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Research on Chaos of Nonlinear Singular Integral Equation

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Abstract

In this paper, one class of nonlinear singular integral equation is discussed through Lagrange interpolation method. We research the connections between numerical solutions of the equations and chaos in the process of solving by iterative method.

Keywords

Lagrange Interpolation Method, Nonlinear Singular Integral Equation, Iterative Method

1. Introduction

Nonlinear singular integral equation originated in the 1920s. In the middle of last century, some scholars used topological methods and variational methods to study Fredholm type nonlinear integral equation systematically, which was often used by later generations. Although compared with the linear singular integral equation, it is more difficult than we thought to solve the nonlinear singular integral equation. We consider one class of nonlinear singular integral equation in the interval $[-1,1]$ as follows

$$a(t)\phi^2(t)+\frac{b(t)}{\pi}\int_{-1}^{1}\frac{(1-\tau^2)\phi(\tau)}{\tau-t}\mathrm{d}\tau=c(t), \tag{1}$$

through Lagrange interpolation method, we choose $t_1=-1$, $t_2=0$, $t_3=1$ as interpolation knots to discredit Equation (1), and then we can get the corresponding nonlinear equations by calculation [1] [2]. In the process of assigning values to $a(t),b(t),c(t)$ to solve the nonlinear equations, we find that in some cases, the number of solutions by program is less than its actual number, when we seek the solutions through iterative method [3], bifurcation on some point appears, we assume that there exists some connection between nonlinear singular integral equation and chaos phenomenon.

2. Preliminary

2.1. Nonlinear Singular Integral Equation

The general form of nonlinear singular integral equation is as follows

$$a(t)\phi^2(t)+\frac{b(t)}{\pi}\int_{-1}^{1}\frac{\phi(\tau)}{\tau-t}\mathrm{d}\tau+c(t)=0 \tag{2}$$

here L denotes the interval $[-1,1]$, $a(t), b(t), c(t)$ are all given polynomials, in order to eliminate the singularity of integrand at −1 and 1, we add $1-\tau^2$ to numerator, *i.e.* the form (1).

2.2. Lagrange Interpolation Method

Lagrange interpolation method is a fundamental method when we find a function approximation. Its basic idea is transfer the constructional problem of interpolation polynomial

$L_n(x)$ to the construction of 11 interpolation basic functions. Specifically speaking, we can determine a line through two different points in the plane, this is Lagrange linear interpolation; for three points not in a same line, we can get a parabola as interpolation polynomial. For continuous function $y=f(x)$, there are $n+1$ different knots $x_i(x)$, $x=0,1,2,\cdots,n$ in $[a,b]$, we can construct a interpolation polynomial $L_n(x)$ with the degree of no more than n which is through these knots, the general formula of Lagrange interpolation method is [4]

$$L_n(x)=\sum_{i=0}^{n} f(x_i)l_i(x) \tag{3}$$

here

$$l_i(x)=\prod_{j=0,\, j\neq i}^{n}\frac{x-x_j}{x_i-x_j}, i=0,1,2,\cdots,n \tag{4}$$

is called Lagrange interpolation basic function, and it satisfies the eigenfunction $l_i(x_i)$, *i.e.*

$$l_i(x_k)=\begin{cases}1, k=i,\\ 0, k\neq i,\end{cases}(i,k=0,1,2,\cdots,n) \tag{5}$$

here the truncation error is

$$R_n(x)=f(x)-L_n(x)=\frac{f^{n+1}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i), \xi\in(a,b). \tag{6}$$

2.3. Chaos

Actually, the essence is accordant although the definitions of chaos vary a lot. Here we give the definition that Devaney described. At first, we need some terms.

Suppose $f:A\to A$ is a self-mapping of set A. A subset E of A is called invariant set, if $f(E)\subset E$.

Suppose $E \subset A$ is a invariant set of mapping $f: A \to A$, obviously, the orbit of an arbitrary point a of *E* locates in *E*.

