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The Most Irrational Number that Shows up Everywhere: The Golden Ratio

Jiwon Choi[1], Agegnehu Atena[2], Wondimu Tekalign[2]

[1]Department of Biology, Savannah State University, Savannah, GA, USA
[2]Department of Mathematics, Savannah State University, Savannah, GA, USA
Email: atenaa@savannahstate.edu

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Abstract

Since the time of the ancient Greeks, humans have been aware of this mathematical idea. Golden ratio is an irrational number that is symbolized by the Greek numeral phi (φ). One can find this ratio everywhere. It is in nature, art, architecture, human body, etc. But this symbolism can result in a strong connection with mathematical nature. In this paper we will be discussing the connection between Fibonacci sequence (a series of numbers where every number is equal to the sum of two numbers before it) and Golden ratio. Secondly, how this mathematical idea shows up in a nature, such as sunflower and human DNA.

Keywords

The Golden Ratio, Fibonacci Sequence, Nature

1. Introduction

Leonardo Pisano Fibonacci, typically known as Fibonacci, was an Italian mathematician. He played major role in the resurrection of ancient mathematics, as well as making significant contributions himself. The book "Liber Abaci" is a basic numerical sequence that serves as the foundation for the mathematical relationship that lies behind phi (φ). Indian mathematicians knew about this sequence as early as the 6th century AD, but it was Fibonacci who introduced it to the western after his travels around the Mediterranean and North Africa [1].

Ancient Greek mathematicians first studied the golden ratio, because of its frequent appearance in geometry; the division of a line into "extreme and mean ratio" (the golden section) is important in the geometry of regular pentagrams and pentagons. 5th-century BC mathematician Hippias discovered that the golden ratio was an irrational number. Euclid's Elements provide several propositions and proofs employing the golden ratio and first known definition; a straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser [1]. Some of the greatest mathematical minds of all ages, from ancient Greece, through the medieval and the Renaissance to present-day people, have spent endless hours over this simple ratio and properties. The purpose of this manuscript is to investigate some real word applications that involve the Golden Ratio.

2. Calculation of Golden Ratio

The "golden ratio" is a unique mathematical relationship. The number Φ/φ is known as the golden ratio. Two positive numbers x and y, with $x > y$, are said to be in the golden ratio if the ratio between the sum of those numbers and the larger one is the same as the ratio between the larger one and the smaller: Assuming $x > y$

$$\frac{x+y}{x} = \frac{x}{y}$$

Two quantities x and y are said to be in the golden ratio φ if

$$\varphi = \frac{x}{y} = \frac{x+y}{x}$$

One method for finding the value of φ is to start with the left fraction. Through simplifying the fraction and substituting in $\frac{y}{x} = \frac{1}{\varphi}$,

$$\frac{x+y}{x} = \frac{x}{x} + \frac{y}{x} = 1 + \frac{y}{x} = 1 + \frac{1}{\varphi}$$

Therefore,

$$\varphi = 1 + \frac{1}{\varphi}$$

Multiplying by φ gives

$$1 + \varphi = \varphi^2$$

which can be rearranged to

$$\varphi^2 - \varphi - 1 = 0$$

Using the quadratic formula,

$$\varphi = \frac{1 \pm \sqrt{1+4}}{2}$$

Therefore,

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618033$$

or

$$\varphi = \frac{1-\sqrt{5}}{2} = -0.618033$$

Because φ is the ratio between positive quantities, φ is necessarily the positive one.

The φ can be expressed by the following series.

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}$$

Finally, we can write φ as,

$$\varphi = 1 + \frac{1}{\varphi}$$

which is another recursive definition that can be continued as

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\varphi \cdots}}}$$

Connection with Fibonacci

Furthermore, the golden ratio is the ratio of consecutive Fibonacci numbers, The Fibonacci number are a sequence of numbers in which each number is the sum of the two numbers before it. As the first two numbers, it starts with 0 and 1. This sequence is one of the most famous mathematical formulas in mathematics. Fibonacci numbers are a sequence of whole numbers arranged as 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, … Every number is the sum of the preceding two numbers. This sequence is called the Fibonacci sequence and it's an infinite sequence. As shown in **Figure 1**, Fibonacci numbers can be represented as a spiral, if we make squares with those widths. In the given **Figure 1**, we can see how the squares fit neatly together. For instance, 5 and 8 add up to 13, 8 and 13 add up to 21, and it goes on.

