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Six New Sets of the Non-Elementary Jef-Family-Functions that Are Giving Solutions to Some Second-Order Nonlinear Autonomous ODEs

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Abstract

In this paper, we define some new sets of non-elementary functions in a group of solutions $x(t)$ that are sine and cosine to the upper limit of integration in a non-elementary integral that can be arbitrary. We are using Abel's methods, described by Armitage and Eberlein. The key is to start with a non-elementary integral function, differentiating and inverting, and then define a set of three functions that belong together. Differentiating these functions twice gives second-order nonlinear ODEs that have the defined set of functions as solutions. We will study some of the second-order nonlinear ODEs, especially those that exhibit limit cycles. Using the methods described in this paper, it is possible to define many other sets of non-elementary functions that are giving solutions to some second-order nonlinear autonomous ODEs.

Keywords

Non-Elementary Functions, Second-Order Nonlinear Autonomous ODE, Limit Cycle

1. Introduction

On page 1 in the book [1], we find the sentences: Very few ordinary differential equations have explicit solutions expressible in finite terms. This is not because ingenuity fails, but because the repertory of standard functions (polynomials, exp, sin and so on) in terms of which solutions may be expressed is too limited to accommodate the variety of differential equations encountered in practice.

This is the main reason for this work. It should be possible to do something with this problem. If we don't have enough tools in our mathematical toolbox, we must make the tools first. For this problem, we will attempt to define some new non-elementary functions. The numbers I have given the ODEs and the integral functions (IF) in the text are the numbers they have in my collection.

Wolfram Math World describes three nonlinear second-order ODEs that have the Jacobi elliptic functions *sn*, *cn* and *dn* as solutions. Define a solution $x(t) = cn(t)$ and differentiate twice, and you will obtain the ODE:

$$\frac{d^2x}{dt^2} = \left(2k^2 - 1\right)x - 2k^2x^3, 0 \le k < 1 \tag{1}$$

And if we use the Jacobi amplitude function $am(t,k)$ as a solution $x(t)$ and differentiate twice, we will obtain the ODE:

$$\frac{d^2x}{dt^2} = -k^2 \sin(x)\cos(x) \tag{2}$$

This causes us to think that other second-order nonlinear ODEs have functions made by the same methods than Jacobi elliptic functions, as their solutions. It should be possible to make more non-elementary functions by changing the non-elementary integral. In this paper, we will work in the same way: First, define some non-elementary functions, and then differentiate them twice in order to see what kind of ODEs these functions are giving solutions to.

We will use the methods described by Armitage and Eberlein [2] in their book Elliptic Functions, especially Section 1.6 and 1.7. They apply what they call the Abel's methods. "Eberlein sought to relate the ideas of Abel to the later work of Jacobi." Here is a brief summary of how they define the Jacobi elliptic functions:

In Section 1.6 they define a function *x*:

1) $$x = x(\psi) = \int_0^{\psi} \frac{du}{\sqrt{1 - k^2 \sin^2 u}}, 0 \le k < 1, -\infty < \psi < \infty \tag{3}$$

Here is *x* a function of ψ. The positive derivative of *x* is:

2) $\frac{dx}{d\psi} = \frac{1}{\sqrt{1 - k^2 \sin^2 \psi}}$ and then inverting $\frac{dx}{d\psi}$, so that (4)

3) $\frac{d\psi}{dx} = \sqrt{1 - k^2 \sin^2 \psi}$ here is ψ a function of *x*. (5)

And then in Section 1.7 they define a set of functions *sn*, *cn* and *dn*, so that

$$sn(x) = \sin\psi, \quad cn(x) = \cos\psi \quad \text{and} \quad dn(x) = \sqrt{1 - k^2 \sin^2 \psi} \tag{6}$$

Armitage and Eberlein are using *x* as variable to the functions *sn*, *cn* and *dn*. In order to avoid misunderstandings with the solution $x(t)$, we will use the variable u instead of *x*, and the amplitude φ instead of ψ.

