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Existence and Concentration of Sign-Changing Solutions of Quasilinear Choquard Equation

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Abstract

In this paper, we study the following quasilinear equation of choquard type:

$$\begin{cases} -div\left(A(x,u)|\nabla u|^{p-2}\nabla u\right)+\frac{1}{p}A_t(x,u)|\nabla u|^p+V(\varepsilon x)|u|^{p-2}u \\ =\left(I_\alpha*|u|^q\right)|u|^{q-2}u,\ x\in\mathbb{R}^N \\ u(x)\to 0,\quad |x|\to\infty \end{cases}$$

where $A(x,t)$ is given real functions on $\mathbb{R}^N\times\mathbb{R}$ and $A_t(x,t)=\frac{\partial}{\partial t}A(x,t)$ with $N\geq 3$, $1<p<N$, $\max\{N-2p,1\}<\alpha<N$, $\frac{p(N+\alpha)}{2N}\leq q<p_\alpha^*=\frac{p(N+\alpha)}{2(N-p)}$, and $\varepsilon>0$ is a small parameter, I_α is the Riesz potential. We establish for small ε the existence of a sequence of sign-changing solutions concentrating near a given local minimum point of the bounded potential function V by using the method of invariant sets of descending flow, perturbation method and truncation technique.

Keywords

Quasilinear Choquard Equation, The Method of Invariant Sets of Descending Flow, Truncation, Sign-Changing Solutions

1. Introduction and Main Results

In this paper, we consider the following quasilinear equation of choquard type

$$\begin{cases} -div\left(A(x,u)|\nabla u|^{p-2}\nabla u\right)+\frac{1}{p}A_t(x,u)|\nabla u|^p+V(\varepsilon x)|u|^{p-2}u \\ =\left(I_\alpha*|u|^q\right)|u|^{q-2}u,\ x\in\mathbb{R}^N \\ u(x)\to 0,\quad |x|\to\infty \end{cases} \tag{1}$$

where $A(x,t)$ is given real functions on $\mathbb{R}^N \times \mathbb{R}$ and $A_t(x,t) = \frac{\partial}{\partial t} A(x,t)$ with $N \geq 3$, $1 < p < N$, $\max\{N-2p,1\} < \alpha < N$, $\frac{p(N+\alpha)}{2N} \leq q < p_\alpha^* = \frac{p(N+\alpha)}{2(N-p)}$, and $\varepsilon > 0$ is a small parameter, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}} := \frac{A_\alpha}{|x|^{N-\alpha}}$$

and Γ is the Gamma function.

In the mathematical literature, very few results are known about Equation (1). Since the classical variational approach fails if $A_t(x,t) \not\equiv 0$. In reference [1], the author considered the following quasilinear equation

$$-div\left(A(x,u)|\nabla u|^{p-2}\nabla u\right) + \frac{1}{p}A_t(x,u)|\nabla u|^p + |u|^{p-2}u = g(x,u), \quad x \in \mathbb{R}^N, \quad (2)$$

In fact, the natural functional associated to (2) is

$$\mathcal{J}(u) = \frac{1}{p}\int_{\mathbb{R}^N} A(x,u)|\nabla u|^p \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N}|u|^p \mathrm{d}x - \int_{\mathbb{R}^N} G(x,u)\mathrm{d}x,$$

which is not defined in $W^{1,p}(\mathbb{R}^N)$ for a general coefficient $A(x,t)$ in the principal part. Moreover, even if $A(x,t)$ is smooth strictly positive bounded function, the corresponding energy functional is well defined in $W^{1,p}(\mathbb{R}^N)$, if $A_t(x,t) \not\equiv 0$ it is Gâteaux differentiable only along directions of $W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ [1]. More recently, if Ω is a bounded subset of $\mathbb{R}^N$ a different approach has been developed which exploits the interaction between two different norm on $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ [2] [3] [4]. So, use the interaction between the norm $\|\cdot\|_X$ and the standard one on $W^{1,p}(\mathbb{R}^N)$, if $G(x,t)$ has a subcritical growth, we state that $\mathcal{J}$ satisfies a weaken version of the Cerami's variant of the Palais-Smale condition in X_r. We note that, in general, $\mathcal{J}$ cannot verify the standard Palais-Smale condition, or its Cerami's variant, as Palais-Smale sequences may converge in the $W^{1,p}(\mathbb{R}^N)$-norm but be unbounded in $L^\infty(\mathbb{R}^N)$ [4]. For this reason, we know that there is a bounded radial solution of Equation (2) under certain assumptions.

For the Choquard equation

$$-\Delta u + V(x)u = \left(I_\alpha * |u|^p\right)|u|^{p-2}u, \quad x \in \mathbb{R}^3. \quad (3)$$

where $p \in \left(\frac{2N-\alpha}{N}, \frac{2N-\alpha}{N-2}\right)$. When the potential V is a positive constant, Lieb [5] obtained the existence and uniqueness of positive radial ground states for (3), Lions [6] established the existence of infinitely many radial solutions, Ma and Zhao [7] studied the radial symmetry and uniqueness of positive ground states for (3) in higher dimension space via the method of moving planes. For more related results, we refer to [2] [8] [9] [10] [11] [12] and references therein.

In this article, we consider the existence of sign-changing solutions for the quasilinear Choquard Equation (1) by using the method of invariant sets of descending flow and the perturbation method. Byeon-Wang type penalization method [13] can be used to deal with multiple localized nodal solutions for semiclassical Schrödinger equations. Additional coercive term [14] can be used to make the perturbed functional has necessary compactness properties in changed space.

From now on, Let $A:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ be such that

(A_0) $A(x,t)$ is a C^1 Carathéodory function, *i.e.*
$A(\cdot,t):x\in\mathbb{R}^N\mapsto A(x,t)\in\mathbb{R}$ is measurable for all $t\in\mathbb{R}$,
$A(x,\cdot):t\in\mathbb{R}\mapsto A(x,t)\in\mathbb{R}$ is C^1 for a.e. $x\in\mathbb{R}^N$;

(A_1) $A(x,t)$ and $A_t(x,t)$ are essentially bounded if t is bounded, *i.e.*

$$\sup_{|t|\le r}|A(\cdot,t)|\in L^\infty(\mathbb{R}^N),\ \sup_{|t|\le r}|A_t(\cdot,t)|\in L^\infty(\mathbb{R}^N)\quad \text{for any } r>0;$$

(A_2) Exists a constant $c_1,c_2>0$ such that

$$c_1\le A(x,t)\le c_2\quad \text{a.e. } x\in\mathbb{R}^N;$$

(A_3) Exist constants $R\ge 1$ and $\alpha_1>0$ such that

$$pA(x,t)+A_t(x,t)t\ge\alpha_1A(x,t)\quad \text{a.e. in } \mathbb{R}^N, \text{ if } |t|\ge R;$$

(A_4) Exist constants $\mu>p$ and $\alpha_2>0$ such that

$$(\mu-p)A(x,t)-A_t(x,t)t\ge\alpha_2A(x,t)\quad \text{a.e. in } \mathbb{R}^N, \text{ for all } t\in\mathbb{R}.$$

The potential function V satisfies the following assumptions:

(V_1) $V\in C^1(\mathbb{R}^N,\mathbb{R})$ and there exist constants $b>a>0$ such that

$$a\le V(x)\le b,\quad x\in\mathbb{R}^N;$$

(V_2) There exists a bounded domain $\mathcal{M}\subset\mathbb{R}^N$ with smooth boundary $\partial\mathcal{M}$ such that

$$\left\langle\overrightarrow{n(x)},\nabla V(x)\right\rangle>0,\quad x\in\partial\mathcal{M}.$$

where $\overrightarrow{n(x)}$ is the outer normal of $\partial\mathcal{M}$ at x. according to condition (V_2) that

$$\mathcal{A}=\{x\in\mathcal{M}\mid\nabla V(x)=0\}\ne\varnothing.\quad \text{(see [15])}$$

and $\mathcal{A}$ is a closed subset of $\mathcal{M}$, without losing generality, we assume that $0\in\mathcal{A}$.

For any set $B\in\mathbb{R}^N$ and any $\delta>0$, we denote

$$B^\delta=\left\{x\in\mathbb{R}^N\mid dist(x,B):=\inf_{y\in B}|x-y|<\delta\right\},$$

$$B_\delta=\left\{x\in\mathbb{R}^N\mid\delta x\in B\right\}.$$

The main results of this paper are as follows:

Theorem 1.1 *Assume that* $A(x,t)$ *satisfy conditions* (A_0)-(A_4), *the potential function* V *satisfies* (V_1) *and* (V_2), *for any positive integer k, there exists* $\varepsilon_k>0$ *such that if* $0<\varepsilon<\varepsilon_k$, *the problem* (1) *has at least k pairs of sign-changing solu-*

tions $\pm u_{j,\varepsilon}$, $j=1,2,\cdots,k$. *In addition, for any* $\delta>0$ *there exist* $\mu>0, c=c_k>0$ *and* $\varepsilon_k(\delta)>0$ *such that if* $0<\varepsilon<\varepsilon_k(\delta)$, *then it holds that*

$$\left|u_{j,\varepsilon}\right| \le c\exp\left\{-\mu\, dist\left(x,\left(\mathcal{A}^{\delta}\right)_{\varepsilon}\right)\right\}, \quad j=1,2,\cdots,k,\ x\in\mathbb{R}^N.$$

2. Preliminaries

We note that Equation (1) corresponding energy functional is

$$I_\varepsilon(u)=\frac{1}{p}\int_{\mathbb{R}^N} A(x,u)|\nabla u|^p\,\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon x)|u|^p\,\mathrm{d}x-\frac{1}{2q}\int_{\mathbb{R}^N}\left(I_\alpha *|u|^q\right)|u|^q\,\mathrm{d}x \quad (4)$$

We use the penalization method due to Byeon and Wang [13]. Let $\zeta\in C^\infty$ be a cut-off function, $\zeta(t)=0$ for $t\le 0$; $\zeta(t)=1$ for $t\ge 1$; $0\le\zeta'(t)\le 2$ and $0\le\zeta(t)\le 1$. Define

$$\chi_\varepsilon(x)=\begin{cases}0, & x\in\mathcal{M}_\varepsilon\\ \varepsilon^{-p}\zeta\left(dist\left(x,\mathcal{M}_\varepsilon\right)\right), & x\notin\mathcal{M}_\varepsilon\end{cases}$$

In order to overcome the lack of compactness condition, we set the workspace as $X_\varepsilon=W^{1,p}\left(\mathbb{R}^N\right)\cap L_\varepsilon^m\left(\mathbb{R}^N\right)$, where $L_\varepsilon^m\left(\mathbb{R}^N\right)$ is a weighted space defined as

$$L_\varepsilon^m\left(\mathbb{R}^N\right)=\left\{u\in L^m\left(\mathbb{R}^N\right)\mid \int_{\mathbb{R}^N}\exp\left\{(m-p)\,dist(\varepsilon x,\mathcal{M})\right\}|u|^m\,\mathrm{d}x<+\infty\right\},$$

The corresponding norm is defined as

$$\|u\|_{L_\varepsilon^m\left(\mathbb{R}^N\right)}=\left(\int_{\mathbb{R}^N}\exp\left\{(m-p)\,dist(\varepsilon x,\mathcal{M})\right\}|u|^m\,\mathrm{d}x\right)^{\frac{1}{m}}.$$

which $p<m<\min\{2,q\}$ if $1<p<2$; $p<m<q$ if $p\ge 2$.

Define

$$\|u\|_{X_\varepsilon}=\|u\|_{W^{1,p}\left(\mathbb{R}^N\right)}+\|u\|_{L_\varepsilon^m\left(\mathbb{R}^N\right)}.$$

Meanwhile, We add the forced disturbance term such that I_ε has the necessary compactness property on X_ε, introduce some auxiliary functions. Let $\xi\in C^\infty(\mathbb{R},[0,1])$ be a smooth, even function, such that $\xi(t)=1$ if $|t|\le 1$, $\xi(t)=0$ if $|t|\ge 2$, and ξ is decreasing in $[1,2]$. For $\varepsilon\in(0,1]$, $x\in\mathbb{R}^N$, $t\in\mathbb{R}$, we define

$$b_\varepsilon(x,t)=\xi\left(\varepsilon\exp\left\{dist(\varepsilon x,\mathcal{M})\right\}t\right),\ m_\varepsilon(x,t)=\int_0^t b_\varepsilon(x,\tau)\,\mathrm{d}\tau,$$

$$k_\varepsilon(x,t)=\left(\frac{t}{m_\varepsilon(x,t)}\right)^{m-p}|t|^{p-2}t,\ K_\varepsilon(x,t)=\int_0^t k_\varepsilon(x,\tau)\,\mathrm{d}\tau.$$

Now, we define the perturbation functional:

$$\begin{aligned}\Gamma_\varepsilon(u)=&\frac{1}{p}\int_{\mathbb{R}^N}A(x,u)|\nabla u|^p\,\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^N}E(\varepsilon x)|u|^p\,\mathrm{d}x+\sigma\int_{\mathbb{R}^N}K_\varepsilon(x,u)\mathrm{d}x\\&+\frac{1}{p\beta}\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^\beta-\frac{1}{2q}\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)|u|^q\,\mathrm{d}x,\quad u\in X_\varepsilon\end{aligned} \quad (5)$$

where $p<p\beta<q$, $E(x)=V(x)-\sigma$, $\sigma>0$ sufficiently small, E satisfies the assumptions (V_1) and (V_2).

Note that for any $v \in X_\varepsilon$

$$\begin{aligned}\left\langle \Gamma'_\varepsilon(u), v\right\rangle &= \int_{\mathbb{R}^N} A(x,u)|\nabla u|^{p-2}\nabla u \nabla v \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} A_t(x,u) v |\nabla u|^p \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} E(\varepsilon x)|u|^{p-2} u v \mathrm{d}x + \sigma \int_{\mathbb{R}^N} k_\varepsilon(x,u) v \mathrm{d}x \\ &\quad + \left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \mathrm{d}x - 1\right)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^{p-2} u v \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} \left(I_\alpha * |u|^q\right)|u|^{q-2} u v \mathrm{d}x\end{aligned} \tag{6}$$

We will use the abstract critical point theorem to prove the existence of the infinitely variable sign solution of Equation (1). However, when verifying the conditions of the theorem, we encounter essential difficulties, that is, we can't guarantee that the positive cone and the negative cone are flow-invariant, so we follow the idea of literature [14] [16] [17] and use truncation method to truncate the nonlocal term. First, we define the following auxiliary functions:

$$b_\lambda(t) = \xi(\lambda t),\ m_\lambda(t) = \int_0^t b_\lambda(\tau)\mathrm{d}\tau,$$

$$g_\lambda(t) = \frac{m_\lambda(t)}{t},\ h_\lambda(t) = g_\lambda(t) + b_\lambda(t).$$

Define the perturbation functional:

$$\begin{aligned}\Gamma_{\varepsilon,\lambda}(u) &= \frac{1}{p}\int_{\mathbb{R}^N} A(x,u)|\nabla u|^p \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} E(\varepsilon x)|u|^p \mathrm{d}x + \sigma\int_{\mathbb{R}^N} K_\varepsilon(x,u)\mathrm{d}x \\ &\quad + \frac{1}{p\beta}\left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \mathrm{d}x - 1\right)_+^{\beta} - \frac{1}{2q} g_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\psi(u), \quad u \in X_\varepsilon\end{aligned} \tag{7}$$

where $\psi(u) = \int_{\mathbb{R}^N}\left(I_\alpha * |u|^q\right)|u|^q \mathrm{d}x$.

