'(x)=\left(\frac{x}{x-1}\right)^x\cdot\left(\ln\frac{x}{x-1}-\frac{1}{x-1}\right)<0.$$

Thus $f_2(x)$ is a strictly deceasing function for $x \ge 3$.

Since $f_2(3) = 3.375$ and $\lim_{x\to\infty} f_2(x) = e \approx 2.718$, for $x \ge 3$,

$$3.375 \ge f_2(x) = \left(\frac{x}{x-1}\right)^x \ge e \tag{2.1.2}$$

Since for $x \ge 3$, $f_1(x)$ has a lower bound of 4 and $f_2(x)$ has an upper bound of 3.375, $f_1(x) = \frac{2(2x-1)}{x-1} > f_2(x) = \left(\frac{x}{x-1}\right)^x$ is proven. (2.1.3)

Lemma 2: For $n \ge 2$ and $\lambda \ge 3$, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ (2.2)

Proof:

When $\lambda \ge 3$ and $n = 2$, $\binom{\lambda n}{n} = \binom{2\lambda}{2} = \frac{2\lambda(2\lambda-1)(2\lambda-2)!}{2(2\lambda-2)!} = \lambda(2\lambda-1)$

(2.2.1)

$$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{\lambda^{2\lambda-\lambda+1}}{2(\lambda-1)^{2(\lambda-1)-\lambda+1}} = \frac{\lambda(\lambda-1)}{2}\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} \tag{2.2.2}$$

In (2.1) when $x = \lambda \ge 3$, we have $\frac{2(2\lambda-1)}{\lambda-1} > \left(\frac{\lambda}{\lambda-1}\right)^{\lambda}$ (2.2.3)

Since $\frac{\lambda(\lambda-1)}{2}$ is a positive number for $\lambda \ge 3$, referring to (2.2.1) and (2.2.2), when $\frac{\lambda(\lambda-1)}{2}$ multiplies both sides of (2.2.3), we have

$$\left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{2(2\lambda-1)}{\lambda-1}\right) = \lambda(2\lambda-1) = \binom{\lambda n}{n}$$
$$> \left(\frac{\lambda(\lambda-1)}{2}\right)\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} = \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}.$$

Thus, $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ when $\lambda \ge 3$ and $n = 2$. (2.2.4)

By induction on *n*, when $\lambda \ge 3$, if $\binom{\lambda n}{n} > \frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$ is true for *n*, then for $n+1$,

$$\binom{\lambda(n+1)}{n+1} = \binom{\lambda n+\lambda}{n+1}$$
$$= \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)}\cdot\binom{\lambda n}{n}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)(\lambda n+1)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)(n+1)}\cdot\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

$$\binom{\lambda(n+1)}{n+1} > \frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)}$$

$$\cdot\frac{\lambda n+1}{n}\cdot\frac{1}{n+1}\cdot\frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Notice $\frac{\lambda n+1}{n}>\lambda$, and

$$\frac{(\lambda n+\lambda)(\lambda n+\lambda-1)\cdots(\lambda n+2)}{(\lambda n+\lambda-n-1)(\lambda n+\lambda-n-2)\cdots(\lambda n-n+1)}>\left(\frac{\lambda}{\lambda-1}\right)^{\lambda-1}$$

because $\frac{\lambda n+\lambda}{\lambda n+\lambda-n-1}=\frac{\lambda}{\lambda-1};\frac{\lambda n+\lambda-1}{\lambda n+\lambda-n-2}>\frac{\lambda}{\lambda-1};\cdots;\frac{\lambda n+2}{\lambda n-n+1}>\frac{\lambda}{\lambda-1}$.

Thus, $\binom{\lambda(n+1)}{n+1}>\frac{\lambda^{\lambda-1}}{(\lambda-1)^{\lambda-1}}\cdot\frac{\lambda}{1}\cdot\frac{1}{n+1}\cdot\frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}}$.

$$\frac{\lambda^{\lambda-1}}{(\lambda-1)^{(\lambda-1)}}\cdot\frac{\lambda}{1}\cdot\frac{1}{(n+1)}\cdot\frac{\lambda^{\lambda n-\lambda+1}}{(\lambda-1)^{(\lambda-1)n-\lambda+1}}=\frac{\lambda^{\lambda(n+1)-\lambda+1}}{(n+1)(\lambda-1)^{(\lambda-1)(n+1)-\lambda+1}}.$$

Hence, $$\binom{\lambda(n+1)}{n+1}>\frac{\lambda^{\lambda(n+1)-\lambda+1}}{(n+1)(\lambda-1)^{(\lambda-1)(n+1)-\lambda+1}} \quad (2.2.5)$$

From (2.2.4) and (2.2.5), we have for $n\geq 2$ and $\lambda\geq 3$,

$$\binom{\lambda n}{n}>\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$$

Thus, **Lemma 2** is proven.