During the last 30 years there has been a lot of progress in finding solutions to nonlinear ODEs and PDEs. The progress is mostly made by using different methods like the Prelle-Singer method [3], Abel's equations [4] [5], the new Jacobi elliptic functions [6] [7], and the old Jacobi elliptic functions [8] [9], a new method [10], revised methods [11], Jacobi elliptic function expansion method [12], the expo-elliptic functions [13] [14], two new groups of non-elementary functions [15].

With exception from the new Jacobi elliptic functions, the expo-elliptic functions and two new groups of non-elementary functions, it seems to me that nobody has tried to make new non-elementary functions that can give solutions to second-order nonlinear ODEs. In this paper we will define some more sets of functions in the Jacobi elliptic function-Family. Some of these functions are giving solutions to ODEs that exhibit limit cycles and asymptotes with holes.

The functions defined in this paper are new to the literature, at least to my knowledge. This paper is a continuation of [15].

A limit cycle is a closed trajectory in phase space having the property that at least one other trajectory spirals into it, either as time approaches to infinity or as time approaches to negative infinity. In other words, the limit cycle is an isolated trajectory (isolated in the sense that neighboring trajectories are not closed); it spirals either toward or away from the limit cycle. If all neighboring trajectories approach the limit cycle, we say that the limit cycle is stable or attractive, that is, in the case where all the neighboring trajectories approach the limit cycle as time approaches to infinity [16].

2. The Jef-Family

I have named this group of functions *The Jef-Family*, after the Jacobi elliptic functions (Jef). Some of the functions in this group have Jacobi elliptic functions as a special case. We will define a general set of functions for this group. Common for all the members of *The Jef-Family* are that they contain $\sin\varphi$ or $\cos\varphi$ or both, where $\varphi=\varphi(u)$ is the upper limit of integration in a non-elementary integral. The applications to these functions are as solutions to some nonlinear ODEs. The members of *The Jef-Family* are sets of three functions, where the two first are the same in every set: $\sin\varphi$ and $\cos\varphi$. Only the third function $\frac{\mathrm{d}\varphi}{\mathrm{d}u}$ is different from set to set in this group of non-elementary functions.

In order to work with these functions, we must give them some symbols. I've used the notations of Jacobi elliptic functions, *sn*, *cn* and *dn* and added a letter in front, as for example *bsn*, *bcn* and *bdn*. Close related functions I have given the same symbol, but different numbers, as for example bsn_2 .

2.1. Definition

Define a set of three functions *asn*, *acn* and *adn*, so that $asn(u)=\sin\varphi$, $acn(u)=\cos\varphi$ and $adn(u)=\frac{\mathrm{d}\varphi}{\mathrm{d}u}=h(\varphi)$, where the function *h* can be arbitrary, with or without square root, with fraction, and even $\sin(\sin\varphi)$ or $\mathrm{e}^{\sin\varphi}$.

$\varphi = \varphi(u)$ is the upper limit of integration in a non-elementary integral. When $\frac{d\varphi}{du} = \sqrt{1-k^2\sin^2\phi}$, is $asn(u) = sn(u,k)$, $acn(u) = cn(u,k)$ and

$$adn(u) = dn(u,k) \tag{7}$$

2.2. Some Connections between These Functions

$$asn^2(u) + acn^2(u) = 1 \tag{8}$$

The connection between the functions $adn(u), asn(u)$ and $acn(u)$ depends on how the function *h* is. They are continuous and differentiable on the whole R, for the limitations of the parameters.

2.3. The Derivatives to These Functions

$$\frac{d}{du} asn(u) = acn(u) adn(u) \tag{9}$$

$$\frac{d}{du} acn(u) = -asn(u) adn(u) \tag{10}$$

$$\frac{d}{du} adn(u) = \frac{dh}{d\varphi} \frac{d\varphi}{du} \tag{11}$$

3. Some Examples

We will divide *The Jef-Family* into some subgroups:

A. $\frac{d\varphi}{du}$ is a square root, or is containing a square root.

A1. The square root is containing $\sin\varphi$, $\cos\varphi$ or both:

$$\frac{d\varphi}{du} = \sqrt{h(\sin\varphi, \cos\varphi)} \tag{12}$$

One example in this subgroup is the set of functions *psn*, *pcn* and *pdn* described in [15].

