

A Proof of Four-Color Theorem

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Abstract

Four-color theorem has only been proved by computer since it was proposed, many people have proposed their mathematical proof of four-color theorem, but their proof is disputed then, what lead to a situation that the mathematical proof of four-color theorem has been lacking to this day. In this article, we have summarized some laws based on previous researches, and proposed a mathematical proof of four-color theorem by using these laws through a recursive method.

Keywords

Four-Color Theorem

1. Introduction

Many people tried to prove the four-color theorem since it was proposed. It could date back to 1852 when Francis Guthrie, while trying to color a map, noticed that four colors suffice. His brother Frederick Guthrie then explained this discovery to August DeMorgan, who in turn showed it to Arthur Cayley. The problem was first published as a puzzle for the public by Cayley in 1878.

A year later, in 1879, Alfred Kempe published his proof, which was not disputed until 1890. In 1880, Peter Tait published a different proof, which was proved incomplete in 1891 by Julius Petersen, he found Alfred Kempe's proof could only prove five-color theorem [1]. Then Philip Franklin in 1922 proved that the four-color theorem is true for maps which contain at most 25 regions [2].

In 1976, Kenneth Appel and Wolfgang Haken published their proof of four-color theorem [3]. Their proof has not been disputed to this day. However, it relies heavily on the use of computer, and not be seen as a mathematical proof, so people are still looking for a mathematical proof of four-color theorem.

In this article, the next two sections include some of the laws we have summarized based on previous researches and a mathematical proof of four-color theorem by using these laws through a recursive method.

2. Related Works

The four-color theorem states that any map, a division of the plane into any number of regions, can be colored using no more than four colors in such a way that no two adjacent regions share the same color. We can use a set of nodes and lines to represent the map, for the node stand for region and the line between two nodes stand for the adjacency relationship of two regions (**Figure 1**).

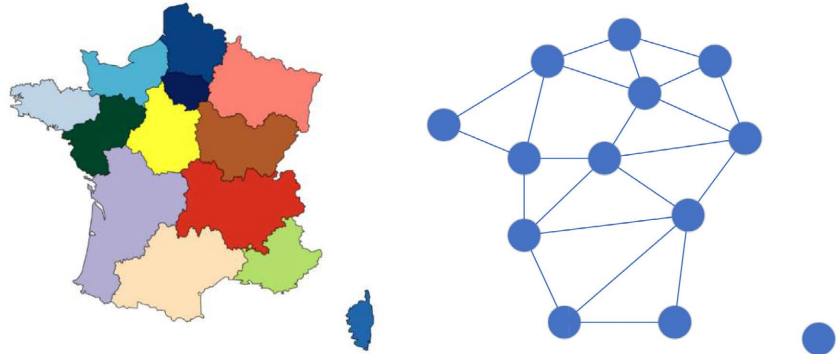


Figure 1. An example of representation.

It's obvious that:

1) Keep add lines to a graph, we could get a new graph. If this new graph could be colored using no more than four colors, the original graph can be colored using no more than four colors.

2) Two lines between the same two nodes could divide all areas into two parts of it, nodes and lines in each area could constitute a graph. If each these two new graphs could be colored using no more than four colors, the original graph can be colored using no more than four colors. For example, if the middle graph and the right graph in **Figure 2** could be colored using no more than four colors, the left graph can be colored using no more than four colors.

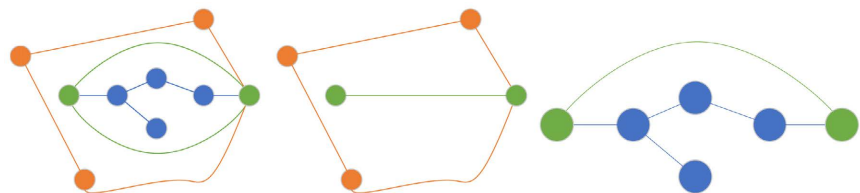


Figure 2. Two lines between same two nodes.

3) Three nodes linked end by end could divide all areas into two parts of it, nodes and lines in each area could constitute a graph. If each these two new graphs could be colored using no more than four colors, the original graph can be colored using no more than four colors. For example, if the middle graph and the right graph in **Figure 3** could be colored using no more than four colors, the left graph can be colored using no more than four colors.

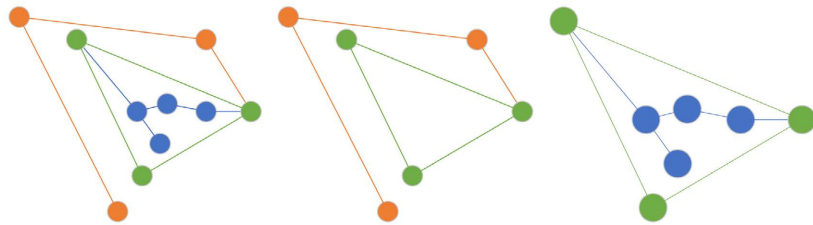


Figure 3. Three nodes linked end-to-end.

We use “**saturate graph**” to call the graph which couldn’t be added any line and not contain the situations (2) or (3). It could be seen from (1) that if any saturate graph could be colored using no more than four colors, then any graph can be colored using no more than four colors.

If a graph could be colored using no more than four colors, we can use “A” to mark any area which is colored by one of any two colors, use “B” to mark any area which is colored by one of any two other colors. For the convenience of the following description, we have made the following definitions:

Unique node: an only node with its color in a graph.

A (or B) loop: a structure formed by connecting nodes which are all marked by A (or B), linked end-to-end.

Odd A (or B) loop: an A (or B) loop containing odd amounts of nodes.

3. Proof of Four-Color Theorem

For any saturate graph G with amounts of nodes greater than 25, assume it contain n nodes, and have been colored abide by the following rule:

- As the five-color theorem have already been confirmed [1], if G couldn’t be colored by using no more than four colors, use five colors. Else use four colors.