The orbit of mapping $f: A \to A$ on its invariant set *E* is topological transitivity, if for two arbitrary points *a* and *b* and their own arbitrary neighbourhoods V_a and V_b, there exists a sufficiently great number *n* such that

$$f^n(V_a) \cap V_b \neq \varnothing \tag{7}$$

Suppose $E \subset A$ is an invariant set of mapping $f: A \to A$ The orbit of mapping $f: A \to A$ on *E* is extremely sensitive to initial values, if there exists a positive number d_0 and a corresponding point $b \in E$ such that

$$d(a,b) < \delta \tag{8}$$

and

$$\overline{\lim_{n\to\infty}} d\left(f^n(a), f^n(b)\right) \geq d_0 \tag{9}$$

for $\forall a \in E$ and $\forall \delta > 0$.

Now we give the definition of chaos that Devaney described.

Suppose $f: A \to A$ is a continuous mapping from *A* to *A*, here *A* is an area in n-dimensional Euclidean space, and $E \subset A$ is an invariant set of *f*. We call the iterative orbit of *f* on *E* is chaos, if the mapping $f: A \to A$ satisfies the three properties as follows on *E*:

1) the periodic point of *f* on *E* is dense;

2) the fall orbit of *f* is topological transitivity on *E*;

3) the iterative orbit of *f* on *E* is sensitive to initial values.

3. Numerical Evaluation

In the interval $[-1,1]$, we discuss the nonlinear singular integral equation

$$a(t)\phi^2(t) + \frac{b(t)}{\pi}\int_{-1}^{1}\frac{(1-\tau^2)\phi(\tau)}{\tau - t}\mathrm{d}\tau = c(t) \tag{10}$$

through Lagrange interpolation method, we regard $t_1 = -1$, $t_2 = 0$, $t_3 = 1$ as interpolation knots to discredit Equation (1), and then by calculation we can get functions as follows

$$\begin{cases} a(t)x^2 + \dfrac{b(t)}{\pi}\left(\dfrac{2}{3}x + \dfrac{4}{3}y\right) = c(t) \\ a(t)y^2 + \dfrac{b(t)}{\pi}\left(-\dfrac{2}{3}x + \dfrac{2}{3}z\right) = c(t) \\ a(t)z^2 + \dfrac{b(t)}{\pi}\left(-\dfrac{4}{3}y - \dfrac{2}{3}z\right) = c(t) \end{cases} \tag{11}$$

We assign values to $a(t) = 1$, $b(t) = 300\pi$, $c(t) = x^2 + 2x + 3$, and then we can get ternary quadratic nonlinear equations as follows

$$\begin{cases} x^2 + 200x + 400y - x^2 - 2x - 3 = 0 \\ y^2 - 200x + 200z - x^2 - 2x - 3 = 0 \\ z^2 - 400y - 200z - x^2 - 2x - 3 = 0 \end{cases} \tag{12}$$

its simplification is

$$\begin{cases} 198x + 400y - 3 = 0 \\ y^2 - 202x + 200z - x^3 - 3 = 0 \\ z^2 - 400y - 200z - x^2 - 2x - 3 = 0 \end{cases} \quad (13)$$

We can get one solution of the equations through program

$$\begin{cases} x = 90.0448 - 0.0000i \\ y = -44.5647 + 0.0000i \\ z = 1.2157e + 002 - 4.3520e - 015i \end{cases} \quad (14)$$

We use the iterated function as follows

$$\begin{cases} x_{k+1} = -\frac{1}{198}(400y_k - 3) \\ y_{k+1} = \frac{1}{400}\left(z_k^2 - 200z_k - x_k^2 - 2x_k - 3\right) \\ z_{k+1} = -\frac{1}{200}\left(y_k^2 - 202x_k - x_k^2 - 3\right) \end{cases} \quad (15)$$

We assign initial values to $x = 0.2, y = 0.2, z = 0.2$, through the program, we can get a image as follows (**Figure 1**).