A2. The square root is containing $\sin(\sin\varphi)$, $e^{\sin\varphi}$ or both:

$$\frac{d\varphi}{du} = \sqrt{h(\sin(\sin\varphi, \cos\varphi)), f\left(e^{\sin\varphi, \cos\varphi}\right)} \tag{13}$$

A3. The square root is numerator in a fraction:

$$\frac{d\varphi}{du} = \frac{\sqrt{h(\sin\varphi, \cos\varphi)}}{g(\sin\varphi, \cos\varphi)} \tag{14}$$

B. $\frac{d\varphi}{du}$ is not a square root.

In this subgroup we find many of the most interesting examples, limit cycles and holes in asymptotes.

B1. $$\frac{d\varphi}{du} = f(\sin\varphi, \cos\varphi) \tag{15}$$

B2. $$\frac{d\varphi}{du} = h\left(\sin(\sin\varphi, \cos\varphi), e^{\sin\varphi, \cos\varphi}\right) \tag{16}$$

3.1. Two Examples of Subgroup A2

The functions *dsn*, *dcn* and *ddn*.

Define an integral function u (IF 42):

$$u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{1 + h\sin(p\sin\theta)}}, \quad -1 < h < 1, \quad -\infty < p < \infty \tag{17}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{1}{\sqrt{1 + h\sin(p\sin\varphi)}} \tag{18}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = \sqrt{1 + h\sin(p\sin\varphi)} \tag{19}$$

Define a set of three functions: *dsn*, *dcn* and *ddn*, so that

$$dsn(u) = \sin\varphi \tag{20}$$

$$dcn(u) = \cos\varphi, \tag{21}$$

$$ddn(u) = \sqrt{1 + h\sin(p\sin\varphi)} = \sqrt{1 + h\sin(p\,dsn(u))} \tag{22}$$

The connection between the functions *dsn*, *dcn* and *ddn*:

$$dsn^2(u) + dcn^2(u) = 1 \tag{23}$$

$$ddn^2(u) - h\sin(p\,dsn(u)) = 1 \tag{24}$$

Periods:

$\sin(p\sin\theta)$ increases from -1 to 1 when $p\sin\theta$ increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Then is $p\sin\theta = \frac{\pi}{2}$ the maximum value to the amplitude φ, which gives a period we may call *D*. $\varphi = \arcsin\left(\frac{\pi}{2p}\right)$.

The period will then be

$$D = D(k, p) = \int_0^{\arcsin\left(\frac{\pi}{2p}\right)} \frac{\mathrm{d}\theta}{\sqrt{1 + h\sin(p\sin\theta)}} \tag{25}$$

D is a quarter-period, and depends on the values of k and p.

$$dsn(u + 4D) = \sin(\varphi + 2\pi) = \sin\varphi = dsn(u) \tag{26}$$

$$dcn(u + 4D) = \cos(\varphi + 2\pi) = \cos\varphi = dcn(u) \tag{27}$$

$$ddn(u + 4D) = \sqrt{1 + h\sin(p\sin(\varphi + 2\pi))} = \sqrt{1 + h\sin(p\sin\varphi)} = ddn(u) \tag{28}$$

The functions $dsn(u)$, $dcn(u)$ and $ddn(u)$ have a real period of $4D$.

Some special cases:

$$dsn(0) = \sin 0 = 0, \quad dsn(D) = \sin\frac{\pi}{2} = 1, \quad dsn(u, h, p) = \sin\varphi, \quad ddn(u, h, p) = \frac{\mathrm{d}\varphi}{\mathrm{d}u} \tag{29}$$

$$dcn(0) = \cos 0 = 1, \quad dcn(D) = \cos\frac{\pi}{2} = 0, \quad dsn(u, 0, 0) = \sin(u), \quad ddn(u, h, 0) = 1 \tag{30}$$

$$ddn(0)=1,\ \ ddn(D)=\sqrt{1+h\sin(p)},\ \ dcn(u,0,0)=\cos(u),\ \ ddn(u,0,p)=1 \quad (31)$$