3. A Prime Number between $(\lambda-1)n$ and λn When $n\geq(\lambda-2)\geq 25$

Proposition:

For $n\geq\lambda-2\geq 25$, there exists at least a prime number p such that

$$(\lambda-1)n<p\leq\lambda n. \quad (3.1)$$

Proof:

Referring to (1.7), when $n\geq(\lambda-2)\geq 3$, if there is a prime number p in $\Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$, then $p\geq n+1=\sqrt{(n+2)n+1}>\sqrt{\lambda n}$. From (1.5),

$0\leq v_p\left(\Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\right)\leq R(p)\leq 1$. Then every prime number in

$$\Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} \text{ has a power of 0 or 1.} \quad (3.2)$$

From (1.7), for $\lambda\geq 3$ and $n\geq 3$,

$$\binom{\lambda n}{n}<\Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\cdot 2^{2n-3}\cdot(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}.$$

Applying this inequality to (2.2), when $n\geq(\lambda-2)\geq 3$,

$$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}}<\binom{\lambda n}{n}<\Gamma_{\lambda n\geq p>n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\cdot 2^{2n-3}\cdot(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}.$$

$\frac{\lambda^{\lambda n-\lambda+1}}{n(\lambda-1)^{(\lambda-1)n-\lambda+1}} < \Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}\cdot 2^{2n-3}\cdot(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}$. Since $(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2} > 1$ and $2^{2n-3} > 1$, $\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{\lambda^{\lambda n-\lambda+1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}\cdot 2^{2n-3}\cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}}$.

$$\frac{\lambda^{\lambda n-\lambda+1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+2}\cdot 2^{2n-3}\cdot n(\lambda-1)^{(\lambda-1)n-\lambda+1}} = \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}.$$

Thus, $\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}$.

Referring to (2.1.2), when $\lambda \ge 3$, $\left(\frac{\lambda}{\lambda-1}\right)^{\lambda} \ge e$. Thus, when $n \ge (\lambda-2) \ge 3$,

$$\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}}.$$

$$\frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot\left(\frac{\lambda}{\lambda-1}\right)^{\lambda}\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} \ge \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot e\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = f_3(n,\lambda)$$

Thus, $\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > \frac{2\lambda^2\cdot\left(\left(\frac{\lambda-1}{4}\right)\cdot e\right)^{(n-1)}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = f_3(n,\lambda)$ (3.3)

Let $x \ge 3$ and $y \ge 5$ both be real numbers.

When $x = y-2$,

$$f_3(x,y) = \frac{2(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{x-1}}{((x+2)\cdot x)^{\frac{\sqrt{x\cdot(x+2)}}{3}+3}} > f_4(x) = \frac{2(x+2)^2\cdot\left(\left(\frac{x+1}{4}\right)\cdot e\right)^{x-1}}{((x+2)\cdot x)^{\frac{x+1}{3}+3}} > 0 \quad (3.4)$$

$$f_4'(x) = f_4(x)\cdot\left(\frac{2}{x+2}+\ln\left(\frac{x+1}{4}\right)+\frac{4}{3}-\frac{2}{x+1}-\frac{1}{3}\ln((x+2)\cdot x)-\frac{10}{3x}-\frac{8}{3(x+2)}\right)$$
$$= f_4(x)\cdot f_5(x)$$

where $f_5(x) = \frac{2}{x+2}+\ln\left(\frac{x+1}{4}\right)+\frac{4}{3}-\frac{2}{x+1}-\frac{1}{3}\ln((x+2)\cdot x)-\frac{10}{3x}-\frac{8}{3(x+2)}$

$$f_5(x) = \frac{2}{x+2}+\ln\left(\frac{x+1}{4}\right)+\frac{4}{3}-\frac{2}{x+1}-\frac{\ln(x)}{3}-\frac{\ln(x+2)}{3}-\frac{10}{3x}-\frac{8}{3(x+2)}$$

$$f_5'(x)=\frac{-2}{(x+2)^2}+\frac{1}{x+1}+\frac{2}{(x+1)^2}-\frac{1}{3x}-\frac{1}{3(x+2)}+\frac{10}{3x^2}+\frac{8}{3(x+2)^2}$$
$$=\frac{-2x^2-4x-2}{(x+1)^2\cdot(x+2)^2}+\frac{2x^2+8x+8}{(x+1)^2\cdot(x+2)^2}+\frac{3x^2+6x}{3x(x+1)(x+2)}$$
$$-\frac{x^2+3x+2}{3x(x+1)(x+2)}-\frac{x^2+x}{3x(x+1)(x+2)}+\frac{10}{3x^2}+\frac{8}{3(x+2)^2}$$
$$f_5'(x)=\frac{4x+6}{(x+1)^2\cdot(x+2)^2}+\frac{x^2+2x-2}{3x(x+1)(x+2)}+\frac{10}{3x^2}+\frac{8}{3(x+2)^2}>0 \quad \text{when} \quad x\ge 3.$$

Thus, $f_5(x)$ is a strictly increasing function for $x\ge 3$.

When $x=9$,

$f_5(x)=\frac{2}{9+2}+\ln\left(\frac{9+1}{4}\right)+\frac{4}{3}-\frac{2}{9+1}-\frac{1}{3}\ln(9)-\frac{1}{3}\ln(9+2)-\frac{10}{27}-\frac{8}{33}>0$. Thus, for $x\ge 9$, $f_5(x)>0$. Then, $f_4'(x)=f_4(x)\cdot f_5(x)>0$.