We can see the bifurcation point is exactly the solution (3); and through its dynamic image, we can get a revelation to research that whether chaos must occurs in the process of solving nonlinear singular integral equation.

We give initial values of x, y, z a small change, assigning $x = 0.20001, y = 0.2, z = 0.2$ (the yellow image) and another case $x = 0, y = 0, z = 0$ (the red image), we adopt the same approach and draw the three cases on one picture, the result is as follows [5] (**Figure 2**).

We can see although the images almost overlap when the initial values of x, y, z have a small change, the three images differ a lot. Design a program called yilaixing.m, taking the case of $x = 0.2, y = 0.2, z = 0.2$ and the case $x = 0.20001, y = 0.2, z = 0.2$ into consideration, to prove the initial value sensitivity of chaos, we research the differences between the two images. The result is as follows [2] (**Figure 3**).

We can observe that for the two cases above, differences of x, y, z are oscillating as the number of iterating is increased. From these results, it is clear that the shape of images is similar, but the difference between corresponding points is great.

Analogously, we assign values to $a(t) = 1, b(t) = 300, c(t) = x^2 + 2x + 3$, and then through (2), we can get ternary quadratic nonlinear equations as follows

$$\begin{cases} \pi x^2 + 200x + 400y = \pi\left(x^2 + 2x + 3\right) \\ \pi y^2 - 200x + 200z = \pi\left(x^2 + 2x + 3\right) \\ \pi z^2 - 400y - 200z = \pi\left(x^2 + 2x + 3\right) \end{cases}$$

its simplification is

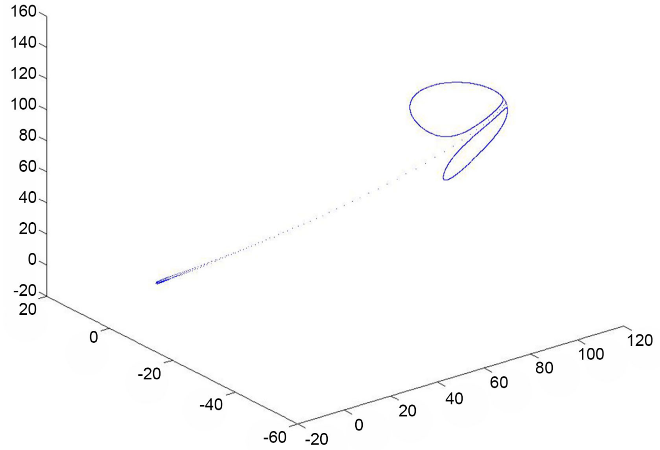

Figure 1. The image obtained by assigning initial values to *x*, *y*, and *z*.

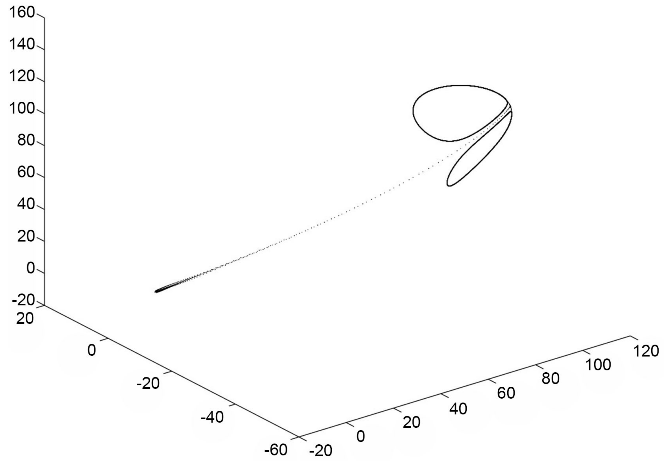

Figure 2. The image of *x*, *y* and *z* given initial values.