By the definition of b_λ, g_λ, it is easy to prove that $g_\lambda(t)$ satisfies the following properties:

Lemma 2.1 [15] *For* $t > 0, 0 < \lambda < 1$, *it holds*

(1) $g_\lambda(t) = 1, g'_\lambda(t) = 0$ if $0 < t < \dfrac{1}{\lambda}$;

(2) $g'_\lambda(t)t + g_\lambda(t) = b_\lambda(t)$;

(3) $g'_\lambda(t)t + g_\lambda(t) \le \dfrac{c_\lambda}{t}, b_\lambda(t)t \le g_\lambda(t)t \le c_\lambda$, where $c_\lambda = \dfrac{\int_0^\infty \xi(\tau)\mathrm{d}\tau}{\lambda}$.

$\forall v \in X_\varepsilon$, since $g'_\lambda(t)t + g_\lambda(t) = b_\lambda(t)$, we have

$$\begin{aligned}\left\langle \Gamma'_{\varepsilon,\lambda}(u), v\right\rangle &= \int_{\mathbb{R}^N} A(x,u)|\nabla u|^{p-2}\nabla u \nabla v \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} A_t(x,u) v |\nabla u|^p \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} E(\varepsilon x)|u|^{p-2} u v \mathrm{d}x + \sigma \int_{\mathbb{R}^N} k_\varepsilon(x,u) v \mathrm{d}x \\ &\quad + \left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \mathrm{d}x - 1\right)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^{p-2} u v \mathrm{d}x \\ &\quad - \frac{1}{2} h_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\int_{\mathbb{R}^N} \left(I_\alpha * |u|^q\right)|u|^{q-2} u v \mathrm{d}x.\end{aligned} \tag{8}$$

According to Hardy-Littlewood-Sobolev inequality and Sobolev inequality

$$\psi^{\frac{1}{2}}(u) \le c\|u\|^q_{W^{1,p}(\mathbb{R}^N)}.$$

Therefore, when $\|u\|_{W^{1,p}(\mathbb{R}^N)} \le \left(\frac{1}{c\lambda}\right)^{\frac{1}{q}}$, there is $\Gamma_{\varepsilon,\lambda}(u) = \Gamma_\varepsilon(u)$ and $\Gamma'_{\varepsilon,\lambda}(u) = \Gamma'_\varepsilon(u)$; when $|u(x)| \le \varepsilon^{-1}\exp\{-dist(\varepsilon x, \mathcal{M})\}$ and $\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,dx - 1\right)_+ = 0$, there is $\Gamma_\varepsilon(u) = I_\varepsilon(u)$ and $\Gamma'_\varepsilon(u) = I'_\varepsilon(u)$. Hence, finding the solution of the Equation (4) translates into finding the critical point of $\Gamma_{\varepsilon,\lambda}(u)$.

We describe the abstract critical point theorem in detail below [15].

Let *X* be a Banach space and *f* be an even C^1 functional on *X*. Let P, Q are two open convex sets of *X*, $Q = -P$. Set

$$W = P \cup Q, \quad \Sigma = \partial P \cap \partial Q.$$

Assume

(I_1) *f* satisfies the (*PS*) condition.

(I_2) $c^* = \inf_{x\in\Sigma} f(x) > 0$. And assume there exists an odd continuous map $F: X \to X$ satisfying the following:

(F_1) given $c_0, b_0 > 0$, there exists $b = b(c_0, b_0) > 0$ such that if $\|f'(x)\| \ge b_0$, $|f(x)| \le c_0$, then

$$\langle f'(x), x - Fx\rangle \ge b\|x - Fx\|_X > 0.$$

(F_2) $F(\partial P) \subset P$, $F(\partial Q) \subset Q$.

Define

$$\Gamma_j = \left\{E \mid E \subset X : E \text{ compact}, -E = E, \gamma\left(E \cap \eta^{-1}(\Sigma)\right) \ge j \text{ for } \eta \in \Lambda\right\},$$

$$\Lambda = \left\{\eta \mid \eta \in C(X, X) : \eta \text{ odd}, \eta(P) \subset P, \eta(Q) \subset Q, \eta(x) = x \text{ if } f(x) < 0\right\}.$$

where γ is the genus of symmetric sets, defined by

$$\gamma(E) = \inf\left\{n \mid \text{there exists an odd map } \eta : E \to \mathbb{R}^n \setminus \{0\}\right\}.$$

Assume

(Γ) $\Gamma_j, j = 1, 2, \cdots$ is nonempty.

We define

$$c_j = \inf_{E\in\Gamma_j} \sup_{x\in E\setminus W} f(x), \; j = 1, 2, \cdots,$$

$$K_c = \{x \mid f'(x) = 0, \; f(x) = c\}, K_c^* = K_c \setminus W.$$

Theorem 2.2 ([18], *Theorem* 2.5) *Assume* (I_1), (I_2), (F_1), (F_2), (Γ) *hold, then*

(1) $c_j \ge c^*$, $K^*_{c_j} \ne \varnothing$,

(2) $c_j \to \infty$, for $j \to \infty$.

(3) $\gamma(K_c^*) \ge k$, if $c_j = c_{j+1} = \cdots = c_{j+k-1} = c$.

3. Existence of Sign-Changing Critical Points

In this section, we first introduce some important properties of auxiliary functions, then prove that $\Gamma_{\varepsilon,\lambda}$ satisfies the (*PS*) condition, and then use Theorem B to prove the existence of the critical point of $\Gamma_{\varepsilon,\lambda}$.

Lemma 3.1 ([19], *Lemma* 2.2), ([20], *Lemma* 2.1) *For* $x\in\mathbb{R}^N$

(1) $0\le b_\varepsilon(x,t)\le\frac{m_\varepsilon(x,t)}{t}\le 1$;

(2) If $|t|<\varepsilon^{-1}\exp\{-dist(\varepsilon x,\mathcal{M})\}$, then $m_\varepsilon(x,t)=t$;

If $\varepsilon^{-1}\exp\{-dist(\varepsilon x,\mathcal{M})\}\le|t|\le 2\varepsilon^{-1}\exp\{-dist(\varepsilon x,\mathcal{M})\}$, then $\varepsilon^{-1}\exp\{-dist(\varepsilon x,\mathcal{M})\}\le|m_\varepsilon(x,t)|\le c\varepsilon^{-1}\exp\{-dist(\varepsilon x,\mathcal{M})\}$;

If $|t|>2\varepsilon^{-1}\exp\{-dist(\varepsilon x,\mathcal{M})\}$, then $|m_\varepsilon(x,t)|=c\varepsilon^{-1}\exp\{-dist(\varepsilon x,\mathcal{M})\}$;

where $c=\int_0^\infty\xi(\tau)\mathrm{d}\tau$;

(3) $$\begin{aligned}&c_1\left(1+\varepsilon^{m-p}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|t|^{m-p}\right)|t|^{p-2}t\le k_\varepsilon(x,t)\\&\le c_2\left(1+\varepsilon^{m-p}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|t|^{m-p}\right)|t|^{p-2}t\end{aligned};$$

(4) $\frac{1}{m}tk_\varepsilon(x,t)\le K_\varepsilon(x,t)\le\frac{1}{p}tk_\varepsilon(x,t)$;

(5) $(k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2))(t_1-t_2)\ge c\varepsilon^{m-p}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|t_1-t_2|^m$, $p\ge 2$;

$$(k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2))(t_1-t_2)\ge c\varepsilon^{m-p}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}\frac{|t_1-t_2|^2}{|t_1|^{2-m}+|t_2|^{2-m}},$$ $1<p<2$;

(6) $$\begin{aligned}&|k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2)|\\&\le c\left(|t_1|^{p-2}+|t_2|^{p-2}+\varepsilon^{m-p}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}\left(|t_1|^{m-2}+|t_2|^{m-2}\right)\right)|t_1-t_2|\end{aligned},$$ $p\ge 2$;

$$\begin{aligned}&|k_\varepsilon(x,t_1)-k_\varepsilon(x,t_2)|\\&\le c\left(|t_1-t_2|^{p-1}+\varepsilon^{m-p}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|t_1-t_2|^{m-1}\right)\end{aligned},\quad 1<p<2.$$

Lemma 3.2 *Let* $\{u_n\}\subset X_\varepsilon$ *be a Palais-Smale sequence of the functional* $\Gamma_{\varepsilon,\lambda}$, *then* $\{u_n\}$ *is bounded in* X_ε.

Proof: According to (7), (8) and assumptions (A_4), (A_2) and Lemma 3.1 (4) (3), we have

$$\begin{aligned}&\mu\Gamma_{\varepsilon,\lambda}(u_n)-\langle\Gamma'_{\varepsilon,\lambda}(u_n),u_n\rangle\\&=\mu\Bigg[\frac{1}{p}\int_{\mathbb{R}^N}A(x,u_n)|\nabla u_n|^p\,\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^N}E(\varepsilon x)|u_n|^p\,\mathrm{d}x+\sigma\int_{\mathbb{R}^N}K_\varepsilon(x,u_n)\,\mathrm{d}x\\&\quad+\frac{1}{p\beta}\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^\beta-\frac{1}{2q}g_\lambda\left(\psi^{\frac{1}{2}}(u_n)\right)\psi(u_n)\Bigg]\\&\quad-\int_{\mathbb{R}^N}A(x,u_n)|\nabla u_n|^p\,\mathrm{d}x-\frac{1}{p}\int_{\mathbb{R}^N}A_t(x,u_n)u_n|\nabla u_n|^p\,\mathrm{d}x-\int_{\mathbb{R}^N}E(\varepsilon x)|u_n|^p\,\mathrm{d}x\\&\quad-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&-\sigma\int_{\mathbb{R}^N} k_\varepsilon(x,u_n)u_n\mathrm{d}x+\frac{1}{2}h_\lambda\left(\psi^{\frac{1}{2}}(u_n)\right)\psi(u_n)\\
&=\frac{1}{p}\int_{\mathbb{R}^N}\left((\mu-p)A(x,u_n)-A_t(x,u_n)u_n\right)|\nabla u_n|^p\,\mathrm{d}x+\frac{\mu-p}{p}\int_{\mathbb{R}^N}E(\varepsilon x)|u_n|^p\,\mathrm{d}x\\
&\quad+\sigma\int_{\mathbb{R}^N}\left(\mu K_\varepsilon(x,u_n)-k_\varepsilon(x,u_n)u_n\right)\mathrm{d}x+\frac{\mu}{p\beta}\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^\beta\\
&\quad-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x\\
&\quad+\frac{1}{2}h_\lambda\left(\psi^{\frac{1}{2}}(u_n)\right)\psi(u_n)-\frac{\mu}{2q}g_\lambda\left(\psi^{\frac{1}{2}}(u_n)\right)\psi(u_n)\\
&\geq\frac{\alpha_2}{p}\int_{\mathbb{R}^N}A(x,u_n)|\nabla u_n|^p\,\mathrm{d}x+\frac{\mu-p}{p}\int_{\mathbb{R}^N}E(\varepsilon x)|u_n|^p\,\mathrm{d}x\\
&\quad+\sigma\left(\frac{\mu}{m}-1\right)\int_{\mathbb{R}^N}k_\varepsilon(x,u_n)u_n\mathrm{d}x+c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^\beta-c\\
&\geq\frac{c_1\alpha_2}{p}\int_{\mathbb{R}^N}|\nabla u_n|^p\,\mathrm{d}x+\frac{\mu-p}{p}\int_{\mathbb{R}^N}E(\varepsilon x)|u_n|^p\,\mathrm{d}x+c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^\beta\\
&\quad-c+c\int_{\mathbb{R}^N}c_1\left(1+\varepsilon^{m-p}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|u_n|^{m-p}\right)|u_n|^p\,\mathrm{d}x\\
&\geq c\left(\|u_n\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u_n\|_{L_\varepsilon^m(\mathbb{R}^N)}^m\right)+c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^\beta-c\\
&\geq c\|u_n\|_{X_\varepsilon(\mathbb{R}^N)}^p+c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^\beta-c
\end{aligned}
$$

Thus, The (*PS*) sequence $\{u_n\}$ of functional $\Gamma_{\varepsilon,\lambda}$ is bounded in X_ε. ■

Lemma 3.3 *The embedding* $X_\varepsilon\hookrightarrow L^r(\mathbb{R}^N)(1\leq r<p^*)$ *is compact.*