Thus, $f_4(x)$ is a strictly increasing function for $x\ge 9$.

Let $x_1=9$ and $y_1=11$. From (3.4), when $x=y-2$, $f_3(x,y)>f_4(x)>0$.

Thus, when $x=(y-2)\ge 9$, then $xy\ge x_1y_1=99$, $f_3(x,y)$ is an increasing function with respect to the product of xy. (3.5)

$$\frac{\partial f_3(x,y)}{\partial x}=f_3(x,y)\cdot\left(\ln\left(\frac{y-1}{4}\right)+1-\frac{\sqrt{y}}{6\sqrt{x}}\cdot\ln(yx)-\frac{\sqrt{y}}{3\sqrt{x}}-\frac{3}{x}\right) \quad (3.6)$$
$$=f_3(x,y)\cdot f_6(x,y)$$

where $f_6(x,y)=\ln\left(\frac{y-1}{4}\right)+1-\frac{\sqrt{y}}{6\sqrt{x}}\cdot\ln(yx)-\frac{\sqrt{y}}{3\sqrt{x}}-\frac{3}{x}$

When $x=y-2$, then

$$f_6(x,y)=f_7(x)=\ln\left(\frac{x+1}{4}\right)+1-\frac{\sqrt{x+2}}{6\sqrt{x}}\cdot(\ln(x+2)+\ln(x)+2)-\frac{3}{x}$$
$$f_7'(x)=\frac{1}{x+1}-\frac{\sqrt{x+2}}{6\sqrt{x}}\cdot\left(\frac{1}{x+2}+\frac{1}{x}\right)+\frac{\ln(x+2)+\ln(x)+2}{6x\sqrt{x(x+2)}}+\frac{3}{x^2}$$

When $x\ge 3$, $f_7'(x)=\left(\frac{1}{x+1}-\frac{1}{3\sqrt{x(x+2)}}\right)+\frac{\ln(x+2)+\ln(x)}{6x\sqrt{x(x+2)}}+\frac{3}{x^2}>0$.

Thus, when $x\ge 3$, $f_7(x)$ is a strictly increasing function. (See **Appendix A2** for more details)

When $x=(y-2)\ge 3$, since $f_6(x,y)=f_7(x)$, $f_6(x,y)$ is an increasing function respect to xy.

When $x=(y-2)=9$, $f_6(x,y)=\ln\left(\frac{11-1}{4}\right)+1-\frac{\sqrt{11}}{6\sqrt{9}}\cdot\ln(99)-\frac{\sqrt{11}}{3\sqrt{9}}-\frac{3}{9}>0$.

$\frac{\partial f_6(x,y)}{\partial x}=\frac{\sqrt{y}}{12x\sqrt{x}}\cdot\ln(y)+\frac{\sqrt{y}}{12x\sqrt{x}}\cdot\ln(x)+\frac{\sqrt{y}}{6x\sqrt{x}}+\frac{\sqrt{y}}{6x\sqrt{x}}+\frac{3}{x^2}>0$ when $x\ge(y-2)\ge 3$.

Thus, when $x\ge(y-2)\ge 9$, $f_6(x,y)>0$, and it is an increasing function with respect to x and to the product of xy, then, $\frac{\partial f_3(x,y)}{\partial x}=f_3(x,y)\cdot f_6(x,y)>0$.

Thus, when $x \ge y-2 \ge 9$, $f_3(x,y)$ is an increasing function with respect to *x*. (3.7)

Referring to (3.5) and (3.7), when $x \ge y-2 \ge 9$, then $xy \ge x_1 y_1 = 99$, $f_3(x,y)$ is an increasing function with respect to the product of xy. (3.8)

Let $x = n$ and $y = \lambda$. Then when $n \ge (\lambda-2) \ge 9$, $f_3(n,\lambda)$ is an increasing function with respect to the product of λn and *n*. (3.9)

When $n = (\lambda-2) = 25$,

$$f_3(n,\lambda) = \frac{2\lambda^2 \cdot \left(\left(\frac{\lambda-1}{4}\right)\cdot e\right)^{n-1}}{(\lambda n)^{\frac{\sqrt{\lambda n}}{3}+3}} = \frac{2\cdot 27^2 \cdot \left(\left(\frac{27-1}{4}\right)\cdot e\right)^{25-1}}{(27\cdot 25)^{\frac{\sqrt{27\cdot 25}}{3}+3}} \approx \frac{1.249\text{E}+33}{9.784\text{E}+32} > 1.$$

Since $f_3(n,\lambda)$ is an increasing function of the product of λn, when $n = \lambda - 2 \ge 25$, $f_3(n,\lambda) > 1$.

Since $f_3(n,\lambda)$ is an increasing function with respect to *n*, when $n \ge \lambda-2 \ge 25$, $f_3(n,\lambda) > 1$.