The derivatives to these functions:

$$\frac{\mathrm{d}}{\mathrm{d}u}dsn(u)=\cos\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}u}=dcn(u)ddn(u) \quad (32)$$

$$\frac{\mathrm{d}}{\mathrm{d}u}dcn(u)=-\sin\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}u}=-dsn(u)ddn(u) \quad (33)$$

$$\frac{\mathrm{d}}{\mathrm{d}u}ddn(u)=\frac{1}{2}hp\,dcn(u)\cos\left(p\,dsn(u)\right) \quad (34)$$

Define a solution $x(t)=dsn(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=dcn(t)ddn(t) \quad (35)$$

Then we become Equation (3140):

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-x\left(1+h\sin(px)\right)+\frac{1}{2}hp\left(1-x^2\right)\cos(px) \quad (36)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\frac{1}{2}hp\left(1-x^2\right)\cos(px)+hx\sin(px)+x=0 \quad (37)$$

See **Figure 1**, where $p=2,h=1$ the equilibrium points are centers and saddles, and the number of them goes to infinity.

The functions *ksn*, *kcn* and *kdn*.

Define an integral function u (IF 46):

$$u=u(\varphi)=\int_0^{\varphi}\frac{\mathrm{d}\theta}{\sqrt{\mathrm{e}^{a\sin\theta}\left(1-k^2\sin^2\left(p\sin\theta\right)\right)}},\quad 0\le k<1,\quad -\infty<a,p<\infty \quad (38)$$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi}=\frac{1}{\sqrt{\mathrm{e}^{a\sin\varphi}\left(1-k^2\sin^2\left(p\sin\varphi\right)\right)}} \quad (39)$$

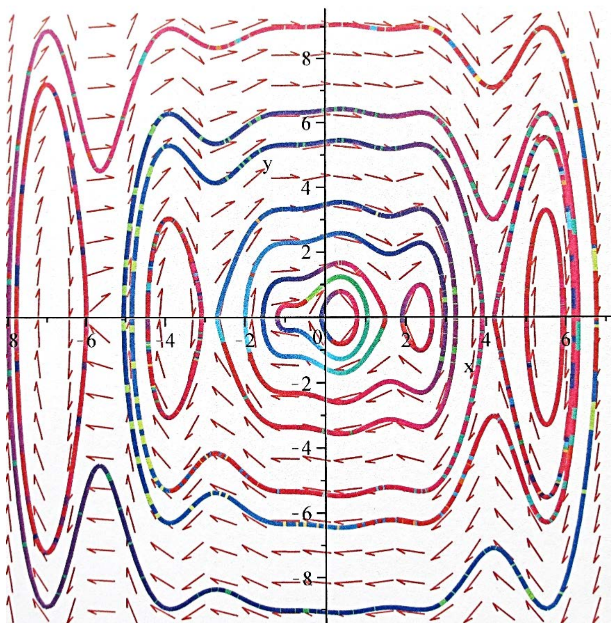

Figure 1. 6 centers and 5 saddle points.

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = \sqrt{\mathrm{e}^{a\sin\varphi}\left(1-k^2\sin^2\left(p\sin\varphi\right)\right)} \tag{40}$$

Define a set of three functions *ksn*, *kcn* and *kdn*, so that

$$ksn(u) = \sin\varphi \tag{41}$$

$$kcn(u) = \cos\varphi \tag{42}$$

$$kdn(u) = \sqrt{\mathrm{e}^{a\sin\varphi}\left(1-k^2\sin^2\left(p\sin\varphi\right)\right)} = \sqrt{\mathrm{e}^{a\,ksn(u)}\left(1-k^2\sin^2\left(p\,ksn(u)\right)\right)} \tag{43}$$

The connection between the functions *ksn*, *kcn* and *kdn*:

$$ksn^2(u) + kcn^2(u) = 1 \tag{44}$$

$$kdn^2(u) = \mathrm{e}^{a\,ksn(u)}\left(1-k^2\sin^2\left(p\,ksn(u)\right)\right) \tag{45}$$