Proof: Let $\{u_n\}$ be the (*PS*) sequence of functional $\Gamma_{\varepsilon,\lambda}$, $u_n\subset X_\varepsilon$ satisfy $\Gamma_{\varepsilon,\lambda}(u_n)\to c$, $\Gamma'_{\varepsilon,\lambda}(u_n)\to 0$ in $(X_\varepsilon)'$, by Lemma 3.2, there is a constant $\hat{\eta}_L>0$ independent of ε such that $\|u_n\|_\varepsilon\leq\hat{\eta}_L$ and $\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^\beta\leq\hat{\eta}_L$. Up to subsequences, suppose $u_n\rightharpoonup u$ in X_ε and $u_n\to u$ in $L_{loc}^r(\mathbb{R}^N)(1\leq r<p^*)$. First prove $u_n\to u$ in $L^1(\mathbb{R}^N)$, for any $R>0$ that

$$
\begin{aligned}
&\int_{\mathbb{R}^N\setminus B(0,R)}|u|\,\mathrm{d}x\\
&=\int_{\mathbb{R}^N\setminus B(0,R)}\exp\left\{\frac{m-p}{m}dist(\varepsilon x,\mathcal{M})\right\}\cdot\exp\left\{-\frac{m-p}{m}dist(\varepsilon x,\mathcal{M})\right\}\cdot|u|\,\mathrm{d}x\\
&\leq\left(\int_{\mathbb{R}^N\setminus B(0,R)}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|u|^m\,\mathrm{d}x\right)^{\frac{1}{m}}\\
&\quad\cdot\left(\int_{\mathbb{R}^N\setminus B(0,R)}\exp\left\{-\frac{m-p}{m-1}dist(\varepsilon x,\mathcal{M})\right\}\mathrm{d}x\right)^{\frac{m-1}{m}}\\
&\leq\|u\|_{L_\varepsilon^M(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N\setminus B(0,R)}\exp\left\{-\frac{m-p}{m-1}dist(\varepsilon x,\mathcal{M})\right\}\mathrm{d}x\right)^{\frac{m-1}{m}}\\
&=o_R(1),
\end{aligned}
$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^N}|u_n-u|\,\mathrm{d}x&=\int_{B(0,R)}|u_n-u|\,\mathrm{d}x+\int_{\mathbb{R}^N\setminus B(0,R)}|u_n-u|\,\mathrm{d}x\\&=o_n(1)+o_R(1)\to 0.\end{aligned}$$

For $1<r<p^*$,

$$\begin{aligned}\int_{\mathbb{R}^N}|u_n-u|^r\,\mathrm{d}x&=\int_{\mathbb{R}^N}|u_n-u|^{r\theta+(1-\theta)r}\,\mathrm{d}x\\&\le\left(\int_{\mathbb{R}^N}|u_n-u|^{r\theta\cdot\frac{1}{r\theta}}\,\mathrm{d}x\right)^{r\theta}\cdot\left(\int_{\mathbb{R}^N}|u_n-u|^{(1-\theta)r\cdot\frac{p^*}{(1-\theta)r}}\,\mathrm{d}x\right)^{\frac{(1-\theta)r}{p^*}}\\&\le c\left(\int_{\mathbb{R}^N}|u_n-u|\,\mathrm{d}x\right)^{r\theta}\to 0.\end{aligned}$$

where $0<\theta<1$, $\frac{1}{r}=\theta+\frac{1-\theta}{p^*}$. ■

Lemma 3.4 *For every* $\varepsilon>0$, $\Gamma_{\varepsilon,\lambda}$ *satisfies the Palais-Smale condition.*

Proof: Let $\{u_n\}\subset X_\varepsilon$ be a Palais-Smale sequence of the functional $\Gamma_{\varepsilon,\lambda}$, so $\left|\Gamma_{\varepsilon,\lambda}(u_n)\right|\le c$ and $\Gamma'_{\varepsilon,\lambda}(u_n)\to 0\,(n\to\infty)$, We will prove that $\{u_n\}$ has a convergent subsequence in X_ε. According to Lemma 3.2, $u_n\rightharpoonup u$ in X_ε, then by assuming A_2, A_1, Hardy-Littlewood-Sobolev inequality, *Hölder* inequality and Lemma 3.3, there are

$$\begin{aligned}o(1)&=\left\langle\Gamma'_{\varepsilon,\lambda}(u_n)-\Gamma'_{\varepsilon,\lambda}(u),u_n-u\right\rangle\\&=\int_{\mathbb{R}^N}\left(A(x,u_n)|\nabla u_n|^{p-2}\nabla u_n-A(x,u)|\nabla u|^{p-2}\nabla u,\nabla u_n-\nabla u\right)\mathrm{d}x\\&\quad+\frac{1}{p}\int_{\mathbb{R}^N}\left(A_t(x,u_n)|\nabla u_n|^p-A_t(x,u)|\nabla u|^p,u_n-u\right)\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}E(\varepsilon x)\left(|u_n|^{p-2}u_n-|u|^{p-2}u,u_n-u\right)\mathrm{d}x\\&\quad+\sigma\int_{\mathbb{R}^N}\left(k_\varepsilon(x,u_n)-k_\varepsilon(x,u)\right)(u_n-u)\,\mathrm{d}x\\&+\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^{p-2}u_n(u_n-u)\,\mathrm{d}x\\&-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^{p-2}u(u_n-u)\,\mathrm{d}x\\&-\frac{1}{2}h_\lambda\left(\psi^{\frac{1}{2}}(u_n)\right)\int_{\mathbb{R}^N}\left(I_\alpha*|u_n|^q\right)\left(|u_n|^{q-2}u_n(u_n-u)\right)\mathrm{d}x\\&+\frac{1}{2}h_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)\left(|u|^{q-2}u(u_n-u)\right)\mathrm{d}x\\&\ge c_1\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u,\nabla u_n-\nabla u\right)\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}E(\varepsilon x)\left(|u_n|^{p-2}u_n-|u|^{p-2}u,u_n-u\right)\mathrm{d}x\\&\quad+\sigma\int_{\mathbb{R}^N}\left(k_\varepsilon(x,u_n)-k_\varepsilon(x,u)\right)(u_n-u)\,\mathrm{d}x+o(1).\end{aligned}$$

Thus

$$\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u,\nabla u_n-\nabla u\right)\mathrm{d}x\to 0;$$

$$\int_{\mathbb{R}^N}\left(|u_n|^{p-2}u_n-|u|^{p-2}u,u_n-u\right)\mathrm{d}x\to 0;$$

$$\int_{\mathbb{R}^N}\left(k_\varepsilon\left(x,u_n\right)-k_\varepsilon\left(x,u\right)\right)\left(u_n-u\right)\mathrm{d}x\to 0.$$

In the following proof process, the following basic inequalities will be used [21].

$\forall\, y,z\in\mathbb{R}^N$, when $p\geq 2$ there is

$$\begin{cases}\left(|y|^{p-2}y-|z|^{p-2}z,y-z\right)\geq c|y-z|^p,\\ \left||y|^{p-2}y-|z|^{p-2}z\right|\leq c\left(|y|^{p-2}+|z|^{p-2}\right)|y-z|;\end{cases}\tag{9}$$

when $1<p<2$ there is

$$\begin{cases}\left(|y|^{p-2}y-|z|^{p-2}z,y-z\right)\geq c|y-z|^2\left(|y|^{2-p}+|z|^{2-p}\right)^{-1},\\ \left||y|^{p-2}y-|z|^{p-2}z\right|\leq c|y-z|^{p-1}.\end{cases}\tag{10}$$

For $p\geq 2$, by (9) we have

$$\int_{\mathbb{R}^N}\left|\nabla\left(u_n-u\right)\right|^p\leq c\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u,\nabla u_n-\nabla u\right)\mathrm{d}x\to 0,$$

$$\int_{\mathbb{R}^N}|u_n-u|^p\leq c\int_{\mathbb{R}^N}\left(|u_n|^{p-2}u_n-|u|^{p-2}u,u_n-u\right)\mathrm{d}x\to 0.$$

In addition, it follows from Lemma 3.1 (5) that

$$\begin{aligned}&\int_{\mathbb{R}^N}\exp\left\{(m-p)dist(\varepsilon x,\mathcal{M})\right\}|u_n-u|^m\\&\leq c\int_{\mathbb{R}^N}\left(k_\varepsilon\left(x,u_n\right)-k_\varepsilon\left(x,u\right)\right)\left(u_n-u\right)\mathrm{d}x\to 0.\end{aligned}$$

For $1<p<2$, by (10) we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\left|\nabla\left(u_n-u\right)\right|^p\\&\leq c\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u,\nabla u_n-\nabla u\right)^{\frac{p}{2}}\left(|\nabla u_n|^{2-p}+|\nabla u|^{2-p}\right)^{\frac{p}{2}}\mathrm{d}x\\&\leq c\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u,\nabla u_n-\nabla u\right)\mathrm{d}x\right)^{\frac{p}{2}}\\&\cdot\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^{2-p}+|\nabla u|^{2-p}\right)^{\frac{p}{2-p}}\mathrm{d}x\right)^{\frac{2-p}{2}}\\&\leq c\left(\int_{\mathbb{R}^N}\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u,\nabla u_n-\nabla u\right)\mathrm{d}x\right)^{\frac{p}{2}}\to 0,\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^N}|u_n-u|^p\leq c\left(\int_{\mathbb{R}^N}\left(|u_n|^{p-2}u_n-|u|^{p-2}u,u_n-u\right)\mathrm{d}x\right)^{\frac{p}{2}}\to 0.$$

It follows from Lemma 3.1 (5) that

$$\begin{aligned}&\left(k_\varepsilon\left(x,u_n\right)-k_\varepsilon\left(x,u\right)\right)\left(u_n-u\right)\\&\geq c\varepsilon^{m-p}\exp\left\{(m-p)dist(\varepsilon x,\mathcal{M})\right\}\frac{|u_n-u|^2}{|u_n|^{2-m}+|u|^{2-m}}.\end{aligned}$$

then

$$\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|u_n-u|^m$$
$$\le \exp\left\{\frac{(2-m)(m-p)}{2}dist(\varepsilon x,\mathcal{M})\right\}\cdot\left(|u_n|^{2-m}+|u|^{2-m}\right)^{\frac{m}{2}}$$
$$\cdot\left[\left(k_\varepsilon(x,u_n)-k_\varepsilon(x,u)\right)(u_n-u)\right]^{\frac{m}{2}}.$$

Thus

$$\int_{\mathbb{R}^N}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|u_n-u|^m$$
$$\le c\left(\int_{\mathbb{R}^N}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}\left(|u_n|^{2-m}+|u|^{2-m}\right)^{\frac{m}{2-m}}\mathrm{d}x\right)^{\frac{2-m}{2}}$$
$$\cdot\left(\int_{\mathbb{R}^N}\left(k_\varepsilon(x,u_n)-k_\varepsilon(x,u)\right)(u_n-u)\mathrm{d}x\right)^{\frac{m}{2}}$$
$$\le c\left(\int_{\mathbb{R}^N}\left(k_\varepsilon(x,u_n)-k_\varepsilon(x,u)\right)(u_n-u)\mathrm{d}x\right)^{\frac{m}{2}}\to 0.$$

So, $u_n\to u$ in X_ε, $\Gamma_{\varepsilon,\lambda}$ satisfies the Palais-Smale condition. ■

Next, we will prove the existence of the critical point of the functional $\Gamma_{\varepsilon,\lambda}$ by using the descending invariant set method Theorem 2.2. Before verifying the condition of Theorem 2.2, we will give a few important Lemmas:

Lemma 3.5 *Let* $\max\{N-2p,1\}<\alpha<N$, $p<q<p_\alpha^*$, *if* $u_n\subset W^{1,p}(\mathbb{R}^N)$, $u_n\rightharpoonup u$ *in* $W^{1,p}(\mathbb{R}^N)$, *and* $u_n(x)\to u(x)$ *a.e.* $x\in\mathbb{R}^N$, *then there is a subcolumn still marked as* $\{u_n\}$, *which satisfies*

(1) $$\int_{\mathbb{R}^N}\left(I_\alpha*|u_n|^q\right)|u_n|^q=\int_{\mathbb{R}^N}\left(I_\alpha*|u_n-u|^q\right)|u_n-u|^q+\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)|u|^q+o_n(1),\tag{11}$$

(2) $$\int_{\mathbb{R}^N}\left[\left(I_\alpha*|u_n|^q\right)|u_n|^{q-2}u_n-\left(I_\alpha*|u_n-u|^q\right)|u_n-u|^{q-2}(u_n-u)-\left(I_\alpha*|u|^q\right)|u|^{q-2}u\right]\varphi =o_n(1)\|\varphi\|_{W^{1,p}(\mathbb{R}^N)}.\tag{12}$$

where $\varphi\in c_o^\infty(\mathbb{R}^N)$, as $n\to\infty$, $o_n(1)\to 0$.

To prove Lemma 3.5, we also need the following results:

Lemma 3.6 ([22], *Theorem* 4.2.7) *Let* $\Omega\subseteq\mathbb{R}^N$ *be a domain and* $\{u_n\}$ *is bounded in* $L^q(\Omega)$ *for some* $q>1$, *If* $u_n(x)\to u(x)$ a.e. $x\in\Omega$, *then* $u_n\rightharpoonup u$ *in* $L^q(\Omega)$.

Lemma 3.7 ([23], *Theorem* 2.5) *If* $\{u_n\}$ *is bounded in* $W^{1,p}(\mathbb{R}^N)$, $u_n\rightharpoonup u$ *in* $W^{1,p}(\mathbb{R}^N)$ *and* $u_n(x)\to u(x)$ a.e. $x\in\mathbb{R}^N$, *then for* $p<q<p_\alpha^*$

(1) $$\int_{\mathbb{R}^N}\left||u_n|^q-|u_n-u|^q-|u|^q\right|^{\frac{2N}{N+\alpha}}\mathrm{d}x\to 0,$$

(2) $$\int_{\mathbb{R}^N}\left||u_n|^q+|u_n-u|^q-|u|^q\right|^{\frac{2N}{N+\alpha}}\mathrm{d}x\to 0,$$

(3) $$\int_{\mathbb{R}^N}\left||u_n|^{q-2}u_n-|u_n-u|^{q-2}(u_n-u)-|u|^{q-2}u\right|^{\frac{2Np}{(N+\alpha)p-2N+2p}}\mathrm{d}x\to 0.$$

Lemma 3.8 ([24], *Theorem* 2.6) *Let* $0<\alpha<N$, $s\in\left(1,\frac{N}{\alpha}\right)$, *and let*

$\{u_n\}\subset L^1(\mathbb{R}^N)\cap L^s(\mathbb{R}^N)$ *be bounded and such that, up to a subsequence, for any bounded domain* $\Omega\subset\mathbb{R}^N$, $u_n\to 0$ in $L^s(\Omega)$ as $n\to\infty$. *Then, up to a subsequence if necessary,* $(I_\alpha * u_n)(x)\to 0$ a.e. $x\in\mathbb{R}^N$ as $n\to\infty$.