Thus, referring to (3.3), when $n \ge \lambda-2 \ge 25$,

$$\Gamma_{\lambda n \ge p > n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > f_3(n,\lambda) > 1.$$

Let integer $m \ge n$. When $m \ge n \ge \lambda-2 \ge 25$,

$$\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > f_3(m,\lambda) > 1. \tag{3.10}$$

$$\begin{aligned}&\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\\ &= \Gamma_{\lambda m \ge p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\cdot \prod_{i=1}^{i=\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \ge p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right.\\ &\left.\cdot\,\Gamma_{\frac{\lambda m}{i+1} \ge p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)\end{aligned}$$

In $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \ge p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$, for every distinct prime number p in these ranges, the numerator $(\lambda m)!$ has the product of $p\cdot 2p \cdot 3p \cdots ip = i!\cdot p^i$. The denominator $((\lambda-1)m)!$ also has the same product of $i!\cdot p^i$. Thus, they cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$.

Referring to (1.2), $\prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{(\lambda-1)m}{i} \ge p > \frac{\lambda m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) = 1$.

Thus,

$$\begin{aligned}&\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\\ &= \Gamma_{\lambda m \ge p > (\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\cdot \prod_{i=1}^{\lambda-2}\left(\Gamma_{\frac{\lambda m}{i+1} \ge p > \frac{(\lambda-1)m}{i+1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)\end{aligned}$$

$$\Gamma_{\lambda m \ge p > m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} = \prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right). \tag{3.11}$$

(See **Appendix A3** for more details)

$\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right)$ is the product of $(\lambda-1)$ sectors from $i=1$ to $i=(\lambda-1)$.

Each of these sectors is the prime number factorization of the product of the consecutive integers between $\frac{(\lambda-1)m}{i}$ and $\frac{\lambda m}{i}$.

From (3.10) and (3.11), when $m \ge n \ge \lambda-2 \ge 25$,

$\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) > 1$.

Referring to (1.1), $\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} \ge 1$. Thus, when $m \ge n \ge \lambda-2 \ge 25$, at least one of the sectors in

$\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right) > 1$.

Let $\Gamma_{\frac{\lambda m}{i} \ge p > \frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\} > 1$ be such a sector and let $m = ni$ where $(\lambda-1) \ge i \ge 1$ from (3.11). Thus, when $m = ni \ge n \ge \lambda-2 \ge 25$,

$$\Gamma_{\frac{\lambda ni}{i} \ge p > \frac{(\lambda-1)ni}{i}}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} = \Gamma_{\lambda n \ge p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} > 1. \tag{3.12}$$

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{(\lambda ni)\cdot(\lambda ni-1)\cdots(\lambda ni-i)\cdots(\lambda ni-2i)\cdots(\lambda ni-(n-1)i)\cdots(\lambda ni-ni+1)\cdot((\lambda-1)ni)!}{((\lambda-1)ni)!}$$

$$\frac{(\lambda ni)!}{((\lambda-1)ni)!} = \frac{i\cdot(\lambda n)\cdot(\lambda ni-1)\cdots i\cdot(\lambda n-1)\cdots i\cdot(\lambda n-2)\cdots i\cdot(\lambda n-n+1)\cdots(\lambda ni-ni+1)\cdot((\lambda-1)ni)!}{((\lambda-1)ni)!}$$

Thus, $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ contains all the factors of

(λn), $(\lambda n-1)$, $(\lambda n-2)$, $\cdots$, $(\lambda n-n+1)$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

These factors make up all the consecutive integers in the range of $\lambda n \ge p > (\lambda-1)n$ in $\frac{(\lambda n)!}{((\lambda-1)n)!}$. Thus, $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ contains $\frac{(\lambda n)!}{((\lambda-1)n)!}$.

Referring to the definition, all prime numbers in $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the ranges of $\lambda ni \ge p > \lambda n$ and $(\lambda-1)n > p$ do not contribute to $\Gamma_{\lambda n \ge p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}$, nor does i for $(\lambda-1) \ge i \ge 1$. Only the prime numbers in the prime factorization of $\frac{(\lambda ni)!}{((\lambda-1)ni)!}$ in the range of $\lambda n \geq p > (\lambda-1)n$ present in $\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\}$. Since $\frac{(\lambda n)!}{((\lambda-1)n)!}$ is the product of all the consecutive integers in this range,

$\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} = \Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\}$.

Referring to (3.12), when $m = ni \geq n \geq \lambda - 2 \geq 25$,

$\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda ni)!}{((\lambda-1)ni)!}\right\} > 1$. Thus, when $n \geq \lambda - 2 \geq 25$,

$\Gamma_{\lambda n \geq p > (\lambda-1)n}\left\{\frac{(\lambda n)!}{((\lambda-1)n)!}\right\} > 1$. Referring to (1.3), there exists at least a prime number *p* such that $(\lambda-1)n < p \leq \lambda n$.

Thus, **Proposition (3.1)** is proven. It becomes a theorem: **Theorem (3.1)**.