The derivatives to these functions:

$$\frac{\mathrm{d}}{\mathrm{d}u} ksn(u) = kcn(u)\,kdn(u) \tag{46}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} kcn(u) = -ksn(u)\,kdn(u) \tag{47}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u} kdn(u) = \mathrm{e}^{a\,ksn(u)} kcn(u)\Big[&\frac{a}{2}\left(1-k^2\sin^2\left(p\,ksn(u)\right)\right) \\ &- pk^2\sin\left(p\,ksn(u)\right)\cos\left(p\,ksn(u)\right)\Big]\end{aligned} \tag{48}$$

Define a solution $x(t) = ksn(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kcn(t)\,kdn(t) \tag{49}$$

Equation (2544):

$$\begin{aligned}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = &-x\mathrm{e}^{ax}\left(1-k^2\sin^2(px)\right) \\ &+\left(1-x^2\right)\mathrm{e}^{ax}\left(\frac{a}{2}\left(1-k^2\sin^2(px)\right) - pk^2\sin(px)\cos(px)\right)\end{aligned} \tag{50}$$

3.2. An Example from the Subgroup A3

The functions *lsn*, *lcn* and *ldn*.

Define an integral function u (IF 15):

$$u = u(\varphi) = \int_0^{\varphi} \frac{1+p\cos\theta + v\sin^2\theta}{\sqrt{n-h^2\cos\theta - k^2\sin^2\theta}}\mathrm{d}\theta \tag{51}$$

$$0 \le h,k < 1,\quad -\infty < p,v < \infty,\quad n \ge 2$$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{1+p\cos\varphi + v\sin^2\varphi}{\sqrt{n-h^2\cos\varphi - k^2\sin^2\varphi}} \tag{52}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u}=\frac{\sqrt{n-h^2\cos\varphi-k^2\sin^2\varphi}}{1+p\cos\varphi+v\sin^2\varphi} \tag{53}$$

Define a set of three functions *lsn*, *lcn* and *ldn*, so that $lsn(u)=\sin\varphi$

$$lcn(u)=\cos\varphi \tag{54}$$

$$ldn(u)=\frac{\sqrt{n-h^2\cos\varphi-k^2\sin^2\varphi}}{1+p\cos\varphi+v\sin^2\varphi}=\frac{\sqrt{n-h^2lcn(u)-k^2lsn^2(u)}}{1+p\,lcn(u)+v\,lsn^2(u)} \tag{55}$$

The connection between the functions *lsn*, *lcn* and *ldn*:

$$lsn^2(u)+lcn^2(u)=1 \tag{56}$$

$$ldn^2(u)\left(1+p\,lcn(u)+v\,lsn^2(u)\right)^2+h^2lcn(u)+k^2lsn^2(u)=n \tag{57}$$

The derivative to these functions:

$$\frac{\mathrm{d}}{\mathrm{d}u}lsn(u)=lcn(u)ldn(u) \tag{59}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}lcn(u)=-lsn(u)ldn(u) \tag{60}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}ldn(u)=lsn(u)\left[\frac{\frac{1}{2}h^2-k^2lcn(u)}{\left(1+p\,lcn(u)+v\,lsn^2(u)\right)^2}-\frac{\left(2v\,lcn(u)-p\right)\left(n-h^2lcn(u)-k^2lsn^2(u)\right)}{\left(1+p\,lcn(u)+v\,lsn^2(u)\right)^3}\right] \tag{61}$$

Define a solution $x(t)=lcn(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-lsn(t)ldn(t) \tag{62}$$

After one more derivation we obtain Equation (2081):

$$\left(1+px+v-vx^2\right)^3\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\left(1+px+v-vx^2\right)\left[x\left(n-h^2x-k^2+k^2x^2\right)+\left(1-x^2\right)\left(\frac{1}{2}h^2-k^2x\right)\right]-\left(1-x^2\right)\left(2vx-p\right)\left(n-h^2x-k^2+k^2x^2\right)=0 \tag{63}$$

This second order ODE has two asymptotes:

$$x=\frac{p}{2v}\pm\frac{\sqrt{p^2+4v^2+4v}}{2v} \tag{64}$$

See **Figure 2**, where $n=2,h=\frac{1}{h},k=\frac{1}{2},p=-1,v=-\frac{1}{4}$ The asymptotes are $x=1$ and $x=3$. The closed green trajectory is almost touching the two asymptotes.