Proof of Lemma 3.5:

(1) We note that:

$$\int_{\mathbb{R}^N}\left(I_\alpha *|u_n|^q\right)|u_n|^q-\left(I_\alpha *|u_n-u|^q\right)|u_n-u|^q-\left(I_\alpha *|u|^q\right)|u|^q$$
$$=\int_{\mathbb{R}^N}\left(I_\alpha *\left(|u_n|^q+|u_n-u|^q\right)\right)\left(|u_n|^q-|u_n-u|^q\right)-\left(I_\alpha *|u|^q\right)|u|^q,$$

By the Hardy-Littlewood-Sobolev inequality, for sufficiently small $\delta>0$, there exists $K_1>0$, which makes

$$\left|\int_{\Omega_1}\left(I_\alpha *|u|^q\right)|u|^q\right|\le\frac{\delta}{6};\ \Omega_1:=\left\{x\in\mathbb{R}^N:|u(x)|\ge K_1\right\}.$$

Using the Hardy-Littlewood-Sobolev inequality again, we have

$$\left|\int_{\Omega_1}\left(I_\alpha *\left(|u_n|^q+|u_n-u|^q\right)\right)\left(|u_n|^q-|u_n-u|^q\right)\right|$$
$$\le C(N,\alpha)\left(\int_{\mathbb{R}^N}\left||u_n|^q+|u_n-u|^q\right|^{\frac{2N}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}}\cdot\left(\int_{\Omega_1}\left||u_n|^q-|u_n-u|^q\right|^{\frac{2N}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}}$$
$$\le C(N,\alpha)\left(\int_{\Omega_1}\left||u_n|^q-|u_n-u|^q\right|^{\frac{2N}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}}$$
$$\le C(N,\alpha)\left(\int_{\Omega_1}|u|^{\frac{2N}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}},$$

For the above δ, when K_1 is large enough, there is

$$\left|\int_{\Omega_1}\left(I_\alpha *\left(|u_n|^q+|u_n-u|^q\right)\right)\left(|u_n|^q-|u_n-u|^q\right)\right|\le\frac{\delta}{6}.$$

Similary, let $\Omega_2:=\left\{x\in\mathbb{R}^N:|x|\ge R\right\}\setminus\Omega_1$, let $R>0$ be large enough so that

$$\left|\int_{\Omega_2}\left(I_\alpha *|u|^q\right)|u|^q\right|\le\frac{\delta}{6},$$

and

$$\left|\int_{\Omega_2}\left(I_\alpha *\left(|u_n|^q+|u_n-u|^q\right)\right)\left(|u_n|^q-|u_n-u|^q\right)\right|\le\frac{\delta}{6}.$$

For $K_2>K_1$, let $\Omega_3(n):=\left\{x\in\mathbb{R}^N:|u(x)|\ge K_2\right\}\setminus(\Omega_1\cup\Omega_2)$. If $\Omega_3(n)\ne\phi$, then $\forall x\in\Omega_3(n)$, there is $|u(x)|<K_1$ and $|x|<R$. Observed, $u_n(x)\to u(x)$ a.e. $x\in\Omega$. By Severini-Egoroff theorem, u_n converges to u in $B_R(0)$, thus $|\Omega_3(n)|\to 0$. So, for a large enough n

$$\left|\int_{\Omega_3(n)}\left(I_\alpha *|u|^q\right)|u|^q\right|\le\frac{\delta}{6},$$

and

$$\left|\int_{\Omega_3(n)}\left(I_\alpha *\left(|u_n|^q+|u_n-u|^q\right)\right)\left(|u_n|^q-|u_n-u|^q\right)\right|\le\frac{\delta}{6}.$$

Finally, we estimate that

$$\int_{\Omega_4(n)}\left(I_\alpha*\left(|u_n|^q+|u_n-u|^q\right)\right)\left(|u_n|^q-|u_n-u|^q\right)-\left(I_\alpha*|u|^q\right)|u|^q$$

where $\Omega_4(n)=\mathbb{R}^N\setminus\left(\Omega_1\cup\Omega_2\cup\Omega_3(n)\right)$, $\Omega_4(n)\subset B_R(0)$.

It follows from Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\int_{\Omega_4(n)}|u_n-u|^{\frac{2Nq}{N+\alpha}}=0 \quad\text{and}\quad \lim_{n\to\infty}\int_{\Omega_4(n)}\left||u_n|^q-|u|^q\right|^{\frac{2N}{N+\alpha}}=0.$$

This means that by the Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned}&\left|\int_{\Omega_4(n)}\left(I_\alpha*\left(|u_n|^q+|u_n-u|^q\right)\right)|u_n-u|^q\right|\\&\le C(N,\alpha)\left(\int_{\Omega_4(n)}|u_n-u|^{\frac{2Nq}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}}\to 0,\\&\left|\int_{\Omega_4(n)}\left(I_\alpha*\left(|u_n|^q+|u_n-u|^q\right)\right)\left(|u_n|^q-|u|^q\right)\right|\\&\le C(N,\alpha)\left(\int_{\Omega_4(n)}\left||u_n|^q-|u|^q\right|^{\frac{2N}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}}\to 0.\end{aligned}$$

Let $H_n=|u_n|^q+|u_n-u|^q-|u|^q$, then

$$\begin{aligned}&\lim_{n\to\infty}\int_{\Omega_4(n)}\left(I_\alpha*\left(|u_n|^q+|u_n-u|^q\right)\right)\left(|u_n|^q-|u_n-u|^q\right)-\left(I_\alpha*|u|^q\right)|u|^q\\&=\lim_{n\to\infty}\int_{\Omega_4(n)}\left(I_\alpha*H_n\right)|u|^q.\end{aligned}$$

Because H_n is bounded in $L^{\frac{2N}{N+\alpha}}\left(\mathbb{R}^N\right)$ and $H_n\to 0$ a.e. $x\in\mathbb{R}^N$, by Lemma 3.6, $H_n\rightharpoonup 0$ in $L^{\frac{2N}{N+\alpha}}\left(\mathbb{R}^N\right)$, thus $I_\alpha*H_n\rightharpoonup 0$ in $L^{\frac{2N}{N-\alpha}}\left(\mathbb{R}^N\right)$, so

$$\lim_{n\to\infty}\int_{\Omega_4(n)}\left(I_\alpha*H_n\right)|u|^q\to 0.$$

Thus, summing up, we obtain that

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}\left(I_\alpha*|u_n|^q\right)|u_n|^q-\left(I_\alpha*|u_n-u|^q\right)|u_n-u|^q-\left(I_\alpha*|u|^q\right)|u|^q\right|\le\delta.$$

By the arbitrariness of δ, conclusion (1) is established.

(2) According to Lemma 3.7, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left||u_n|^q-|u_n-u|^q-|u|^q\right|^{\frac{2N}{N+\alpha}}\mathrm{d}y=0,$$

thus

$$\left|\int_{\mathbb{R}^N}\left(I_\alpha*\left(|u_n|^q-|u_n-u|^q-|u|^q\right)\right)V_n\varphi\right|=o_n(1)\|\varphi\|_{W^{1,p}\left(\mathbb{R}^N\right)},\tag{13}$$

where $V_n=|u_n|^{q-2}u_n, |u_n-u|^{q-2}(u_n-u)$ or $|u|^{q-2}u$.

Similarly,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left||u_n|^{q-2}u_n-|u_n-u|^{q-2}(u_n-u)-|u|^{q-2}u\right|^{\frac{2Np}{(N+\alpha)p-2N+2p}}\mathrm{d}x=0,$$

thus

$$\left|\int_{\mathbb{R}^N}\left(I_\alpha * W_n\right)\left(\left|u_n\right|^{q-2}u_n-\left|u_n-u\right|^{q-2}\left(u_n-u\right)-\left|u\right|^{q-2}u\right)\varphi\right|=o_n(1)\|\varphi\|_{W^{1,p}(\mathbb{R}^N)},\quad(14)$$

where $W_n=\left|u_n\right|^q,\left|u_n-u\right|^q$ or $|u|^q$.

The direct calculation of (13) + (14) is

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[\left(I_\alpha*\left|u_n\right|^q\right)\left|u_n\right|^{q-2}u_n-\left(I_\alpha*\left|u_n-u\right|^q\right)\left|u_n-u\right|^{q-2}\left(u_n-u\right)-\left(I_\alpha*|u|^q\right)|u|^{q-2}u\right]\varphi\\&=\int_{\mathbb{R}^N}\left(I_\alpha*\left|u_n-u\right|^q\right)|u|^{q-2}u\varphi+\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)\left|u_n-u\right|^{q-2}\left(u_n-u\right)\varphi\\&\quad+o_n(1)\|\varphi\|_{W^{1,p}(\mathbb{R}^N)}.\end{aligned}$$

According to Rellich theorem, there is a subcolumn, which may as well still be recorded as $\{u_n\}$, for any bounded region $\Omega\subset\mathbb{R}^N$, we have $\left|u_n-u\right|^q\to 0$ in $L^{\frac{2N}{N+\alpha}}(\Omega)$. By Lemma 3.8, $I_\alpha*\left|u_n-u\right|^q\to 0$ a.e. $x\in\mathbb{R}^N$, from Hardy-Littlewood-Sobolev inequality it follows that

$$\sup_n\left\|I_\alpha*\left|u_n-u\right|^q\right\|_{L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)}\le\sup_n\left\|u_n-u\right\|_{L^{\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)}<\infty.$$

By Lemma 3.6, we have $\left\|I_\alpha*\left|u_n-u\right|^q\right\|^{\frac{Np}{Np-N+p}}\rightharpoonup 0$ in $L^{\frac{2Np-2N+2p}{(N-\alpha)p}}(\mathbb{R}^N)$, with $\dfrac{2Np-2N+2p}{(N-\alpha)p}>1$.

Because $|u|^{\frac{Np(q-1)}{Np-N+p}}\in L^{\frac{2Np-2N+2p}{(N+\alpha)p-2N+2p}}(\mathbb{R}^N)$, so

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(I_\alpha*\left|u_n-u\right|^q\right)|u|^{q-2}u\varphi\\&\le\left(\int_{\mathbb{R}^N}\left(I_\alpha*\left|u_n-u\right|^q\right)^{\frac{Np}{Np-N+p}}|u|^{\frac{Np(q-1)}{Np-N+p}}\mathrm{d}x\right)^{\frac{Np-N+p}{Np}}\cdot\|\varphi\|_{W^{1,p}(\mathbb{R}^N)}\\&=o_n(1)\|\varphi\|_{W^{1,p}(\mathbb{R}^N)}.\end{aligned}$$

In a similar way, we can verify that

$$\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)\left|u_n-u\right|^{q-2}\left(u_n-u\right)\varphi=o_n(1)\|\varphi\|_{W^{1,p}(\mathbb{R}^N)}.$$

Therefore, conclusion (2) is established. ■

Let

$$\mathcal{J}_\varepsilon(u)=\frac{1}{p}\int_{\mathbb{R}^N}A(x,u)|\nabla u|^p\,\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^N}E(\varepsilon x)|u|^p\,\mathrm{d}x+\sigma\int_{\mathbb{R}^N}K_\varepsilon(x,u)\mathrm{d}x.$$

The definition operator $F:X_\varepsilon\to X_\varepsilon$, $v=Fu\in X_\varepsilon$ is the only solution of the following equation.

$$\begin{aligned}&\frac{1}{2}\left\langle\mathcal{J}_\varepsilon'(u)+\mathcal{J}_\varepsilon'(v)-d\mathcal{J}_\varepsilon(u-v),\eta\right\rangle+\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^{p-2}v\eta\mathrm{d}x\\&=\frac{1}{2}h_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)|u|^{q-2}u\eta\mathrm{d}x,\quad\forall\,\eta\in X_\varepsilon\end{aligned}\quad(15)$$

Lemma 3.9 $\forall\, u, v \in X_\varepsilon$

(1) For $p \ge 2$,

$$\begin{aligned}&\left\langle \mathcal{J}_\varepsilon'(u) - \mathcal{J}_\varepsilon'(v), \varphi \right\rangle \\ &\le c\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^{p-2} + \|v\|_{W^{1,p}(\mathbb{R}^N)}^{p-2} \right) \|u-v\|_{W^{1,p}(\mathbb{R}^N)} \|\varphi\|_{W^{1,p}(\mathbb{R}^N)} \\ &\quad + c\left(\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-2} + \|v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-2} \right) \|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)} \|\varphi\|_{L_\varepsilon^m(\mathbb{R}^N)},\end{aligned}$$

For $1 < p < 2$,

$$\begin{aligned}&\left\langle \mathcal{J}_\varepsilon'(u) - \mathcal{J}_\varepsilon'(v), \varphi \right\rangle \\ &\le c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p-1} \|\varphi\|_{W^{1,p}(\mathbb{R}^N)} + \|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1} \|\varphi\|_{L_\varepsilon^m(\mathbb{R}^N)} \right).\end{aligned}$$

(2) For $p > 1$,

$$\left\langle \mathcal{J}_\varepsilon'(u), \varphi \right\rangle \le c\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^{p-1} \|\varphi\|_{W^{1,p}(\mathbb{R}^N)} + \|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1} \|\varphi\|_{L_\varepsilon^m(\mathbb{R}^N)} \right).$$

(3) For $p \ge 2$,

$$\left\langle \mathcal{J}_\varepsilon'(u) - \mathcal{J}_\varepsilon'(v), u-v \right\rangle \ge c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p} + \|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m} \right),$$

For $1 < p < 2$,

$$\begin{aligned}&\left\langle \mathcal{J}_\varepsilon'(u) - \mathcal{J}_\varepsilon'(v), u-v \right\rangle \\ &\ge c\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{2} \left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^{2-p} + \|v\|_{W^{1,p}(\mathbb{R}^N)}^{2-p} \right)^{-1} \\ &\quad + c\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{2} \left(\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^{2-m} + \|v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{2-m} \right)^{-1}.\end{aligned}$$

Proof: (1) From V_1, A_1 and *Hölder* inequality it follows that

$$\begin{aligned}&\left\langle \mathcal{J}_\varepsilon'(u) - \mathcal{J}_\varepsilon'(v), \varphi \right\rangle \\ &= \int_{\mathbb{R}^N} \left(A(x,u)|\nabla u|^{p-2}\nabla u - A(x,v)|\nabla v|^{p-2}\nabla v \right)\nabla\varphi \mathrm{d}x \\ &\quad + \frac{1}{p}\int_{\mathbb{R}^N} \left(A_t(x,u)|\nabla u|^{p} - A_t(x,v)|\nabla v|^{p} \right)\varphi \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} E(\varepsilon x)\left(|u|^{p-2}u - |v|^{p-2}v \right)\varphi \mathrm{d}x + \sigma\int_{\mathbb{R}^N} \left(k_\varepsilon(x,u) - k_\varepsilon(x,v) \right)\varphi \mathrm{d}x \\ &\le c\int_{\mathbb{R}^N} \left| |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right| \cdot |\nabla\varphi| \mathrm{d}x + c\int_{\mathbb{R}^N} \left| |\nabla u|^{p} - |\nabla v|^{p} \right| \cdot |\varphi| dx \\ &\quad + c\int_{\mathbb{R}^N} \left| |u|^{p-2}u - |v|^{p-2}v \right| \cdot |\varphi| \mathrm{d}x + \sigma\int_{\mathbb{R}^N} \left(k_\varepsilon(x,u) - k_\varepsilon(x,v) \right)\varphi \mathrm{d}x \\ &\le c\left(\int_{\mathbb{R}^N} \left| |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v \right|^{\frac{p}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \|\varphi\|_{W^{1,p}(\mathbb{R}^N)} \\ &\quad + c\left(\int_{\mathbb{R}^N} \left| |u|^{p-2}u - |v|^{p-2}v \right|^{\frac{p}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \|\varphi\|_{W^{1,p}(\mathbb{R}^N)} \\ &\quad + \sigma\int_{\mathbb{R}^N} \left(k_\varepsilon(x,u) - k_\varepsilon(x,v) \right)\varphi \mathrm{d}x.\end{aligned}$$