4. Proof of Legendre's Conjecture

Legendre's conjecture states that there is a prime number between n^2 and $(n+1)^2$ for every positive integer *n*. (4.1)

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k - 2 \geq 25$, there exists at least a prime number *p* such that $j(k-1) < p \leq jk$. (4.2)

When $k = j + 1 \geq 27$, then $j = k - 1 \geq 26$,

Applying $k = j + 1$ into (4.2), then $j^2 < p \leq j(j+1) < (j+1)^2$.

Let $n = j \geq 26$, then we have $n^2 < p < (n+1)^2$. (4.3)

For $1 \leq n \leq 26$, we have a table, Table 1, that shows Legendre's conjecture valid. (4.4)

Combining (4.3) and (4.4), we have proven Legendre's conjecture.

Extension of Legendre's conjecture

There are at least two prime numbers, p_n and p_m, between j^2 and $(j+1)^2$ for every positive integer *j* such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ where p_n is the *n*th prime number, p_m is the *m*th prime number, and $m \geq n+1$. (4.5)

Proof:

Referring to **Theorem (3.1)**, for integers $j \geq k - 2 \geq 25$, there exists at least a prime number *p* such that $j(k-1) < p \leq jk$.

When $k - 1 = j \geq 26$, then $j(k-1) = j^2 < p_n \leq jk = j(j+1)$. Thus, there is at least a prime number p_n such that $j^2 < p_n \leq j(j+1)$ when $j = k - 1 \geq 26$.

When $j = k - 2 \geq 26$, then $k = j + 2$.

$j(k-1) = j(j+1) < p_m \leq jk = j(j+2) < (j+1)^2$. Thus, there is at least another prime number p_m such that $j(j+1) < p_m < (j+1)^2$ when $j = k - 2 \geq 26$.

Thus, when $j \geq 26$, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \le j(j+1) < p_m < (j+1)^2$ where $m \ge n+1$ for $p_m > p_n$. (4.6)

For $1 \le j \le 26$, we have a table, **Table 2**, that shows that (4.5) is valid. (4.7)

Combining (4.6) and (4.7), we have proven (4.5). It becomes a theorem: **Theorem (4.5)**.

Table 1. For $1 \le n \le 26$, there is a prime number between n^2 and $(n+1)^2$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
n^2	1	4	9	16	25	36	49	64	81	100	121	144	169
p	3	5	11	19	29	41	53	67	83	103	127	149	173
$(n+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169	196
n	14	15	16	17	18	19	20	21	22	23	24	25	26
n^2	196	225	256	289	324	361	400	441	484	529	576	625	676
p	199	229	263	307	331	373	409	449	491	541	587	641	683
$(n+1)^2$	225	256	289	324	361	400	441	484	529	576	625	676	729

Table 2. For $1 \le j \le 26$, there are 2 primes such that $j^2 < p_n \le j(j+1) < p_m < (j+1)^2$.

j	1	2	3	4	5	6	7	8	9	10	11	12	13
j^2	1	4	9	16	25	36	49	64	81	100	121	144	169
p_n	2	5	11	19	29	41	53	67	83	103	127	149	173
$j(j+1)$	2	6	12	20	30	42	56	72	90	110	132	156	182
p_m	3	7	13	23	31	43	59	73	97	113	137	163	191
$(j+1)^2$	4	9	16	25	36	49	64	81	100	121	144	169	196
j	14	15	16	17	18	19	20	21	22	23	24	25	26
j^2	196	225	256	289	324	361	400	441	484	529	576	625	676
p_n	199	229	263	293	331	373	409	449	491	541	587	641	683
$j(j+1)$	210	240	272	306	342	380	420	462	506	552	600	650	702
p_m	211	251	277	311	349	389	431	467	521	557	613	659	709
$(j+1)^2$	225	256	289	324	361	400	441	484	529	576	625	676	729

5. The Proofs of Three Related Conjectures

Oppermann's conjecture was proposed in March 1877 by Ludvig Oppermann (1817-1883). It states that for every integer $x > 1$, there is at least one prime number between $x(x-1)$ and x^2, and at least another prime number between x^2 and $x(x+1)$. [4] (5.1)

Proof:

Theorem (4.5) states that there are at least two prime numbers, p_n and p_m, between j^2 and $(j+1)^2$ for every positive integer *j* such that $j^2 < p_n \le j(j+1) < p_m < (j+1)^2$ where $m \ge n+1$ for $p_m > p_n$.

$j(j+1)$ is a composite number except $j = 1$. Since $j^2 < p_n \le j(j+1)$ is valid for every positive integer *j*, when we replace *j* with $j+1$, we have $(j+1)^2 < p_v < (j+1)(j+2)$.

Thus, we have $j(j+1) < p_m < (j+1)^2 < p_v < (j+1)(j+2)$. (5.2)

When $x > 1$, then $(x-1) \geq 1$. Substituting *j* with $(x-1)$ in (5.2), we have

$$x(x-1) < p_m < x^2 < p_v < x(x+1) \quad (5.3)$$

Thus, we have proven Oppermann's conjecture.