G must have a structure like **Figure 4**, and meet following constraint:

- (N_1 or N_2 is not unique node) and (N_3 or N_4 is not unique node).

Where N_1 and N_2 must be different colors, we use A to mark all nodes with these two colors, N_3 and N_4 might be different colors, we use B to mark all nodes with these two colors if they really be different colors, else we use B to mark all nodes with N_3 ’s color and an any other color.

There are two conditions:

- Condition 1: N_1 and N_2 are not contained by any A loop (**Figure 5**).
- Condition 2: N_1 and N_2 are contained by at least one A loop (**Figure 6**).

For condition 1, if we merge N_1, N_2 into a new node N_1^2 , every line which previously linked with N_1, N_2 are now linked with N_1^2 , every node keeping their mark, if there is a unique node in N_1 and N_2 , we set the N_1^2 ’s color to be same as that unique node’s color, else we set N_1^2 ’s color to be same as N_1 ’s color or N_2 ’s color. We can find there is no generation of odd A loop or odd B loop. Therefore, the new graph is in accord with the color law. For ease of description, we call the original graph G_n , call the new graph G_{n-1} . We use $X(G)$ to stand for the amount color type of G , it can be concluded that $X(G_n) = X(G_{n-1})$ (**Figure 7**).

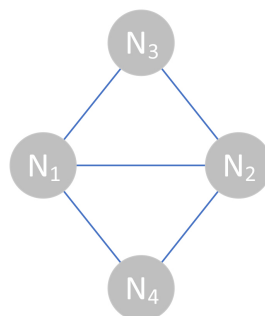


Figure 4. Specified structure.

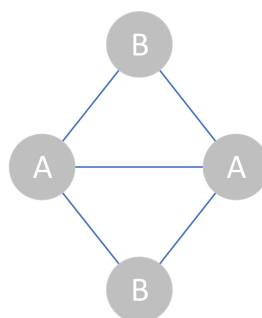


Figure 5. Condition 1.

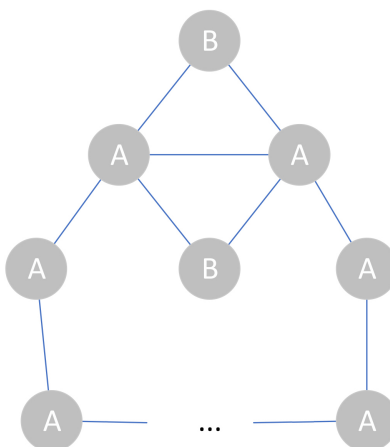


Figure 6. Condition 2.

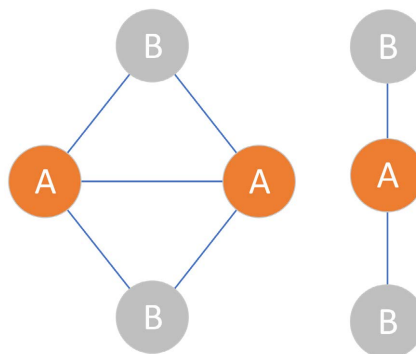


Figure 7. Nodes merged structure.

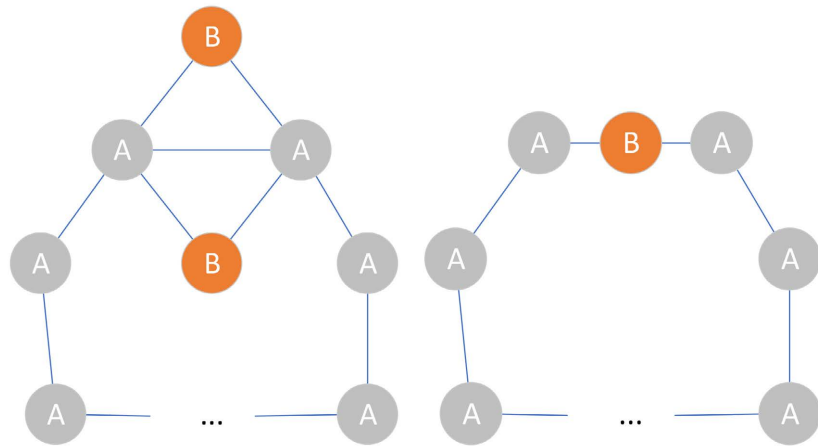


Figure 8. Nodes merged structure.

For condition 2, if we merge N_3, N_4 into a new node N_3^4 , every line which previously linked with N_3, N_4 are now linked with N_3^4 , every node keeping their mark, if there is a unique node in N_3 and N_4 , we set the N_3^4 's color to be same as that unique node's color, else we set N_3^4 's color to be same as N_3 's color or N_4 's color. We can find there is no generation of odd A loop or odd B loop. Therefore, the new graph is in accord with the color law. In this condition, it can be concluded that $X(G_n) = X(G_{n-1})$ too (**Figure 8**).

If G_{n-1} is not a saturate graph, we can still know that G_{n-1} can be colored using no more than four colors only if we can proof the two subgraphs which are generated by following (2) and (3) of it can be colored using no more than four colors.

Similarly, it can be concluded that $X(G_{n-1}) = X(G_{n-2})$, Iterating multiple times, it can be concluded that $X(G_n) = X(G_{2^s})$, As we know $X(G_{2^s})$ equals to four [2], so $X(G_n)$ equals to four.

4. Conclusion

We just proved that for any saturate graph, it could be colored using no more than four colors. As we know from previous words "if any saturate graph could be colored using no more than four colors, then any graph can be colored using no more than four colors", so we can infer that any graph could be colored using no more than four colors. Therefore, four-color theorem has been proven.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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