For $p \ge 2$, from (9) we have

$$\left(\int_{\mathbb{R}^N}\left||\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right|^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}$$
$$\le c\left(\int_{\mathbb{R}^N}\left(|\nabla u|^{p-2}+|\nabla v|^{p-2}\right)^{\frac{p}{p-1}}\cdot|\nabla u-\nabla v|^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}$$
$$\le c\left(\left(\int_{\mathbb{R}^N}|\nabla u|^p\mathrm{d}x\right)^{\frac{p-2}{p}}+\left(\int_{\mathbb{R}^N}|\nabla v|^p\mathrm{d}x\right)^{\frac{p-2}{p}}\right)\cdot\left(\int_{\mathbb{R}^N}|\nabla u-\nabla v|^p\mathrm{d}x\right)^{\frac{1}{p}}$$
$$\le c\left(\|u\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}+\|v\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\right)\|u-v\|_{W^{1,p}(\mathbb{R}^N)},$$

similarly,

$$\left(\int_{\mathbb{R}^N}\left||u|^{p-2}u-|v|^{p-2}v\right|^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\le c\left(\|u\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}+\|v\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\right)\|u-v\|_{W^{1,p}(\mathbb{R}^N)}.$$

For $1<p<2$, from (10) we have

$$\left(\int_{\mathbb{R}^N}\left||\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right|^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}$$
$$\le c\left(\int_{\mathbb{R}^N}|\nabla u-\nabla v|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\le c\|u-v\|^{p-1}_{W^{1,p}(\mathbb{R}^N)},$$
$$\left(\int_{\mathbb{R}^N}\left||u|^{p-2}u-|v|^{p-2}v\right|^{\frac{p}{p-1}}\mathrm{d}x\right)^{\frac{p-1}{p}}\le c\left(\int_{\mathbb{R}^N}|u-v|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\le c\|u-v\|^{p-1}_{W^{1,p}(\mathbb{R}^N)}.$$

In addition, by Lemma 3.1 (6) and *Hölder* inequality, for $p\ge 2$,

$$\sigma\int_{\mathbb{R}^N}\left(k_\varepsilon(x,u)-k_\varepsilon(x,v)\right)\varphi\mathrm{d}x$$
$$\le c\int_{\mathbb{R}^N}\exp\left\{(m-p)\,dist(\varepsilon x,\mathcal{M})\right\}\left(|u|^{m-2}+|v|^{m-2}\right)|u-v||\varphi|\mathrm{d}x$$
$$\le c\left(\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}+\|v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\right)\|u-v\|_{L^m_\varepsilon(\mathbb{R}^N)}\|\varphi\|_{L^m_\varepsilon(\mathbb{R}^N)}.$$

For $1<p<2$,

$$\sigma\int_{\mathbb{R}^N}\left(k_\varepsilon(x,u)-k_\varepsilon(x,v)\right)\varphi\mathrm{d}x$$
$$\le c\int_{\mathbb{R}^N}\exp\left\{(m-p)\,dist(\varepsilon x,\mathcal{M})\right\}|u-v|^{m-1}|\varphi|\mathrm{d}x$$
$$\le c\,\|u-v\|^{m-1}_{L^m_\varepsilon(\mathbb{R}^N)}\|\varphi\|_{L^m_\varepsilon(\mathbb{R}^N)}.$$

(2) Take $v=0$ in (1) to get (2).

(3) From assume A_1, A_2, we have

$$\left\langle\mathcal{J}'_\varepsilon(u)-\mathcal{J}'_\varepsilon(v),u-v\right\rangle$$
$$=\int_{\mathbb{R}^N}\left(A(x,u)|\nabla u|^{p-2}\nabla u-A(x,v)|\nabla v|^{p-2}\nabla v\right)(\nabla u-\nabla v)\mathrm{d}x$$
$$+\frac{1}{p}\int_{\mathbb{R}^N}\left(A_t(x,u)|\nabla u|^p-A_t(x,v)|\nabla v|^p\right)(u-v)\mathrm{d}x$$
$$+\int_{\mathbb{R}^N}E(\varepsilon x)\left(|u|^{p-2}u-|v|^{p-2}v\right)(u-v)\mathrm{d}x$$
$$+\sigma\int_{\mathbb{R}^N}\left(k_\varepsilon(x,u)-k_\varepsilon(x,v)\right)(u-v)\mathrm{d}x$$

$$\geq c_1\int_{\mathbb{R}^N}\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)(\nabla u-\nabla v)\mathrm{d}x$$
$$+c\int_{\mathbb{R}^N}\left(|u|^{p-2}u-|v|^{p-2}v\right)(u-v)\mathrm{d}x$$
$$+\sigma\int_{\mathbb{R}^N}\left(k_\varepsilon(x,u)-k_\varepsilon(x,v)\right)(u-v)\mathrm{d}x.$$

For $p\geq 2$, from (9) we have

$$\int_{\mathbb{R}^N}\left(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\right)(\nabla u-\nabla v)\mathrm{d}x$$
$$\geq c\int_{\mathbb{R}^N}|\nabla u-\nabla v|^p\,\mathrm{d}x\geq c\|u-v\|^p_{W^{1,p}(\mathbb{R}^N)},$$
$$\int_{\mathbb{R}^N}\left(|u|^{p-2}u-|v|^{p-2}v\right)(u-v)\mathrm{d}x\geq c\|u-v\|^p_{W^{1,p}(\mathbb{R}^N)}.$$

From Lemma 3.1 (5), we have

$$\int_{\mathbb{R}^N}\left(k_\varepsilon(x,u)-k_\varepsilon(x,v)\right)(u-v)\mathrm{d}x$$
$$\geq c\int_{\mathbb{R}^N}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|u-v|^m\,\mathrm{d}x=c\|u-v\|^m_{L^m_\varepsilon(\mathbb{R}^N)}.$$

Similarly, from (10) and Lemma 3.1 (5) can prove the case of $1<p<2$, so (3) holds. ■

Lemma 3.10 *If* $\|u\|_{X_\varepsilon}$ *is bounded, then* $\|Fu\|_{X_\varepsilon}=\|v\|_{X_\varepsilon}$ *is bounded.*

Proof: By (15)

$$\|Fu\|^p_{W^{1,p}(\mathbb{R}^N)}+\|Fu\|^m_{L^m_\varepsilon(\mathbb{R}^N)}$$
$$\leq c\langle\mathcal{J}'_\varepsilon(v),v\rangle\leq ch_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)|u|^{q-2}uv\mathrm{d}x$$
$$\leq c\|u\|^{2q-1}_{W^{1,p}(\mathbb{R}^N)}\|Fu\|_{W^{1,p}(\mathbb{R}^N)}\leq c\|Fu\|_{W^{1,p}(\mathbb{R}^N)}$$

thus $\|Fu\|_{X_\varepsilon}$ is bounded. ■

Lemma 3.11 *F is odd, well defined, and continuous on* X_ε.

Proof: From the definition of operator *F*, it is easy to know that *F* is an odd operator. Definition

$$G(v)=\frac{1}{2}\mathcal{J}_\varepsilon(v)+\frac{1}{2}\mathcal{J}_\varepsilon(v-u)+\frac{1}{2}\langle\mathcal{J}'_\varepsilon(u),v\rangle$$
$$+\frac{1}{p}\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^p\,\mathrm{d}x$$
$$-\frac{1}{2}h_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)|u|^{q-2}uv\mathrm{d}x,\quad\forall\, v\in X_\varepsilon$$

Equation (15) has a unique solution $v=Fu$, which can be obtained by solving the minimization problem $\inf\{G(v)\,|\,v\in X_\varepsilon\}$. Since

$$G(v)\geq\frac{1}{2}J_\varepsilon(v)+\frac{1}{2}\langle J'_\varepsilon(u),v\rangle-c_\lambda\geq c_1\|v\|^p_{W^{1,p}(\mathbb{R}^N)}-c_2\|v\|_{W^{1,p}(\mathbb{R}^N)}-c_\lambda$$

thus G is coercive.

Let $\{v_n\}\subset X_\varepsilon$ be a minimizing sequence for the functional *G*, $v_n\to v$ in X_ε. By the lower semicontinuity

$$G(v)\leq\liminf_{n\to\infty}G(v_n)=\inf\{G(v)\,|\,v\in X_\varepsilon\}$$

so v is a solution of (15). Assume v_1, v_2 are solutions of (15), taking $v_1 - v_2$ as the test function, we have

$$\frac{1}{2}\left\langle \mathcal{J}'_\varepsilon(v_1)-\mathcal{J}'_\varepsilon(v_2), v_1-v_2\right\rangle+\frac{1}{2}\left\langle \mathcal{J}'_\varepsilon(v_1-u)-\mathcal{J}'_\varepsilon(v_2-u), v_1-v_2\right\rangle$$
$$+\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)\left(|v_1|^{p-2}v_1-|v_2|^{p-2}v_2\right)(v_1-v_2)\,\mathrm{d}x=0$$

Hence

$$\left\langle \mathcal{J}'_\varepsilon(v_1)-\mathcal{J}'_\varepsilon(v_2), v_1-v_2\right\rangle=0.$$

By Lemma 3.9 (3), for $p\ge 2$

$$\left\langle \mathcal{J}'_\varepsilon(v_1)-\mathcal{J}'_\varepsilon(v_2), v_1-v_2\right\rangle\ge c\left(\|v_1-v_2\|^p_{W^{1,p}(\mathbb{R}^N)}+\|v_1-v_2\|^m_{L^m_\varepsilon(\mathbb{R}^N)}\right),$$

for $1<p<2$

$$\begin{aligned}&\left\langle \mathcal{J}'_\varepsilon(v_1)-\mathcal{J}'_\varepsilon(v_2), v_1-v_2\right\rangle\\&\ge c\|v_1-v_2\|^2_{W^{1,p}(\mathbb{R}^N)}\left(\|v_1\|^{2-p}_{W^{1,p}(\mathbb{R}^N)}+\|v_2\|^{2-p}_{W^{1,p}(\mathbb{R}^N)}\right)^{-1}\\&\quad+c\|v_1-v_2\|^2_{L^m_\varepsilon(\mathbb{R}^N)}\left(\|v_1\|^{2-m}_{L^m_\varepsilon(\mathbb{R}^N)}+\|v_2\|^{2-m}_{L^m_\varepsilon(\mathbb{R}^N)}\right)^{-1}.\end{aligned}$$

Then $v_1 = v_2$, we prove that Equation (15) has a unique solution of $v = Fu$.

The following proves that F is continuous, taking $\eta = v_n - v$ in (15), we have

$$\begin{aligned}&\frac{1}{2}\left\langle \mathcal{J}'_\varepsilon(u-v)-\mathcal{J}'_\varepsilon(u_n-v_n),(u-v)-(u_n-v_n)\right\rangle+\frac{1}{2}\left\langle \mathcal{J}'_\varepsilon(v_n)-\mathcal{J}'_\varepsilon(v), v_n-v\right\rangle\\&+\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)\left(|v_n|^{p-2}v_n-|v|^{p-2}v\right)(v_n-v)\,\mathrm{d}x\\&=\frac{1}{2}\left\langle \mathcal{J}'_\varepsilon(u-v)-\mathcal{J}'_\varepsilon(u_n-v_n),u-u_n\right\rangle+\frac{1}{2}\left\langle \mathcal{J}'_\varepsilon(u_n)-\mathcal{J}'_\varepsilon(u), v-v_n\right\rangle\\&\quad+\left[\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u_n|^p\,\mathrm{d}x-1\right)_+^{\beta-1}-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\right]\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^{p-2}v(v-v_n)\,\mathrm{d}x\\&\quad+\frac{1}{2}h_\lambda\left(\psi^{\frac{1}{2}}(u_n)\right)\int_{\mathbb{R}^N}\left(\left(I_\alpha*|u_n|^q\right)|u_n|^{q-2}u_n-\left(I_\alpha*|u|^q\right)|u|^{q-2}u\right)(v_n-v)\,\mathrm{d}x\\&\quad+\frac{1}{2}\left(h_\lambda\left(\psi^{\frac{1}{2}}(u_n)\right)-h_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\right)\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)|u|^{q-2}u(v_n-v)\,\mathrm{d}x.\end{aligned}\tag{16}$$

Suppose $u_n \to u$ in X_ε, by Lemma 3.9 and Lemma 3.10, for $p \ge 2$, we have

$$\begin{aligned}&\left\langle \mathcal{J}'_\varepsilon(u-v)-\mathcal{J}'_\varepsilon(u_n-v_n),u-u_n\right\rangle+\left\langle \mathcal{J}'_\varepsilon(u_n)-\mathcal{J}'_\varepsilon(u), v-v_n\right\rangle\\&\le c\left(\|u-v\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}+\|u_n-v_n\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\right)\|u-v-u_n+v_n\|_{W^{1,p}(\mathbb{R}^N)}\|u-u_n\|_{W^{1,p}(\mathbb{R}^N)}\\&\quad+c\left(\|u-v\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}+\|u_n-v_n\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\right)\|u-v-u_n+v_n\|_{L^m_\varepsilon(\mathbb{R}^N)}\|u-u_n\|_{L^m_\varepsilon(\mathbb{R}^N)}\\&\quad+c\left(\|u_n\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}+\|u\|^{p-2}_{W^{1,p}(\mathbb{R}^N)}\right)\|u_n-u\|_{W^{1,p}(\mathbb{R}^N)}\|v-v_n\|_{W^{1,p}(\mathbb{R}^N)}\\&\quad+c\left(\|u_n\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}+\|u\|^{m-2}_{L^m_\varepsilon(\mathbb{R}^N)}\right)\|u_n-u\|_{L^m_\varepsilon(\mathbb{R}^N)}\|v-v_n\|_{L^m_\varepsilon(\mathbb{R}^N)}\\&\le c\|u_n-u\|_{X_\varepsilon}=o_n(1),\end{aligned}\tag{17}$$