Brocard's conjecture is based on Henri Brocard (1845-1922). It states that there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$, where p_n is the n^{th} prime number, for every $n > 1$. [5] (5.4)

Proof:

Theorem (4.5) states that there are at least two prime numbers, p_n and p_m, between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1)$ and $j(j+1) < p_m < (j+1)^2$ for every positive integer *j*, where $m \geq n+1$ for $p_m > p_n$. When $j > 1$, $j(j+1)$ is a composite number. Then **Theorem (4.5)** can be written as $j^2 < p_n < j(j+1)$ and $j(j+1) < p_m < (j+1)^2$.

In the series of prime numbers: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, ... all prime numbers except p_1 are odd numbers. Their gaps are two or more. Thus, when $n > 1$, $(p_{n+1} - p_n) \geq 2$. Thus, we have $p_n < (p_n + 1) < (p_n + 2) \leq p_{n+1}$ when $n > 1$. (5.5)

Applying **Theorem (4.5)** to **(5.5)**, when $n > 1$, we have at least two prime numbers p_{m1}, and p_{m2} in between $(p_n)^2$ and $(p_n+1)^2$ such that $(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2$, and at least two more prime numbers p_{m3}, p_{m4} in between $(p_n+1)^2$ and $(p_n+2)^2$ such that $(p_n+1)^2 < p_{m3} < (p_n+1)(p_n+2) < p_{m4} < (p_n+2)^2 \leq (p_{n+1})^2$.

Thus, there are at least 4 prime numbers between $(p_n)^2$ and $(p_{n+1})^2$ for $n > 1$ such that

$$(p_n)^2 < p_{m1} < p_n(p_n+1) < p_{m2} < (p_n+1)^2 < p_{m3} < (p_n+1)(p_n+2) < p_{m4} < (p_{n+1})^2 \quad (5.6)$$

Thus, Brocard's conjecture is proven.

Andrica's conjecture is named after Dorin Andrica [6]. It is a conjecture regarding the gaps between prime numbers. The conjecture states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all *n* where p_n is the n^{th} prime number. If $g_n = p_{n+1} - p_n$ denotes the n^{th} prime gap, then Andrica's conjecture can also be rewritten as $g_n < 2\sqrt{p_n} + 1$. (5.7)

Proof:

From **Theorem (4.5)**, for every positive integer *j*, there are at least two prime numbers p_n and p_m between j^2 and $(j+1)^2$ such that $j^2 < p_n \leq j(j+1) < p_m < (j+1)^2$ where $m \geq n+1$ for $p_m > p_n$. Since $m \geq n+1$, we have $p_m \geq p_{n+1}$.

Thus, we have $j^2 < p_n$ (5.8)

$$\text{and } p_{n+1} \leq p_m < (j+1)^2. \quad (5.9)$$

Since *j*, p_n, p_{n+1} and $(j+1)$ are positive integers,

$$j < \sqrt{p_n} \tag{5.10}$$

$$\text{and} \quad \sqrt{p_{n+1}} < j+1 \tag{5.11}$$

Applying (5.10) to (5.11), we have $\sqrt{p_{n+1}} < \sqrt{p_n} + 1$. (5.12)

Thus, $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all *n* since in **Theorem (4.5)**, *j* holds for all positive integers.

Using the prime gap to prove the conjecture, from (5.8) and (5.9), we have $g_n = p_{n+1} - p_n < (j+1)^2 - j^2 = 2j+1$. From (5.10), $j < \sqrt{p_n}$.

Thus, $g_n = p_{n+1} - p_n < 2\sqrt{p_n} + 1$. (5.13)

Thus, Andrica's conjecture is proven.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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[4] Wikipedia. https://en.wikipedia.org/wiki/Oppermann%27s_conjecture

[5] Wikipedia. https://en.wikipedia.org/wiki/Brocard%27s_conjecture

[6] Wikipedia. https://en.wikipedia.org/wiki/Andrica%27s_conjecture

[7] Wikipedia. https://en.wikipedia.org/wiki/Proof_of_Bertrand%27s_postulate

Appendix

A1

At the 1912 International Congress of Mathematicians, Edmund Landau listed four basic problems about primes. These problems were characterized in his speech as "unattackable at the present state of mathematics" and are now known as Landau's problems. They are as follows:

1) Goldbach's conjecture: Can every even integer greater than 2 be written as the sum of two primes?

2) Twin prime conjecture: Are there infinitely many primes p such that $p + 2$ is prime?

3) Legendre's conjecture: Does there always exist at least one prime between consecutive perfect squares?

4) Are there infinitely many primes p such that $p - 1$ is a perfect square? In other words: Are there infinitely many primes of the form $m^2 + 1$?

As of 2023, all four problems are unresolved.

The above content is quoted from Wikipedia.

https://en.wikipedia.org/wiki/Landau%27s_problems

The author of this paper has browsed the articles about the proof of Legendre's conjecture posted on and published in different media in recent years, and made some brief comments, as below.

1) Kouji Takaki, *Proof of Legendre's Conjecture*, (Mar. 2023)

https://vixra.org/abs/2110.0102

Comment: Using the empirical method to prove Legendre's conjecture is considered insufficient evidence.

2) Zhi Li and Hua Li, *Proof of Legendre Conjecture*, (Sep. 2022)

https://vixra.org/pdf/2209.0070v1.pdf

Comment: The method of probability and statistics introduces the uncertainty in a certain range. It is not the right choice to prove Legendre's conjecture.