This sequence ties directly into the Golden ratio because if you take any two successive Fibonacci numbers, their ratio is very close to the Golden ratio. As the numbers get higher, the ratio becomes even closer to 1.618. For example, the ratio of 3 to 5 is 1.666. But the ratio of 13 to 21 is 1.625. The definition of Fibonacci number as the sum of the previous two terms is as follows: $F(n+1) = F(n) + F(n-1)$ [3].

The definition of Fibonacci number:

$$F(n+1) = F(n) + F(n-1)$$

Then express the ratio of successive Fibonacci terms as follows:

$$\frac{F(n+1)}{F(n)} = \frac{F(n) + F(n-1)}{F(n)} = 1 + \frac{F(n-1)}{F(n)}$$

Now, define a quanity x as:

$$x = \lim_{n \to \infty} \frac{F(n+1)}{F(n)}$$

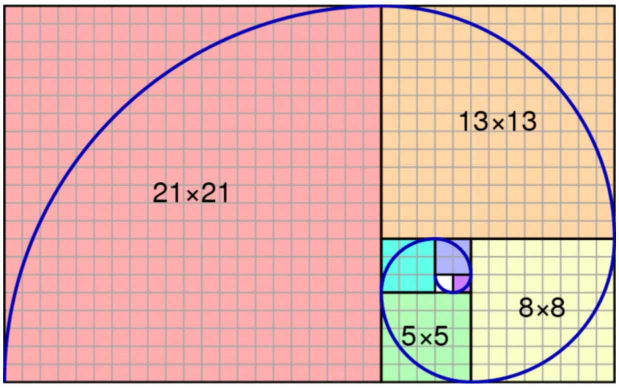

Figure 1. A Fibonacci spiral which approximates the golden spiral, using Fibonacci sequence square sizes up to 34. The spiral is drawn starting from the inner 1 × 1 square and continues outwards to successively larger squares [2].

The inverse of this quanity is

$$\frac{1}{x} = \lim_{n\to\infty}\frac{F(n)}{F(n+1)}$$

This expression means that take the limit as n approaches infinity of a Fibonacci term and the next term. Now, take the limit as n approaches infinity of a Fibonacci term and the next term.

Therefore:

$$\frac{1}{x} = \lim_{n\to\infty}\frac{F(n)}{F(n+1)} = \lim_{n\to\infty}\frac{F(n-1)}{F(n)}$$

Apply the limit as $n \to \infty$ to both sides:

$$\lim_{n\to\infty}\frac{F(n-1)}{F(n)} = \lim_{n\to\infty}\left[1+\frac{F(n-1)}{F(n)}\right] = 1+\lim_{n\to\infty}\frac{F(n-1)}{F(n)}$$

The right-hand side is what defined as x and the right-hand side is to be $\frac{1}{x}$ therefore:

$$x = 1+\frac{1}{x}$$

Multiplying by x and set equal to 0 (zero) gives:

$$x^2 - x - 1 = 0$$

Solve quadratic equation then it will give us

$$x = \frac{1\pm\sqrt{5}}{2}$$

where $x = 1.618033\cdots$, $x = -0.618033\cdots$

The positive root is equal to φ, therefore we can say:

$$x = \lim_{n\to\infty}\frac{F(n+1)}{F(n)} = \varphi$$

3. Some Applications

3.1. Application 1: The Sunflower

The mysterious Golden Ratio can be seen all over the place in nature. They are, found across the cosmos. From the human ear's curve to the human bone structure, to the growth pattern of leaves and flowers, to the spirals of a nautilus seashell or the spirals of galaxies, spirals can be seen everywhere [4]. The Golden Ratio's generality has led to recognized as the geometrical blueprint for life. It was even named the "key to the physics of the cosmos" by a great Greek philosopher.

Artists, philosophers, mathematicians, and architects exploited this ratio in their work after discovering it in nature. They passed down this secret formula for creating the most harmonious interaction between aesthetic components across the generations. There's something about it that appeals to us. It's a dynamic shape, in which the brain attempts but fails to map out this rectangle using basic numbers. This is due to the fact that it is based on an irrational number.