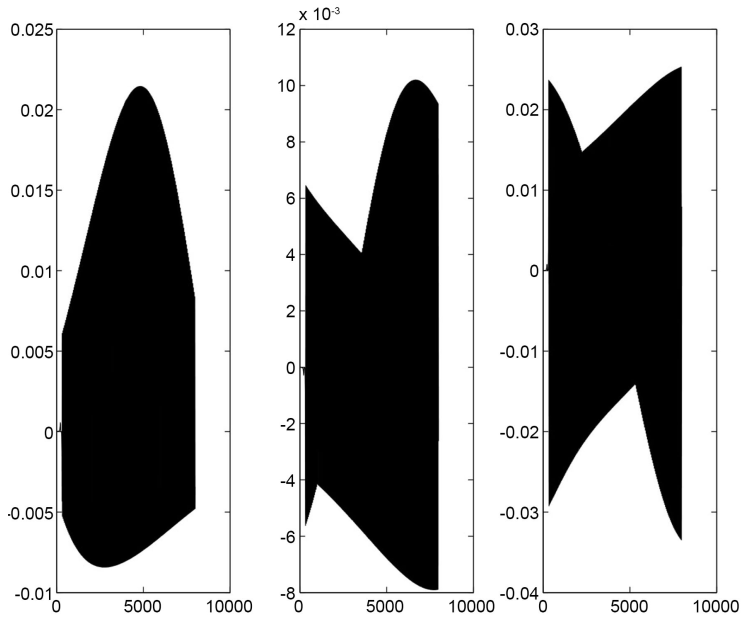

Figure 3. The initial value sensitivity of *x*, *y*, *z*.

$$\begin{cases} 200x+400y=2\pi x+3\pi \\ \pi y^2-200x+200z=\pi\left(x^2+2x+3\right) \\ \pi z^2-400y-200z=\pi\left(x^2+2x+3\right) \end{cases}$$

In this case, the computer cannot solve the nonlinear functions. We adopt iteration similarly with (4)

$$\begin{cases} x_{k+1}=\dfrac{3\pi-400y_k}{200-2\pi} \\ y_{k+1}=\pi\dfrac{z_k^2-\dfrac{200}{\pi}z_k-x_k^2-2x_k-3}{400} \\ z_{k+1}=-\pi\dfrac{y_k^2-\dfrac{200}{\pi}x_k-x_k^2-2x_k-3}{400} \end{cases}$$

assign values to $x=0.2, y=0.2, z=0.2$, through the program we can draw a image about the process of iteration as follows (**Figure 4**).

From the image (**Figure 5**), we can see that in the case of changing the value of $b(t)$ only, the iterative image becomes very complex, and it has many bifurcation points.

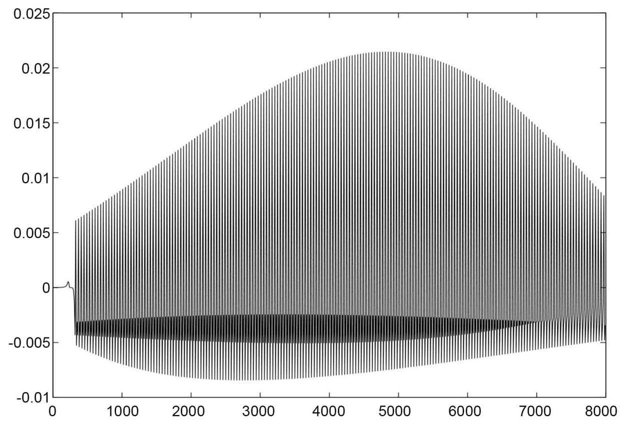

Figure 4. The initial value sensitivity of *x*.