3) Kouider Mohammed Ridha, *On Legendre's Conjecture*, (Sep. 2022)

https://vixra.org/pdf/2209.0051.pdf

Comment: The prime number formular $p(n) = n^2 + n - 1$ is not correct.

4) Kenneth Watanabe, *Definitive Proof of Legendre's Conjecture*, (2019)

https://vixra.org/pdf/1901.0436v1.pdf

Comment: The asymptotic method is not the right choice to prove Legendre's conjecture.

5) Ahmed Telfah, *A Proof of Legendre's Conjecture and Andrica's Conjecture*, (Dec. 2018)

https://www.researchgate.net/publication/329844915_A_proof_of_Legendre's_conjecture_and_Andrica's_conjecture

Comment: Using the distribution function to deal with discrete primes creates bound problems. The empirical method is not sufficient to solve these problems.

6) Samuel Bonaya Buya, *Proof of Legendre's Conjecture*, Research & Reviews: Journal of Applied Science and Innovations, RRJASI, Volum2, Issue 2, Janu-

ary-March, 2018

Comment: There is a contradiction between Equation (2) and Equation (9), which is a fatal error.

7) Samuel Bonaya Buya, *A Simple Proof of Legendre's Conjecture*, (2018) https://www.academia.edu/35702628/A_simple_proof_of_Legendres_conjecture

Comment: Using the prime number function to estimate the prime number between n^2 and $(n+1)^2$ is not the right way to prove Legendre's conjecture.

A2

Derivation details for $f_7'(x) > 0$

When $x = y - 2 \ge 3$, then

$$f_6(x,y) = f_7(x) = \ln\left(\frac{x+1}{4}\right) + 1 - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot \left(\ln(x+2) + \ln(x) + 2\right) - \frac{3}{x}$$

$$\frac{\mathrm{d}\left(\frac{\sqrt{x+2}}{6\sqrt{x}}\right)}{\mathrm{d}x} = \frac{1}{6x} \cdot \frac{\sqrt{x}}{2\sqrt{x+2}} - \frac{1}{6x} \cdot \frac{\sqrt{x+2}}{2\sqrt{x}} = \frac{x - x - 2}{12x\sqrt{x(x+2)}} = \frac{-1}{6x\sqrt{x(x+2)}}$$

$$f_7'(x) = \frac{1}{x+1} - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot \left(\frac{1}{x+2} + \frac{1}{x}\right) + \frac{\ln(x+2) + \ln(x) + 2}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}$$

$$= \frac{1}{x+1} - \frac{\sqrt{x+2}}{6\sqrt{x}} \cdot \frac{x + x + 2}{x(x+2)} + \frac{\ln(x+2) + \ln(x) + 2}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}$$

$$= \frac{1}{x+1} - \frac{1}{3\sqrt{x}} \cdot \frac{x+1}{x\sqrt{x+2}} + \frac{2}{6x\sqrt{x(x+2)}} + \frac{\ln(x+2) + \ln(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}$$

$$= \frac{1}{x+1} - \frac{x+1}{3x\sqrt{x(x+2)}} + \frac{1}{3x\sqrt{x(x+2)}} + \frac{\ln(x+2) + \ln(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}$$

$$= \frac{1}{x+1} - \frac{x}{3x\sqrt{x(x+2)}} + \frac{\ln(x+2) + \ln(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}$$

$$= \left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}}\right) + \frac{\ln(x+2) + \ln(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2}$$

When $x \ge 1$, then $3\sqrt{x(x+2)} + (x+1) > 0$

$$\left(3\sqrt{x(x+2)} + (x+1)\right) \cdot \left(3\sqrt{x(x+2)} - (x+1)\right)$$

$$= \left(3\sqrt{x(x+2)}\right)^2 - (x+1)^2 = 8x^2 + 16x - 1 > 0$$

Thus, $\left(3\sqrt{x(x+2)} + (x+1)\right) \cdot \left(3\sqrt{x(x+2)} - (x+1)\right) > 0$

$$3\sqrt{x(x+2)} - (x+1) > 0$$

$$3\sqrt{x(x+2)} > (x+1) \quad \text{then} \quad \frac{1}{x+1} > \frac{1}{3\sqrt{x(x+2)}}$$

When $x \ge 1$, $\left(\frac{1}{x+1} - \frac{1}{3\sqrt{x(x+2)}}\right) > 0$, and $\frac{\ln(x+2) + \ln(x)}{6x\sqrt{x(x+2)}} + \frac{3}{x^2} > 0$.

Then $f_7'(x)=\left(\frac{1}{x+1}-\frac{1}{3\sqrt{x(x+2)}}\right)+\frac{\ln(x+2)+\ln(x)}{6x\sqrt{x(x+2)}}+\frac{3}{x^2}>0$.

Thus, when $x\geq 3$, $f_7(x)$ is a strictly increasing function.