Notice the apparent spirals in the florets extending out from the center to the edge in the shot of a sunflower in **Figure 2**. These spirals appear to revolve in both counter and counter-clockwise directions. There are 34 clockwise spirals and 21 counterclockwise spirals. The numbers 21 and 34 are remarkable because they are golden ratio when we divide 34 and 21.

If you've ever wondered how Fibonacci flowers manage to make such perfect floret arrangements, the explanation is auxin, a plant hormone. Auxin, a growth hormone, aids in the formation and growth of leaves, flowers, stems, and other plant components. Auxin travels through the plant and interacts with other proteins. Because the hormone circulates in a spiral in the plant, it causes the plant to develop spirally, resulting in "Fibonacci spirals" in sunflowers. Sunflowers can pack a lot of seeds onto their heads because of the Fibonacci pattern. The head pushes the seeds to the perimeter as the individual seed expands, making place for additional seeds. Thus, by following a Fibonacci pattern, sunflowers ensure that their seed growth can continue indefinitely [6].

Figure 2. The head of a sunflower [5].

3.2. Application 2: The DNA Molecule

In addition, recently scientists have concluded that the height of one unit of the DNA helix shows the Golden ratio. The DNA molecule measures 34 angstroms long by 21 angstroms wide for each full cycle of its double helix spiral. These numbers, 34 and 21, are numbers in the Fibonacci series, and their ratio 1.6190. 34 and 21, are numbers in the Fibonacci series and their ratio, 1.6190476 closely approximates phi, 1.6180339.

The Golden ratio or the Fibonacci form the bridge from the number patterns found in the double helix of DNA and the a cross-sectional of the DNA double helix shows the two pentagons called decagon shown in **Figure 3** and **Figure 4**. A decagon is formed by a two pentagons, with one rotated by 36 degree from the other. The crystallographic structure of DNA, stress patterns in nanomaterials, the stability of atomic nuclides and the periodicity of atomic matter depend on the Golden ratio [9]. The golden ratio, or the mathematical ratio of the Phi, has been discovered to be the only mathematical arrangement that can reproduce itself indefinitely without variation. One theory is that this golden spiral reflects a mathematically imprinted instructional pattern in our DNA that defines who we are, what we are, how we appear, and how we live, as revealed by Dr. Leonard Horowitz. However, this application is still on developing.

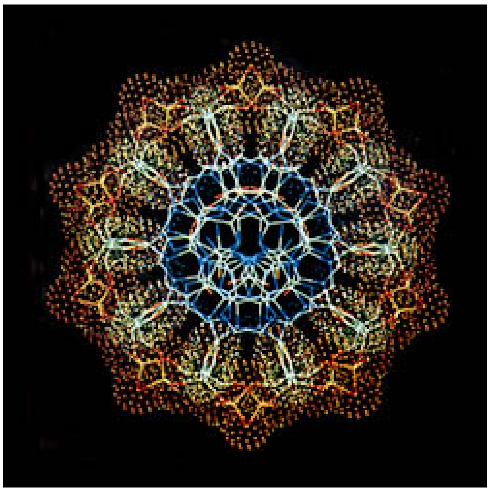

Figure 3. The cross-sectional view from the top of the DNA [7].

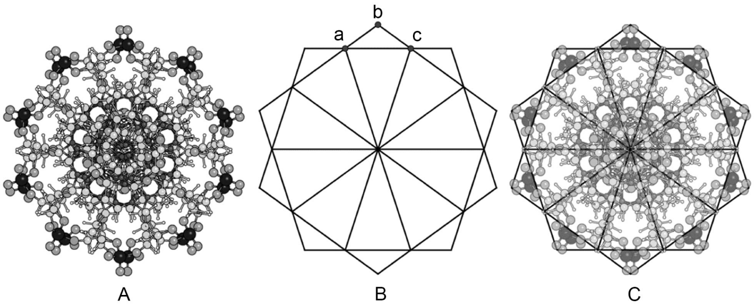

Figure 4. The cross-section of DNA and Decagon [8].

3.3. Application 3: The Spiral Phyllotaxy

The phyllotaxis is arrangement of leaves in some plants. This pattern follows a number of subtle mathematical relationships, which are the Golden ratio or Fibonacci numbers. For instance, aloe polyphylla, sunflower, follows phyllotaxis pattern. Several authors described the phyllotaxis arrangement of leaves on plant stems in such a way that the overall vertical configuration of leaves is optimized to receive rainwater, sunshine, and air, according to a golden angle function of Fibonacci numbers. However, this view is not uniformly accepted [10].