Similarly, for $1 < p < 2$, we have

$$\begin{aligned}
&\left\langle \mathcal{J}_\varepsilon'(u-v) - \mathcal{J}_\varepsilon'(u_n - v_n), u - u_n \right\rangle + \left\langle \mathcal{J}_\varepsilon'(u_n) - \mathcal{J}_\varepsilon'(u), v - v_n \right\rangle \\
&\le c\left(\|u - v - u_n + v_n\|_{W^{1,p}(\mathbb{R}^N)}^{p-1} \|u - u_n\|_{W^{1,p}(\mathbb{R}^N)} + \|u - v - u_n + v_n\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1} \|u - u_n\|_{L_\varepsilon^m(\mathbb{R}^N)} \right) \\
&\quad + c\left(\|u_n - u\|_{W^{1,p}(\mathbb{R}^N)}^{p-1} \|v - v_n\|_{W^{1,p}(\mathbb{R}^N)} + \|u_n - u\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1} \|v - v_n\|_{L_\varepsilon^m(\mathbb{R}^N)} \right) \\
&= o_n(1).
\end{aligned} \tag{18}$$

By Lemma 3.10 and Hardy-Littlewood-Sobolev inequality, we obtain that

$$\begin{aligned}
&\left[\left(\int_{\mathbb{R}^N} \chi_\varepsilon(x) |u_n|^p \mathrm{d}x - 1 \right)_+^{\beta-1} - \left(\int_{\mathbb{R}^N} \chi_\varepsilon(x) |u|^p \mathrm{d}x - 1 \right)_+^{\beta-1} \right] \int_{\mathbb{R}^N} \chi_\varepsilon(x) |v|^{p-2} v(v - v_n) \mathrm{d}x \\
&+ \frac{1}{2}\left(h_\lambda\left(\psi^{\frac{1}{2}}(u_n) \right) - \left(\psi^{\frac{1}{2}}(u) \right) \right) \int_{\mathbb{R}^N} \left(I_\alpha * |u|^q \right) |u|^{q-2} u(v_n - v) \mathrm{d}x = o_n(1).
\end{aligned} \tag{19}$$

By Lemma 3.5 and Lemma 3.10, we have

$$\begin{aligned}
&\frac{1}{2} h_\lambda\left(\psi^{\frac{1}{2}}(u_n) \right) \int_{\mathbb{R}^N} \left(\left(I_\alpha * |u_n|^q \right) |u_n|^{q-2} u_n - \left(I_\alpha * |u|^q \right) |u|^{q-2} u \right) (v_n - v) \mathrm{d}x \\
&= \frac{1}{2} h_\lambda\left(\psi^{\frac{1}{2}}(u_n) \right) \int_{\mathbb{R}^N} \left(I_\alpha * |u_n - u|^q \right) |u_n - u|^{q-2} (u_n - u)(v_n - v) \mathrm{d}x \\
&= o_n(1).
\end{aligned} \tag{20}$$

Thus the right-hand side of (16) satisfies

$$RHS = o_n(1). \tag{21}$$

Next, we estimate the left-hand side of (16), for $p \ge 2$, by (9)

$$\begin{aligned}
LHS &\ge \left\langle \mathcal{J}_\varepsilon'(v_n) - \mathcal{J}_\varepsilon'(v), v_n - v \right\rangle \\
&\ge c\left(\|v_n - v\|_{W^{1,p}(\mathbb{R}^N)}^p + \|v_n - v\|_{L_\varepsilon^m(\mathbb{R}^N)}^m \right),
\end{aligned} \tag{22}$$

for $1 < p < 2$, by (10)

$$\begin{aligned}
LHS &\ge \left\langle \mathcal{J}_\varepsilon'(v_n) - \mathcal{J}_\varepsilon'(v), v_n - v \right\rangle \\
&\ge c\|v_n - v\|_{W^{1,p}(\mathbb{R}^N)}^2 \left(\|v_n\|_{W^{1,p}(\mathbb{R}^N)}^{2-p} + \|v\|_{W^{1,p}(\mathbb{R}^N)}^{2-p} \right)^{-1} \\
&\quad + \|v_n - v\|_{L_\varepsilon^m(\mathbb{R}^N)}^2 \left(\|v_n\|_{L_\varepsilon^m(\mathbb{R}^N)}^{2-m} + \|v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{2-m} \right)^{-1}.
\end{aligned} \tag{23}$$

From (21) to (23), for $p > 1$, we obtain

$$\|v_n - v\|_{X_\varepsilon} \to 0, \text{ as } n \to \infty.$$

Therefore, F is continuous. ■

We verify the condition (F_1) in Theorem 2.2

Lemma 3.12 *If* $u \in X_\varepsilon$, $v = Fu$, *then*

(1) $\left\langle \Gamma_{\varepsilon,\lambda}'(u), u - v \right\rangle \ge c\left(\|u - v\|_{W^{1,p}(\mathbb{R}^N)}^p + \|u - v\|_{L_\varepsilon^m(\mathbb{R}^N)}^m \right)$

(2) $\left\|\Gamma'_{\varepsilon,\lambda}(u)\right\| \le c\left(1+\left|\Gamma_{\varepsilon,\lambda}(u)\right|+\|u-v\|_{X_\varepsilon}\right)^{\alpha}\|u-v\|_{X_\varepsilon}, \quad p\ge 2$

$\left\|\Gamma'_{\varepsilon,\lambda}(u)\right\| \le c\|u-v\|_{X_\varepsilon}^{\alpha-1}, \quad 1<p<2$

where $\alpha=\max\{p,m,p\beta\}$

Proof: (1) By (15), $\forall\,\eta\in X_\varepsilon$ we have

$$\begin{aligned}
&\left\langle\Gamma'_{\varepsilon,\lambda}(u),\eta\right\rangle\\
&=\left\langle\mathcal{J}'_\varepsilon(u),\eta\right\rangle+\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^{p-2}u\eta\,\mathrm{d}x\\
&\quad-\frac{1}{2}\left\langle\mathcal{J}'_\varepsilon(u)-\mathcal{J}'_\varepsilon(v)+\mathcal{J}'_\varepsilon(v-u),\eta\right\rangle\\
&\quad-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^{p-2}v\eta\,\mathrm{d}x\\
&=\frac{1}{2}\left\langle\mathcal{J}'_\varepsilon(u-v),u-v\right\rangle+\frac{1}{2}\left\langle\mathcal{J}'_\varepsilon(u)-\mathcal{J}'_\varepsilon(v),u-v\right\rangle\\
&\quad+\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)\left(|u|^{p-2}u-|v|^{p-2}v\right)\eta\,\mathrm{d}x.
\end{aligned}\tag{24}$$

Hence

$$\begin{aligned}
&\left\langle\Gamma'_{\varepsilon,\lambda}(u),u-v\right\rangle\\
&=\frac{1}{2}\left\langle\mathcal{J}'_\varepsilon(u-v),u-v\right\rangle+\frac{1}{2}\left\langle\mathcal{J}'_\varepsilon(u)-\mathcal{J}'_\varepsilon(v),u-v\right\rangle\\
&\quad+\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)\left(|u|^{p-2}u-|v|^{p-2}v\right)(u-v)\,\mathrm{d}x\\
&\ge c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^m\right).
\end{aligned}$$

(2) For $p\ge 2$, by (7) (15) we have

$$\begin{aligned}
&\mu\Gamma_{\varepsilon,\lambda}(u)-\frac{1}{2}\left\langle\mathcal{J}'_\varepsilon(u)-\mathcal{J}'_\varepsilon(v)+\mathcal{J}'_\varepsilon(u-v),u\right\rangle\\
&=\mu\Gamma_{\varepsilon,\lambda}(u)-\frac{1}{q}\left\langle\mathcal{J}'_\varepsilon(u),u\right\rangle+\frac{1}{2}h_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\int_{\mathbb{R}^N}\left(I_\alpha*|u|^q\right)|u|^q\,\mathrm{d}x\\
&\quad-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^{p-2}vu\,\mathrm{d}x\\
&=\frac{1}{p}\int_{\mathbb{R}^N}\left((\mu-p)A(x,u)-A_t(x,u)\right)|\nabla u|^p\,\mathrm{d}x+\frac{\mu-p}{p}\int_{\mathbb{R}^N}E(\varepsilon x)|u|^p\,\mathrm{d}x\\
&\quad+\sigma\mu\int_{\mathbb{R}^N}K_\varepsilon(x,u)\,\mathrm{d}x-\sigma\int_{\mathbb{R}^N}k_\varepsilon(x,u)u+\frac{\mu}{p\beta}\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta}\\
&\quad-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^{p-2}vu\,\mathrm{d}x\\
&\quad+\frac{1}{2}h_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\psi(u)\,\mathrm{d}x-\frac{\mu}{2q}g_\lambda\left(\psi^{\frac{1}{2}}(u)\right)\psi(u)\,\mathrm{d}x\\
&\ge c\|u\|_{W^{1,p}(\mathbb{R}^N)}^p+\frac{\sigma(\mu-m)}{m}\int_{\mathbb{R}^N}k_\varepsilon(x,u)u\,\mathrm{d}x+\frac{\mu}{p\beta}\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta}\\
&\quad-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^{p-2}vu\,\mathrm{d}x,
\end{aligned}$$

while

$$
\begin{aligned}
&\frac{\sigma(\mu-m)}{m}\int_{\mathbb{R}^N} k_\varepsilon(x,u)u\mathrm{d}x \\
&\geq c\int_{\mathbb{R}^N}\exp\{(m-p)dist(\varepsilon x,\mathcal{M})\}|u|^m\,\mathrm{d}x \geq c\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^m .
\end{aligned}
$$

By *Höder* inequality and Young inequality, we obtain that

$$
\begin{aligned}
&\frac{\mu}{p\beta}\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^\beta-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|v|^{p-2}vu\mathrm{d}x \\
&=\frac{\mu}{p\beta}\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^\beta-\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x \\
&\quad+\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^{\beta-1}\int_{\mathbb{R}^N}\chi_\varepsilon(x)\left(|u|^{p-2}u-|v|^{p-2}v\right)u\mathrm{d}x \\
&\geq c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^\beta-c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)\left(|u|^{p-2}+|v|^{p-2}\right)|u-v||u|\mathrm{d}x\right)^\beta-c \\
&\geq c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^\beta-c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u-v|^p\,\mathrm{d}x\right)^\beta-c,
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\mu\Gamma_{\varepsilon,\lambda}(u)-\frac{1}{2}\left\langle\mathcal{J}_\varepsilon'(u)-\mathcal{J}_\varepsilon'(v)+\mathcal{J}_\varepsilon'(u-v),u\right\rangle \\
&\geq c\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^m\right)+c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^\beta-c\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p\beta}-c.
\end{aligned}
\tag{25}
$$

In addition, by Lemma 3.9

$$
\begin{aligned}
&\mu\Gamma_{\varepsilon,\lambda}(u)-\frac{1}{2}\left\langle\mathcal{J}_\varepsilon'(u)-\mathcal{J}_\varepsilon'(v)+\mathcal{J}_\varepsilon'(u-v),u\right\rangle \\
&\leq c|\Gamma_{\varepsilon,\lambda}(u)|+c\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^{p-2}+\|v\|_{W^{1,p}(\mathbb{R}^N)}^{p-2}\right)\|u-v\|_{W^{1,p}(\mathbb{R}^N)}\|u\|_{W^{1,p}(\mathbb{R}^N)} \\
&\quad+c\left(\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-2}+\|v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-2}\right)\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}\|u\|_{L_\varepsilon^m(\mathbb{R}^N)} \\
&\quad+c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p-1}\|u\|_{W^{1,p}(\mathbb{R}^N)}+\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1}\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}\right) \\
&\leq c|\Gamma_{\varepsilon,\lambda}(u)|+c\|u\|_{W^{1,p}(\mathbb{R}^N)}^{p-1}\|u-v\|_{W^{1,p}(\mathbb{R}^N)}+c\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1}\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)} \\
&\quad+c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p-1}\|u\|_{W^{1,p}(\mathbb{R}^N)}+\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1}\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}\right) \\
&\leq c|\Gamma_{\varepsilon,\lambda}(u)|+c\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^m\right)+c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^m\right),
\end{aligned}
\tag{26}
$$

that is

$$
\begin{aligned}
&c\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^m\right)+c\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^\beta-c\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p\beta}-c \\
&\leq c|\Gamma_{\varepsilon,\lambda}(u)|+c\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^m\right)+c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^m\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|u\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^m+\left(\int_{\mathbb{R}^N}\chi_\varepsilon(x)|u|^p\,\mathrm{d}x-1\right)_+^\beta \\
&\leq c\left(1+|\Gamma_{\varepsilon,\lambda}(u)|+\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^p+\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^m+\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p\beta}\right).
\end{aligned}
\tag{27}
$$

By (24) and Lemma 3.9(1), (27) and Young inequality, we have

$$\begin{aligned}
\left\|\Gamma'_{\varepsilon,\lambda}(u)\right\| &\le \frac{1}{2}\left\|\mathcal{J}'_{\varepsilon}(u-v)\right\| + \frac{1}{2}\left\|\mathcal{J}'_{\varepsilon}(u) - \mathcal{J}'_{\varepsilon}(v)\right\| \\
&\quad + \left|\left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \,\mathrm{d}x - 1\right)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)\left(|u|^{p-2}u - |v|^{p-2}v\right)\mathrm{d}x\right| \\
&\le c\left(1 + \left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \,\mathrm{d}x - 1\right)_+^{\beta-1}\right)\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^{p-2} + \|v\|_{W^{1,p}(\mathbb{R}^N)}^{p-2}\right)\|u-v\|_{W^{1,p}(\mathbb{R}^N)} \\
&\quad + c\left(\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-2} + \|v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-2}\right)\|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)} + c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p-1} + \|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1}\right) \\
&\le c\left(1 + \left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \,\mathrm{d}x - 1\right)_+^{\beta-1}\right)\left(\|u\|_{W^{1,p}(\mathbb{R}^N)}^{p-2} + \|v\|_{W^{1,p}(\mathbb{R}^N)}^{p-2}\right)\|u-v\|_{X_\varepsilon} \\
&\quad + c\left(\|u\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-2} + \|v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-2}\right)\|u-v\|_{X_\varepsilon} \\
&\le c\left(1 + \left|\Gamma_{\varepsilon,\lambda}(u)\right| + \|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p} + \|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m} + \|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p\beta}\right)^2 \|u-v\|_{X_\varepsilon} \\
&\le c\left(1 + \left|\Gamma_{\varepsilon,\lambda}(u)\right| + \|u-v\|_{X_\varepsilon}\right)^\alpha \|u-v\|_{X_\varepsilon}.
\end{aligned}$$

where $\alpha = \max\{p\beta, p, m\}$.