A3

Derivation details for (3.11)

$$\begin{aligned}\Gamma_{\lambda m\geq p>m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}&=\Gamma_{\lambda m\geq p>(\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\cdot\Gamma_{(\lambda-1)m\geq p>\frac{\lambda m}{2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\\&\cdot\Gamma_{\frac{\lambda m}{2}\geq p>\frac{(\lambda-1)m}{2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\cdot\Gamma_{\frac{(\lambda-1)m}{2}\geq p>\frac{\lambda m}{3}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\\&\cdot\Gamma_{\frac{\lambda m}{3}\geq p>\frac{(\lambda-1)m}{3}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\cdot\Gamma_{\frac{(\lambda-1)m}{3}\geq p>\frac{\lambda m}{4}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\\&\cdots\Gamma_{\frac{(\lambda-1)m}{\lambda-3}\geq p>\frac{\lambda m}{\lambda-2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\cdot\Gamma_{\frac{\lambda m}{\lambda-2}\geq p>\frac{(\lambda-1)m}{\lambda-2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\\&\cdot\Gamma_{\frac{(\lambda-1)m}{\lambda-2}\geq p>\frac{\lambda m}{\lambda-1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\cdot\Gamma_{\frac{\lambda m}{\lambda-1}\geq p>\frac{(\lambda-1)m}{\lambda-1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\end{aligned}$$

In $\Gamma_{(\lambda-1)m\geq p>\frac{\lambda m}{2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$, for every distinct prime number *p* in this range, the numerator $(\lambda m)!$ has the prime number *p*. The denominator $((\lambda-1)m)!$ also has the same *p*. Thus, they cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$. Referring to (1.2), $\Gamma_{(\lambda-1)m\geq p>\frac{\lambda m}{2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}=1$.

In $\Gamma_{\frac{(\lambda-1)m}{2}\geq p>\frac{\lambda m}{3}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$, for every distinct prime number *p* in this range, the numerator $(\lambda m)!$ has the product of $p\cdot 2p$. The denominator $((\lambda-1)m)!$ also has the same product of $p\cdot 2p$. Thus, they cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$.

Referring to (1.2), $\Gamma_{\frac{(\lambda-1)m}{2}\geq p>\frac{\lambda m}{3}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}=1$.

In $\Gamma_{\frac{(\lambda-1)m}{3}\geq p>\frac{\lambda m}{4}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$, for every distinct prime number *p* in this range, the numerator $(\lambda m)!$ has the product of $p\cdot 2p\cdot 3p$. The denominator $((\lambda-1)m)!$ also has the same product of $p\cdot 2p\cdot 3p$. Thus, they cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$.

Referring to (1.2), $\Gamma_{\frac{(\lambda-1)m}{3}\geq p>\frac{\lambda m}{4}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}=1$.

⋮

In $\Gamma_{\frac{(\lambda-1)m}{\lambda-3}\geq p>\frac{\lambda m}{\lambda-2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$, for every distinct prime number p in this range, the numerator $(\lambda m)!$ has the product of $p\cdot 2p\cdot 3p\cdots(\lambda-3)p$. The denominator $((\lambda-1)m)!$ also has the same product of $p\cdot 2p\cdot 3p\cdots(\lambda-3)p$. Thus, they cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$.

Referring to (1.2), $\Gamma_{\frac{(\lambda-1)m}{\lambda-3}\geq p>\frac{\lambda m}{\lambda-2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}=1$.

In $\Gamma_{\frac{(\lambda-1)m}{\lambda-2}\geq p>\frac{\lambda m}{\lambda-1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}$, for every distinct prime number p in this range, the numerator $(\lambda m)!$ has the product of $p\cdot 2p\cdot 3p\cdots(\lambda-2)p$. The denominator $((\lambda-1)m)!$ also has the same product of $p\cdot 2p\cdot 3p\cdots(\lambda-2)p$. Thus, they cancel each other in $\frac{(\lambda m)!}{((\lambda-1)m)!}$.

Referring to (1.2), $\Gamma_{\frac{(\lambda-1)m}{\lambda-2}\geq p>\frac{\lambda m}{\lambda-1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}=1$.

Thus,

$$\begin{aligned}\Gamma_{\lambda m\geq p>m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}&=\Gamma_{\lambda m\geq p>(\lambda-1)m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\cdot\Gamma_{\frac{\lambda m}{2}\geq p>\frac{(\lambda-1)m}{2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\\&\cdot\Gamma_{\frac{\lambda m}{3}\geq p>\frac{(\lambda-1)m}{3}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\cdots\Gamma_{\frac{\lambda m}{\lambda-2}\geq p>\frac{(\lambda-1)m}{\lambda-2}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\\&\cdot\Gamma_{\frac{\lambda m}{\lambda-1}\geq p>\frac{(\lambda-1)m}{\lambda-1}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\end{aligned}$$

$$\Gamma_{\lambda m\geq p>m}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}=\prod_{i=1}^{i=\lambda-1}\left(\Gamma_{\frac{\lambda m}{i}\geq p>\frac{(\lambda-1)m}{i}}\left\{\frac{(\lambda m)!}{((\lambda-1)m)!}\right\}\right). \quad (3.11)$$