Fibonacci spiral patterns were produced artificially by manipulating the stress on inorganic microstructures made of a silver core and a silicon dioxide shell. It was found that an elastically mismatched bi-layer structure may cause stress patterns that give rise to Fibonacci spirals [11]. As the result, each new leaf on a plant stem is positioned at a certain angle to the previous one and that this angle is constant between leaves: usually about 137.5 degrees, which is indicative of a pattern that follows a Fibonacci series (**Figure 5**).

3.4. Application 4: The Black Hole

A black hole is a region of space where gravity is so strong that even light cannot escape. Because stuff has been compressed into a small space, gravity is extremely powerful. This can occur when a star dies. People can't perceive black holes because no light can escape. They are undetectable. Space telescopes equipped with specialized equipment can aid in the discovery of black holes. Special technologies can see how stars in close proximity to black holes behave differently from other stars. Black holes warp space in their vicinity so much that in classical General Relativity, nothing, not even light, can escape. However, when quantum effects are included, black holes can lose energy via a process known as Hawking radiation [9]. The golden ratio is the precise point at which the modified heat of a black hole goes from positive to negative, and it is part of the equation for the lower constraint on black hole entropy. The golden ratio is the relationship between the loop quantum gravity parameter and black hole entropy. In binary matrices, which Heisenberg used to explain quantum physics, the golden ratio and negative one over the golden ratio have the highest likelihood nontrivial eigenvalues. The golden ratio is fundamentally important to quantum physics and black holes (**Figure 6**).

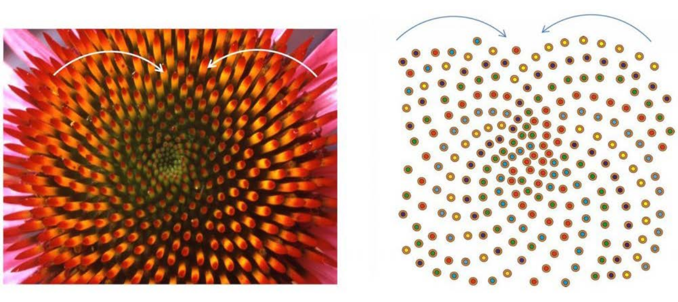

Figure 5. The spiral phyllotaxy patterns having both clockwise and anticlockwise spiral patterns [12].

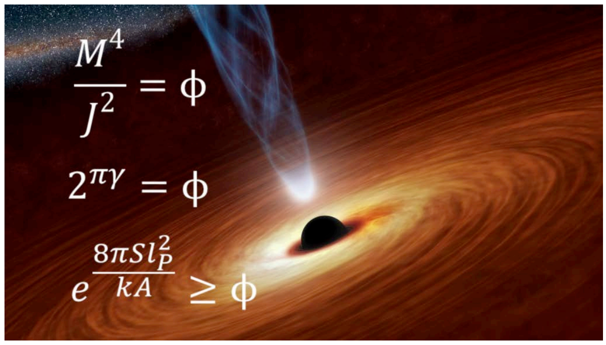

Figure 6. Three formulas that show a relationship between Golden ratio and Black hole [13].

4. Conclusion

The golden ratio has close relationship with Fibonacci sequence. By taking the ratio of Fibonacci number, the value of that ratio gets closer to phi (φ). According to Mario Livio, Some of the greatest mathematical minds of all ages, from Pythagoras and Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa and the Renaissance astronomer Johannes Kepler, to present-day scientific figures, have spent endless hours over this simple ratio and its properties. Biologists, artists, musicians, historians, architects, psychologists, and even mystics have pondered and debated the basis of its ubiquity and appeal. In fact, it is probably fair to say that the Golden Ratio has inspired thinkers of all disciplines like no other number in the history of mathematics [14]. The reason φ and Fibonacci numbers sometime show up (approximately) in nature has to do with constraints of geometry upon the way organisms grow in size. Irrational numbers (those that cannot be expressed as a ratio of integers) are often revealed in this process. Fibonacci numbers, such as five or eight, are common in flower petals, and pine cones develop their seeds outward in spirals of Fibonacci numbers. However, there are just as many plants that do not follow to this principle as there are those that do. The authors would like to explore more real world applications involving the Golden Ratio in the future work.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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