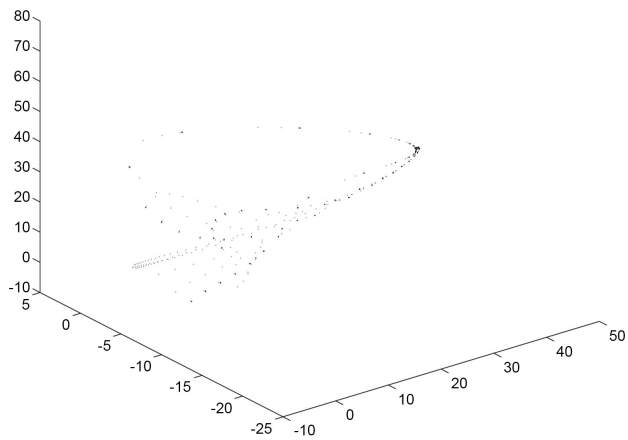

Figure 5. The image after iteration of *x*, *y*, *z*.

We do not have enough evidence to prove that the chaos should follow the process of solving nonlinear singular integral equation so far. Furthermore, we consider whether the similar cases will occur if we increase the number of interpolation knots.

Adopting the Lagrange interpolation method as above, we choose the interpolation knots $t_1 = -1$, $t_2 = -0.8$, $t_3 = -0.6$, $t_4 = -0.4$, $t_5 = -0.2$, $t_6 = 0$, $t_7 = 0.2$, $t_8 = 0.4$, $t_9 = 0.6$, $t_{10} = 0.8$, $t_{11} = 1$, to discredit the Equation (1) (the corresponding Lagrange interpolation basic function is clear, and will not be described here), by calculating, we can get a coefficient matrix as follows to substitute the integral $\int_{-1}^{1} \frac{(1-\tau^2)\phi(\tau)}{\tau - t} \mathrm{d}\tau$

$$A = \begin{bmatrix} 0.1073 & 0.6391 & -0.2593 & 1.2739 & -1.0444 & 1.4275 & -0.6963 & 0.5460 & -0.0648 & 0.0710 & 0 \\ -0.0639 & -0.3692 & 2.3574 & -1.9240 & 3.0814 & -2.2421 & 1.8489 & -0.6413 & 0.3368 & 0 & 0.0710 \\ -0.0058 & -0.5239 & -0.4788 & 2.4384 & -0.8785 & 1.4110 & -0.4393 & 0.4877 & 0 & 0.0748 & 0.0014 \\ -0.0106 & -0.1603 & -0.9144 & -0.5470 & 2.5102 & -0.4438 & 0.8367 & 0 & 0.1829 & 0.0534 & 0.0046 \\ -0.0050 & -0.1467 & -0.1883 & -1.4344 & -0.2247 & 2.1247 & 0 & 0.4781 & 0.0941 & 0.0880 & 0.0033 \\ -0.0057 & -0.0890 & -0.2520 & -0.2113 & -1.7706 & 0 & 1.7706 & 0.2113 & 0.2520 & 0.0890 & 0.0057 \\ 0.0033 & -0.0880 & -0.0941 & -0.4781 & 0 & -2.1247 & 0.2247 & 1.4344 & 0.1883 & 0.1467 & 0.0050 \\ -0.0046 & -0.0534 & -0.1829 & 0 & -0.8367 & 0.4438 & -2.5102 & 0.5470 & 0.9144 & 0.1603 & 0.0160 \\ -0.0014 & -0.0748 & 0 & -0.4877 & 0.4393 & -1.4110 & 0.8785 & -2.4384 & 0.4788 & 0.5239 & 0.0058 \\ -0.0071 & 0 & -0.3368 & 0.6413 & -1.8489 & 2.2421 & -3.0814 & 1.9240 & -2.3574 & 0.3692 & 0.0639 \\ 0 & -0.0710 & 0.0648 & -0.5460 & 0.6963 & -1.4275 & 1.0444 & -1.2739 & 0.2593 & -0.6391 & -0.1073 \end{bmatrix}$$