For $1 < p < 2$, by (24) and Lemma 3.9 (2), we have

$$\begin{aligned}
\left\|\Gamma'_{\varepsilon,\lambda}(u)\right\| &\le \frac{1}{2}\left\|\mathcal{J}'_{\varepsilon}(u-v)\right\| + \frac{1}{2}\left\|\mathcal{J}'_{\varepsilon}(u) - \mathcal{J}'_{\varepsilon}(v)\right\| \\
&\quad + \left|\left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \,\mathrm{d}x - 1\right)_+^{\beta-1} \int_{\mathbb{R}^N} \chi_\varepsilon(x)\left(|u|^{p-2}u - |v|^{p-2}v\right)\mathrm{d}x\right| \\
&\le c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p-1} + \|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1}\right) + c\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p-1}\left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \,\mathrm{d}x - 1\right)_+^{\beta-1} \\
&\le c\left(\|u-v\|_{W^{1,p}(\mathbb{R}^N)}^{p-1} + \|u-v\|_{L_\varepsilon^m(\mathbb{R}^N)}^{m-1}\right) \\
&\le c\|u-v\|_{X_\varepsilon}^{\alpha-1}.
\end{aligned}$$

■

From Lemma 3.12, we can get the following corollary:

Corollary 3.13 *For all* $b_0, c_0 > 0$, *there exists* $b = b(b_0, c_0) > 0$ *such that if*

$$\left|\Gamma_{\varepsilon,\lambda}(u)\right| \le c_0,\ \left\|\Gamma'_{\varepsilon,\lambda}(u)\right\| \ge b_0$$

then

$$\left\langle \Gamma'_{\varepsilon,\lambda}(u), u - Fu \right\rangle \ge b\|u - Fu\|_{X_\varepsilon} > 0,\ u - Fu \ne 0.$$

we denote

$$D(f,g) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}}\mathrm{d}x\mathrm{d}y$$

Lemma 3.14 ([5], *Theorem* 9.8) *If* $N \ge 3, 0 < \alpha < N$, *and* $D(f,f), D(g,g) < \infty$, *then*

$$\left|D(f,g)\right|^2 \le D(f,f)\cdot D(g,g)$$

only when $g \ne 0,\ f = cg$, the equal sign holds, where c is a constant.

For $\delta > 0$, define convex cones P and Q.

$$P := \left\{ u \mid u \in X_\varepsilon\left(\mathbb{R}^N\right), \left\| u^+ \right\|_{W^{1,p}\left(\mathbb{R}^N\right)} < \delta \right\} \quad Q := \left\{ u \mid u \in X_\varepsilon\left(\mathbb{R}^N\right), \left\| u^- \right\|_{W^{1,p}\left(\mathbb{R}^N\right)} < \delta \right\}$$

We verify the condition (F_2) in Theorem 2.2

Lemma 3.15 *For* $0 < \lambda < 1$, *there exists* $\delta_\lambda > 0$, *such that for* $0 < \delta < \delta_\lambda$, *we have*

$$F(\partial P) \subset P, F(\partial Q) \subset Q.$$

Proof: We only prove $F(\partial P) \subset P, \forall u \in \partial P$, let $v = Fu$, taking $\eta = v^+$ in (15), by Hardy-Littlewood-Sobolev inequality and Lemma 3.14 have

$$\begin{aligned}
&\left\| v^+ \right\|_{W^{1,p}\left(\mathbb{R}^N\right)}^p \le \left\| v^+ \right\|_{X_\varepsilon}^p \le \left\langle \mathcal{J}'(v), v^+ \right\rangle \\
&\le ch_\lambda\left(\psi^{\frac{1}{2}}(u)\right) \int_{\mathbb{R}^N} \left(I_\alpha * |u|^q\right) |u|^{q-2} u v^+ \mathrm{d}x \\
&\le ch_\lambda\left(\psi^{\frac{1}{2}}(u)\right) \int_{\mathbb{R}^N} \left(I_\alpha * |u|^q\right) \left|u^+\right|^{q-2} u^+ v^+ \mathrm{d}x \\
&\le ch_\lambda\left(\psi^{\frac{1}{2}}(u)\right) \left(\int_{\mathbb{R}^N} \left(I_\alpha * \left(\left|u^+\right|^{q-2} u^+ v^+\right)\right) \left|u^+\right|^{q-2} u^+ v^+ \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \left(I_\alpha * |u|^q\right) |u|^q \mathrm{d}x \right)^{\frac{1}{2}} \\
&= ch_\lambda\left(\psi^{\frac{1}{2}}(u)\right) \psi^{\frac{1}{2}}(u) \left(\int_{\mathbb{R}^N} \left(I_\alpha * \left(\left|u^+\right|^{q-2} u^+ v^+\right)\right) \left|u^+\right|^{q-2} u^+ v^+ \mathrm{d}x \right)^{\frac{1}{2}} \\
&\le c_\lambda \left\| u^+ \right\|_{L^{qr}\left(\mathbb{R}^N\right)}^{q-1} \left\| v^+ \right\|_{L^{qr}\left(\mathbb{R}^N\right)}
\end{aligned}$$

then $\left\| v^+ \right\|_{L^{qr}\left(\mathbb{R}^N\right)}^{p-1} \le c_\lambda \left\| u^+ \right\|_{L^{qr}\left(\mathbb{R}^N\right)}^{q-1}$. Just taking $\delta_\lambda = \left(\frac{1}{c_\lambda}\right)^{\frac{1}{q-p}}$, then for $0 < \delta < \delta_\lambda$

$$\left\| v^+ \right\|_{L^{qr}\left(\mathbb{R}^N\right)} \le \left(c_\lambda \left\| u^+ \right\|_{L^{qr}\left(\mathbb{R}^N\right)}^{q-1} \right)^{\frac{1}{p-1}} \le \left(c_\lambda \delta^{q-1} \right)^{\frac{1}{p-1}} < \delta$$

Hence $F(\partial P) \subset P$

Similarly available: $F(\partial Q) \subset Q$ ■

We verify the condition (I_2) in Theorem 2.2

Lemma 3.16 *There exist* $\delta_0 > 0$ *and* $0 < c^* = c^*(\delta)$, *such that for any* $0 < \delta < \delta_0$ *have*

$$\Gamma_{\varepsilon,\lambda}(u) \ge c^*, \quad \forall u \in \partial P \cap \partial Q.$$

Proof: $\forall u \in \partial P \cap \partial Q$, we have

$$\begin{aligned}
\Gamma_{\varepsilon,\lambda}(u) &\ge \frac{1}{p} \int_{\mathbb{R}^N} A(x,u) |\nabla u|^p \mathrm{d}x + \frac{1}{p} \int_{\mathbb{R}^N} E(\varepsilon x) |u|^p \mathrm{d}x - \frac{1}{2q} g_\lambda\left(\psi^{\frac{1}{2}}(u)\right) \psi(u) \\
&\ge \frac{\min\{\alpha_0, a\}}{p} \|u\|_{W^{1,p}\left(\mathbb{R}^N\right)}^p - c_0 \|u\|_{L^{qr}\left(\mathbb{R}^N\right)}^{2q} \\
&\ge \frac{\min\{\alpha_0, a\}}{p} \|u\|_{W^{1,p}\left(\mathbb{R}^N\right)}^p - c_0 \delta^{2q-p} \|u\|_{L^{qr}\left(\mathbb{R}^N\right)}^p \\
&\ge \left(\frac{\min\{\alpha_0, a\}}{cp} - c_0 \delta^{2q-p} \right) \|u\|_{W^{1,p}\left(\mathbb{R}^N\right)}^p
\end{aligned}$$

Taking δ_0 is small enough to make $c_0\delta_0^{2q-p} \le \frac{\min\{\alpha_0, a\}}{2cp}$, then for $0<\delta<\delta_0$, we have

$$\Gamma_{\varepsilon,\lambda}(u) \ge \frac{\min\{\alpha_0, a\}}{2cp}\|u\|_{W^{1,p}(\mathbb{R}^N)}^p \ge \frac{\min\{\alpha_0, a\}}{2cp}\delta_0^p := c^*$$

■

Finally, we verify the condition Γ in Theorem 2.2

Let

$$\mathcal{J}_0(u) = \frac{1}{p}\int_{B_1(0)}\left(a|\nabla u|^p + b|u|^p\right)\mathrm{d}x + \sigma\int_{B_1(0)}\exp\{(m-p)|x|\}|u|^m\,\mathrm{d}x$$
$$-\frac{\tilde{c}}{2q}g_\lambda\left(\psi^{\frac{1}{2}}(\tilde{u})\right)\psi(\tilde{u})$$

where $\tilde{c} = \int_0^\infty \xi(\tau)\mathrm{d}\tau$, $B_1(0) = \{x\in\mathbb{R}^N \mid |x|<1\}$, $u\in X_\varepsilon(B_1(0))$, $\tilde{u}\equiv u$ in $B_1(0)$, $\tilde{u}\equiv 0$ in $(B_1(0))^c$. Let $\{e_n\}_{n=1}^\infty$ be a family of linearly independent functions in $C_0^\infty(B_1(0))$, exist an increasing sequence R_n so that

$$\mathcal{J}_0(u)<0,\quad \forall\, u\in H_n, \|u\|\ge R_n$$

where $H_n := span\{e_1,\cdots,e_n\}$, choose the appropriate ε to make $B_1(0)\subset\mathcal{M}_\varepsilon$.

Define

$$\varphi_n \in c\left(B_n, C_0^\infty(B_1(0))\right)$$

$$\varphi_n(t) = R_n\sum_{i=1}^n t_i e_i,\quad t=(t_1,\cdots,t_n)\in B_n = \left\{t \mid t\in\mathbb{R}^N, |t|\le 1\right\}$$

Note

$$\Gamma_j = \left\{E \mid E\subset X_\varepsilon, E \text{ is compact set}, -E=E, \text{for } \eta\in\Lambda, \gamma\left(E\cap\eta^{-1}(\Sigma)\right)\ge j\right\},$$

$$\Lambda = \left\{\eta \mid \eta\in C(X_\varepsilon, X_\varepsilon), \eta \text{ is odd function}, \eta(P)\subset P, \eta(Q)\subset Q, \eta(u)=u \text{ if } \Gamma_{\varepsilon,\lambda}(u)\le 0\right\}.$$

Lemma 3.17 *The set* Γ_j, $j=1,2,\cdots$ *is nonempty.*

Proof: The proof is similar to ([25], Lemma 4.2).

So far, we have verified all the conditions of theorem 2.2 and obtained the following existence theorem.

Theorem 3.18 *Assuming that the conditions* V_1 *and* V_2 *hold, then there exist* $0<\tilde{\varepsilon}<1$ *and* $0<\tilde{\lambda}<1$, *such that when* $0<\varepsilon<\tilde{\varepsilon}$, $0<\lambda<\tilde{\lambda}$, *the functional* $\Gamma_{\varepsilon,\lambda}$ *has infinitely many sign-changing critical points, the corresponding critical values are*

$$c_j(\varepsilon,\lambda) = \inf_{E\in\Gamma_j}\sup_{u\in E\setminus W}\Gamma_{\varepsilon,\lambda}(u),\ j=1,2,\cdots \tag{28}$$

Moreover,

(1) Existence of m_j, $j=1,2,\cdots$ independent of ε,λ such that

$$c_j(\varepsilon,\lambda)\le m_j,\ j=1,2,\cdots \tag{29}$$

(2) If $c_j(\varepsilon,\lambda)=\cdots=c_{j+k-1}(\varepsilon,\lambda)=c$, then $\gamma\left(K_c^*\right)\ge k$, with

$$K_c^* = K_c \setminus W, K_c = \left\{x \mid d\Gamma_{\varepsilon,\lambda}(u) = 0, \Gamma_{\varepsilon,\lambda}(u) = c\right\}.$$

Proof: All the assumptions of Theorem 2.2 are satisfied, by Theorem 2.2 we know (2) is true. We only need to prove (29). It's easy to verify that $\{c_j\}$ is incremental, and $E_j = \varphi_{j+1}(B_{j+1}) \in \Gamma_j$, for $t \in B_{j+1}$, $u \in \varphi_{j+1}(t)$, exists $0 < \tilde{\varepsilon} < 1$, $0 < \tilde{\lambda} < 1$, such that $\left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p \,\mathrm{d}x - 1\right)_+^\beta = 0$, then if $0 < \varepsilon < \tilde{\varepsilon}$, $0 < \lambda < \tilde{\lambda}$, for $u \in \varphi_{j+1}(B_{j+1})$ has $\Gamma_{\varepsilon,\lambda} \le \mathcal{J}_0(u)$, thus

$$c_j(\varepsilon, \lambda) \le m_j := \sup_{u \in E_j} \mathcal{J}_0(u)$$

■

4. The Proof of Theorem 1.1

In this section, we will prove the main result of this paper, that is, when ε, λ is small enough, the critical point of $\Gamma_{\varepsilon,\lambda}$ is also the critical point of I_ε. In order to prove the perturbation functional $\Gamma_{\varepsilon,\lambda}$ and the original functional I_ε have a common critical point, we will prove the following theorem.

Theorem 4.1 (1) *Suppose* $\Gamma_{\varepsilon,\lambda}(u) \le L, \Gamma'_{\varepsilon,\lambda}(u) = 0$, *then there is a constant* $H = H(L)$ *such that* $\|u\|_{W^{1,p}(\mathbb{R}^N)} \le H$;

(2) *Suppose* $\Gamma_\varepsilon(u) \le L, \Gamma'_\varepsilon(u) = 0$, *then exists constant* $\mu > 0$, $c = c(L)$, *For any* $\delta > 0$, *there is* $\varepsilon = \varepsilon(\delta) > 0$, *and when* $0 < \varepsilon < \varepsilon(\delta)$, *we have*

$$|u(x)| \le c \exp\left\{-\mu\, dist\left(x, \left(\mathcal{A}^\delta\right)_\varepsilon\right)\right\}, \quad \forall\, x \in \mathbb{R}^N.$$

Proof (1) It is easily obtained by Lemma 3.2.