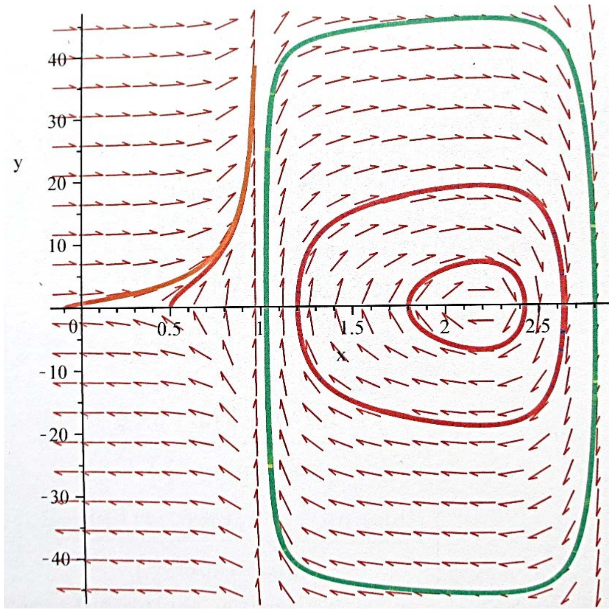

Figure 2. Two asymptotes with a center between them.

4. Some Examples from Subgroup B, Where $\frac{\mathrm{d}\varphi}{\mathrm{d}u}$ Is Not a Square Root

In this group we find functions with the most interesting behavior in the Jef-Family, such as limit cycles, holes in 2D asymptotes and cracks in 3D asymptotes, where the trajectories can pass. These opportunities come from the fact that the second-order ODE contain $\frac{\mathrm{d}x}{\mathrm{d}t}$.

Subgroup B1. $\frac{\mathrm{d}\varphi}{\mathrm{d}u} = f(\sin\varphi, \cos\varphi)$

4.1. The Functions *rsn*, *rcn* and *rdn*

Define an integral function u (IF 56 B):

$$u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{1 + k\sin\theta\cos\theta + h\sin^3\theta\cos^3\theta} \tag{65}$$

$$-2 < k < 2, \quad -8 < h < 8$$

For these values av the parameters, the denominator never become zero. When $h = 0$, the integral is elementary.

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{1}{1 + k\sin\varphi\cos\varphi + h\sin^3\varphi\cos^3\varphi} \tag{66}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = 1 + k\sin\varphi\cos\varphi + h\sin^3\varphi\cos^3\varphi \tag{67}$$

Define a set of three functions *rsn*, *rcn* and *rdn*, so that

$$rsn(u) = \sin\varphi \tag{68}$$

$$rcn(u) = \cos\varphi \tag{69}$$

$$\begin{aligned} rdn(u) &= 1 + k\sin\varphi\cos\varphi + h\sin^3\varphi\cos^3\varphi \\ &= 1 + k\,rsn(u)rcn(u) + h\,rsn^3(u)rcn^3(u) \end{aligned} \tag{70}$$

The connection between these functions:

$$rsn^2(u) + rcn^2(u) = 1 \tag{71}$$

$$rdn(u) - k\,rsn(u)rcn(u) - h\,rsn^3(u)rcn^3(u) = 1 \tag{72}$$

The derivatives to the functions *rsn*, *rcn* and *rdn*:

$$\frac{\mathrm{d}}{\mathrm{d}u} rsn(u) = rcn(u)rdn(u) \tag{73}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} rcn(u) = -rsn(u)rdn(u) \tag{74}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} rdn(u) &= \left(1 + k\,rsn(u)rcn(u) + h\,rsn^3(u)rcn^3(u)\right)\Big[k\,rcn^2(u) \\ &\quad - k\,rsn^2(u) + 3h\,rsn^2(u)rcn^2(u)\left(rcn^2(u) - rsn^2(u)\right)\Big] \end{aligned} \tag{75}$$

Periods:

$\sin\theta\cos\theta$ increases from $-\frac{1}{2}$ to $\frac{1}{2}$ when θ increases from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$. $\sin\theta\cos\theta$ decreases from $\frac{1}{2}$ to $-\frac{1}{2}$ when θ increases from $\frac{\pi}{4}$ to $\frac{3}{4}\pi$. $\sin\theta\cos\theta$ has maximum value $\frac{1}{2}$ when $\theta = \frac{\pi}{4} + n\pi$, where n is an integer.