We can observe that $A(i, j) = -A(j, i)$ from the marix *A*. Assign values to $a(t) = 1, b(t) = 300\pi, c(t) = x^2 + 2x + 3$ as the first case. Then, we get functions as follows

$$\begin{cases} a^2 + 32.1a + 191.7b - 77.7c + 382.2d - 313.2e + 428.4f - 208.8g + 163.8h - 19.5i + 21.3j - a^2 - 2a - 3 = 0 \\ b^2 - 19.2a - 110.8b + 707.1c - 577.2d + 924.3e - 672.6f + 554.7g - 192.3h + 101.1i + 2.13k - a^2 - 2a - 3 = 0 \\ c^2 - 1.74a - 157.2b - 143.6c + 731.4d - 263.4e + 423.3f - 131.7g + 146.4h + 22.5j + 0.42k - a^2 - 2a - 3 = 0 \\ d^2 - 3.18a - 48b - 274.2c - 164.1d + 753e - 133.2f - 131.2f + 251.1g + 54.9i + 15.9j + 1.38k - a^2 - 2a - 3 = 0 \\ e^2 - 1.5a - 44.1b - 56.4c - 430.2d - 67.4e + 637.5f + 143.4h + 28.2i + 26.7j + 0.99k - a^2 - 2a - 3 = 0 \\ f^2 - 1.71a - 26.7b - 75.6c - 63.3d - 531e + 531g + 63.3h + 75.6i + 26.7j + 1.71k - a^2 - 2a - 3 = 0 \\ g^2 - 0.99a - 26.7b - 28.2c - 143.4d - 637.5f + 67.4g + 430.2h + 54.6i + 44.1j + 1.5k - a^2 - 2a - 3 = 0 \\ h^2 - 1.38a - 15.9b - 54.9c - 251.1e + 133.2f - 753g + 164.1h + 274.2i + 48j + 3.18k - a^2 - 2a - 3 = 0 \\ i^2 - 0.42a - 22.5b - 146.4d + 131.7e - 423.3f + 263.4g - 731.4h + 143.6i + 157.2j + 1.74k - a^2 - 2a - 3 = 0 \\ j^2 - 2.13a - 101.1c + 192.3d - 554.7e + 672.6f - 924.3g + 577.2h - 707.1i + 110.8j + 19.2k - a^2 - 2a - 3 = 0 \\ k^2 - 21.3b + 19.5c - 163.8d + 208.8e - 428.4f + 313.2g - 382.2h + 77.7i - 191.7j - 32.1k - a^2 - 2a - 3 = 0 \end{cases}$$

The original idea is to detail calculation progress by increasing the number of interpolation knots, but the increase of computation beyond our imagination, at the same time, we cannot find the solution of the equations above by program. It is a pity that the four iterations we adopt are all divergent, what's more, the speed of divergence is very quick, and they have a strong dependence on initial value, for us, it is difficult to calculate further.

4. Conclusion

We use Lagrange interpolation method to solve a class of nonlinear singular integral equation $a(t)\phi^2(t)+\frac{b(t)}{\pi}\int_{-1}^{1}\frac{(1-\tau^2)\phi(\tau)}{\tau-t}\mathrm{d}\tau=c(t)$. In the process of assigning values to $a(t),b(t),c(t)$, some interesting phenomena occur as we adopt different iterations, it promotes us to research the connection between the solutions of nonlinear singular integral equation and the chaos. When giving the initial values a small change, although the shape of images is similar, the difference between corresponding points is great, which explain the sensitivity of iteration to initial values. Under the condition of assurance of unchangeable iteration, initial values and the values of $a(t),c(t)$, we alter the value of $b(t)$ only, the change of images beyond our imagination. Then we want to detail the process of solving above through increasing the number of interpolation knots, but it is difficult to research further because of the increase of computation and the blindness of choice of iteration, which proves that it is more complex than we imagine to solve nonlinear singular integral equation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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