(2) Refer to ([19], Lemma 5.1-Lemma 5.5), results (2). It is necessary to estimate the uniform boundedness of the solution of the truncation problem. With the help of the technique of section decomposition ([26], Theorem 2.1, Theorem 3.3), through Moser iteration [14] [15] [16] [17], we can get, if $\Gamma_{\varepsilon,\lambda}(u_n) \le L$, $\Gamma'_{\varepsilon,\lambda}(u_n) = 0$, then exists c, μ independent of *n* makes:

$$|u_n(x)| \le c \exp\{-\mu R\}, x \in \Omega_R^{(n)}.$$

we note $R_n(x) = \min\left\{\left|x - y_{n,k}\right| \mid k \in \Lambda\right\}$, then

$$|u_n(x)| \le c \exp\{-\mu R_n(x)\}, \forall\, x \in \Omega_{R_n}^{(n)}.$$

Because $\varepsilon_n y_{n,k} \to y_k^* \in \mathcal{A}$, there is $\varepsilon(\delta) > 0$ for any $\delta > 0$, when $\varepsilon_n \le \varepsilon(\delta)$, we have $\varepsilon_n y_{n,k} \in \mathcal{A}^\delta$, thus

$$|u_n(x)| \le c \exp\{-\mu R_n\} \le c \exp\left\{-\mu\, dist\left(x, \left(\mathcal{A}^\delta\right)_\varepsilon\right)\right\}, \quad \forall\, x \in \mathbb{R}^N. \quad ■$$

Corollary 4.2 (1) *Suppose* $\Gamma_{\varepsilon,\lambda}(u) \le L, \Gamma'_{\varepsilon,\lambda}(u) = 0$, *then there is a constant* $\bar{\lambda} = \bar{\lambda}(L)$ *such that* $\Gamma_{\varepsilon,\lambda}(u) = \Gamma_\varepsilon(u)$ *and* $\Gamma'_\varepsilon(u) = 0$ *if* $0 < \lambda < \bar{\lambda}$;

(2) *Suppose* $\Gamma_\varepsilon(u) \le L, \Gamma'_\varepsilon(u) = 0$, *then there is a constant* $\bar{\varepsilon} = \bar{\varepsilon}(L)$ *such that* $\Gamma_\varepsilon(u) = I_\varepsilon(u)$ *and* $I'_\varepsilon(u) = 0$ *if* $0 < \varepsilon < \bar{\varepsilon}$.

Proof (1) By Theorem 4.1(1), if $0 < \lambda < \bar{\lambda}(L) = \dfrac{1}{cH^q}$, then

$$\|u\|_{W^{1,p}(\mathbb{R}^N)} \le H = \left(\frac{1}{c\bar{\lambda}(L)}\right)^{\frac{1}{q}} \le \left(\frac{1}{c\lambda}\right)^{\frac{1}{q}}$$

thus $\Gamma_{\varepsilon,\lambda}(u) = \Gamma_\varepsilon(u)$ and $\Gamma'_\varepsilon(u) = 0$.

(2) By Theorem 4.1(2), there exists $\mu > 0$, $c = c(L)$ such that for any $\delta > 0$, there is $\varepsilon = \varepsilon(\delta) > 0$, when $0 < \varepsilon < \varepsilon(\delta)$ have

$$|u_n(x)| \le c\exp\left\{-\mu\, dist\left(x, \left(\mathcal{A}^\delta\right)_\varepsilon\right)\right\} \le c\exp\left\{-\mu\, dist\left(x, \mathcal{M}_\varepsilon\right)\right\}$$

we note $d = dist\left(\mathcal{A}^\delta, \partial\mathcal{M}\right)$, then for $x \notin \mathcal{M}_\varepsilon$, we have

$$dist\left(x, \left(\mathcal{A}^\delta\right)_\varepsilon\right) \ge dist\left(x, \mathcal{M}_\varepsilon\right) + d\varepsilon^{-1}$$

thus, as $\varepsilon \to 0$

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\,\mathrm{d}x \le c\varepsilon^{-p}\int_{\mathbb{R}^N\setminus\mathcal{M}_\varepsilon}|u|^p\,\mathrm{d}x \\
&\le c\varepsilon^{-p}\int_{\mathbb{R}^N\setminus\mathcal{M}_\varepsilon}\exp\left\{-\mu p\, dist\left(x, \left(\mathcal{A}^\delta\right)_\varepsilon\right)\right\}\mathrm{d}x \\
&\le c\varepsilon^{-p}\exp\left\{-\mu d\varepsilon^{-1}\right\}\int_{\mathbb{R}^N\setminus\mathcal{M}_\varepsilon}\exp\left\{-\mu(p-1)\, dist\left(x, \left(\mathcal{A}^\delta\right)_\varepsilon\right)\right\}\mathrm{d}x \\
&\le c\varepsilon^{-p}\exp\left\{-\mu d\varepsilon^{-1}\right\}\int_{\mathbb{R}^N\setminus\mathcal{M}}\exp\left\{-\mu(p-1)\, dist\left(x, \mathcal{A}^\delta\right)\right\}\mathrm{d}x \\
&\le c\varepsilon^{-p}\exp\left\{-\mu d\varepsilon^{-1}\right\} \to 0
\end{aligned}$$

In particular, there is a $\bar{\varepsilon}$ so that when $0 < \varepsilon \le \bar{\varepsilon}$, we have

$$\left(\int_{\mathbb{R}^N} \chi_\varepsilon(x)|u|^p\,\mathrm{d}x - 1\right)_+ = 0$$

thus $I_\varepsilon(u) = \Gamma_\varepsilon(u)$ and $\Gamma'_\varepsilon(u) = I'_\varepsilon(u) = 0$. ■

Proof of Theorem 1.1 Given a positive integer *k*, by Theorem 3.18, there is $0 < \tilde{\varepsilon} < 1$ and $0 < \tilde{\lambda} < 1$, such that when $0 < \varepsilon < \tilde{\varepsilon}$, $0 < \lambda < \tilde{\lambda}$, the functional $\Gamma_{\varepsilon,\lambda}$ has *k* pairs of sign-changing critical points $\pm u_{j,\varepsilon}$, $j = 1, 2, \cdots, k$, and the corresponding critical values satisfy:

$$0 < c_1(\varepsilon,\lambda) \le \cdots \le c_k(\varepsilon,\lambda) \le m_k$$

By Corollary 4.2(2), there is $\varepsilon_k = \varepsilon_k(m_k) > 0$, when $0 < \varepsilon < \tilde{\varepsilon}_k = \min\{\varepsilon_k, \tilde{\varepsilon}\}$, $\Gamma_\varepsilon(u) \le m_k$ and $\Gamma'_\varepsilon(u) = 0$, we have

$$\Gamma_\varepsilon(u) = I_\varepsilon(u), I'_\varepsilon(u) = 0.$$

By Corollary 4.2(1), there is $\lambda_k = \lambda_k(m_k) > 0$, when $0 < \lambda < \tilde{\lambda}_k = \min\{\lambda_k, \tilde{\lambda}\}$, $\Gamma_{\varepsilon,\lambda}(u) \le m_k$ and $\Gamma'_{\varepsilon,\lambda}(u) = 0$, we have

$$\Gamma_{\varepsilon,\lambda}(u) = \Gamma_\varepsilon(u), \Gamma'_\varepsilon(u) = 0.$$

Hence, when $0 < \varepsilon < \tilde{\varepsilon}_k, 0 < \lambda < \tilde{\lambda}_k$, which $u_{j,\varepsilon} = u_j(\varepsilon,\lambda), j = 1, 2, \cdots, k$ is also the critical point of functional I_ε. Further, according to Theorem 4.1, there is a constant $\mu > 0$, $c = c(m_k)$, so that for any $\delta > 0$, there is $\varepsilon_k(\delta) > 0$, when $0 < \varepsilon < \varepsilon_k(\delta)$, we have

$$\left|u_{j,\varepsilon}\right| \le c\exp\left\{-\mu\, dist\left(x, \left(\mathcal{A}^\delta\right)_\varepsilon\right)\right\}, \quad \forall x \in \mathbb{R}^N.$$

■

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Candela, A.M. and Salvatore, A. (2020) Existence of Radial Bounded Solutions for Some Quasilinear Elliptic Equations in $\mathbb{R}^n$. *Nonlinear Analysis*, **191**, Article ID: 111625. https://doi.org/10.1016/j.na.2019.111625

[2] Candela, A. and Palmieri, G. (2006) Multiple Solutions of Some Nonlinear Variational Problems. *Advanced Nonlinear Studies*, **9**, 269-286. https://doi.org/10.1515/ans-2006-0209

[3] Candela, A.M. and Palmieri, G. (2009) Infinitely Many Solutions of Some Nonlinear Variational Equations. *Calculus of Variations and Partial Differential Equations*, **34**, 495-530. https://doi.org/10.1007/s00526-008-0193-2

[4] Chen, K. (2004) Multiplicity Results for Some Nonlinear Elliptic Problems. *Journal of the Australian Mathematical Society*, **76**, 247-268. https://doi.org/10.1017/S1446788700008934

[5] Lieb, E.H. (1977) Existence and Uniqueness of the Minimizing Solution of Choquard's Nonlinear Equation. *Studies in Applied Mathematics*, **57**, 93-105. https://doi.org/10.1002/sapm197757293

[6] Moroz, J. and Van Schaftingen, V. (2015) Existence of Groundstates for a Class of Nonlinear Choquard Equations. *Transactions of the American Mathematical Society*, **367**, 6557-6579. https://doi.org/10.1090/S0002-9947-2014-06289-2

[7] Huang, Z.H., Yang, J.F. and Yu, W.L. (2017) Multiple Nodal Solutions of Nonlinear Choquard Equations. *Electronic Journal of Differential Equations*, **2017**, 1-18.

[8] Poppenberg, M., Schmitt, K. and Wang, Z.Q. (2002) On the Existence of Soliton Solutions to Quasilinear Schrödinger Equations. *Calculus of Variations and Partial Differential Equations*, **14**, 329-344. https://doi.org/10.1007/s005260100105

[9] Colin, M. and Jeanjean, L. (2004) Solutions for a Quasilinear Schrödinger Equation: A Dual Approach. *Nonlinear Analysis: Theory, Methods and Applications*, **56**, 213-226. ttps://doi.org/10.1016/j.na.2003.09.008

[10] Chen, S. and Wu, X. (2019) Existence of Positive Solutions for a Class of Quasilinear Schrödinger Equations of Choquard Type. *Journal of Mathematical Analysis and Applications*, **475**, 1754-1777. https://doi.org/10.1016/j.jmaa.2019.03.051

[11] Liu, J. and Wang, Z.Q. (2003) Soliton Solutions for Quasilinear Schrödinger Equations, I. *Proceedings of the American Mathematical Society*, **131**, 441-448. https://doi.org/10.1090/S0002-9939-02-06783-7

[12] Liu, J., Wang, Y. and Wang, Z. (2004) Solutions for Quasilinear Schrödinger Equations via the Nehari Method. *Communications in Partial Differential Equations*, **29**, 879-901. https://doi.org/10.1081/PDE-120037335

[13] Byeon, J. and Wang, Z.Q. (2003) Standing Waves with a Critical Frequency for Nonlinear Schrödinger Equations, II. *Calculus of Variations and Partial Differential Equations*, **18**, 207-219. ttps://doi.org/10.1007/s00526-002-0191-8

[14] Liu, X., Liu, J. and Wang, Z.Q. (2019) Localized Nodal Solutions for Quasilinear Schrödinger Equations. *Journal of Differential Equations*, **267**, 7411-7461. https://doi.org/10.1016/j.jde.2019.08.003

[15] He, R. and Liu, X. (2021) Localized Nodal Solutions for Semiclassical Choquard Equations. *Journal of Mathematical Physics*, **62**, Article 091511. https://doi.org/10.1063/5.0058380

[16] Liu, J., Liu, X. and Wang, Z.Q. (2014) Multiple Sign-Changing Solutions for Quasilinear Elliptic Equations via Perturbation Method. *Communications in Partial Differential Equations*, **39**, 2216-2239. https://doi.org/10.1080/03605302.2014.942738

[17] Liu, X., Liu, J. and Wang, Z.Q. (2013) Quasilinear Elliptic Equations with Critical Growth via Perturbation Method. *Journal of Differential Equations*, **254**, 102-124. https://doi.org/10.1016/j.jde.2012.09.006

[18] Liu, X., Liu, J. and Wang, Z.Q. (2015) Multiple Mixed States of Nodal Solutions for Nonlinear Schrödinger Systems. *Calculus of Variations and Partial Differential Equations*, **52**, 565-586. https://doi.org/10.1007/s00526-014-0724-y

[19] Zhang, B. and Liu, X. (2022) Localized Nodal Solutions for Semiclassical Quasilinear Choquard Equations with Subcritical Growth. *Electronic Journal of Differential Equations*, **2022**, 1-29.

[20] Zhao, J., Liu, X. and Liu, J. (2019) Infinitely Many Sign-Changing Solutions for System of p-Laplace Equations in $\mathbb{R}^n$. *Nonlinear Analysis*, **182**, 113-142. https://doi.org/10.1016/j.na.2018.12.005

[21] Damascelli, L. (1998) Comparison Theorems for Some Quasilinear Degenerate Elliptic Operators and Applications to Symmetry and Monotonicity Results. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, **15**, 493-516. https://doi.org/10.1016/s0294-1449(98)80032-2

[22] Willem, M. (2013) Functional Analysis: Fundamentals and Applications (Cornerstones). Birkhäuser, New York.

[23] Zhang, J.J., do Ó, J.M. and Squassina, M. (2017) Schrödinger-Poisson Systems with a General Critical Nonlinearity. *Communications in Contemporary Mathematics*, **19**, Article 1650028. https://doi.org/10.1142/S0219199716500280

[24] Cassani, D. and Zhang, J. (2019) Choquard-Type Equations with Hardy-Littlewood-Sobolev Upper-Critical Growth. *Advances in Nonlinear Analysis*, **8**, 1184-1212. https://doi.org/10.1515/anona-2018-0019

[25] Liu, J., Liu, X. and Wang, Z.Q. (2016) Sign-Changing Solutions for Coupled Nonlinear Schrödinger Equations with Critical Growth. *Journal of Differential Equations*, **261**, 7194-7236. https://doi.org/10.1016/j.jde.2016.09.018

[26] Tintarev, C. (2013) Concentration Analysis and Cocompactness. In: Adimurthi, Sandeep, K., Schindler, I. and Tintarev, C., Eds., *Concentration Analysis and Applications to PDE*, Birkhäuser, Basel, 117-141. https://doi.org/10.1007/978-3-0348-0373-1_7