$\sin^3\theta\cos^3\theta$ has its maximum value $\frac{1}{8}$ when $\theta = \frac{\pi}{4} + n\pi$. The period that we may call R, has its maximum value when $\theta = \frac{\pi}{4}$. R is an eighth period since $\frac{\pi}{4} = \frac{1}{8}2\pi$.

$$R = R(k,h) = \int_0^{\frac{\pi}{4}} \frac{\mathrm{d}\theta}{1 + k\sin\theta\cos\theta + h\sin^3\theta\cos^3\theta} \tag{76}$$

The functions $rsn(u)$, $rcn(u)$ and $rdn(u)$ have a real period of $8R$, that depends on the parameters k and h.

$$rsn(u + 8R) = \sin(\varphi + 2\pi) = \sin\varphi = rsn(u) \tag{77}$$

$$rcn(u + 8R) = \cos(\varphi + 2\pi) = \cos\varphi = rcn(u) \tag{78}$$

$$\begin{aligned} rdn(u + 8R) &= 1 + k\sin(\varphi + 2\pi)\cos(\varphi + 2\pi) + h\sin^3(\varphi + 2\pi)\cos^3(\varphi + 2\pi) \\ &= rdn(u) \end{aligned} \tag{79}$$

$$rsn(R) = \frac{1}{2}\sqrt{2}, \quad rcn(R) = \frac{1}{2}\sqrt{2}, \quad rdn(R) = 1 + \frac{k}{2} + \frac{h}{8} \tag{80}$$

$$rsn(0) = 0, \quad rcn(0) = 1, \quad rdn(0) = 1$$

Define a solution $x(t) = rcn(t) = \cos\varphi$

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t}&=-rsn(t)\left(1+k\,rsn(t)rcn(t)+h\,rsn^3(t)rcn^3(t)\right)\\&=-rsn(t)-kx\left(1-x^2\right)-hx^3\left(1-x^2\right)^2\end{aligned}\tag{81}$$

Equation (3425):

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}&=k\frac{\mathrm{d}x}{\mathrm{d}t}\left(4x^2-1\right)+hx^2\left(1-x^2\right)\left(8x^2-3\right)\frac{\mathrm{d}x}{\mathrm{d}t}+k^2x^3\left(1-x^2\right)\\&\quad+2hkx^5\left(1-x^2\right)^2+h^2x^7\left(1-x^2\right)^3-x\end{aligned}\tag{82}$$

Notice the many terms $\left(1-x^2\right)$. It is because of $\sin\varphi$ and $\cos\varphi$ in the solution function.

See **Figure 3**, showing a stable limit cycle containing an unstable limit cycle. Here is $k=1, h=-6$.

See **Figure 4**, showing two stable limit cycles and one unstable LC between them. Here is $k=-1, h=6$. Bifurcation with the parameters *h* and *k*.

We have two special cases from Equation (3425):

When $k=0$, we get Equation (3425 B):

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-hx^2\frac{\mathrm{d}x}{\mathrm{d}t}\left(1-x^2\right)\left(8x^2-3\right)-h^2x^7\left(1-x^2\right)^3+x=0\tag{83}$$

See **Figure 5**, where $h=-6$, giving an irregular LC.

Linearization near (0, 0) is showing complex eigenvalues with zero real part, although we can see spirals outside (0, 0). I have named this kind of equilibrium point, unstable center. This is an example of when linearization fails. A stable limit cycle may have an unstable center as equilibrium point.

When $h=0$, we get Equation (3425 C):

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-k\frac{\mathrm{d}x}{\mathrm{d}t}\left(4x^2-1\right)-k^2x^3\left(1-x^2\right)+x=0\tag{84}$$

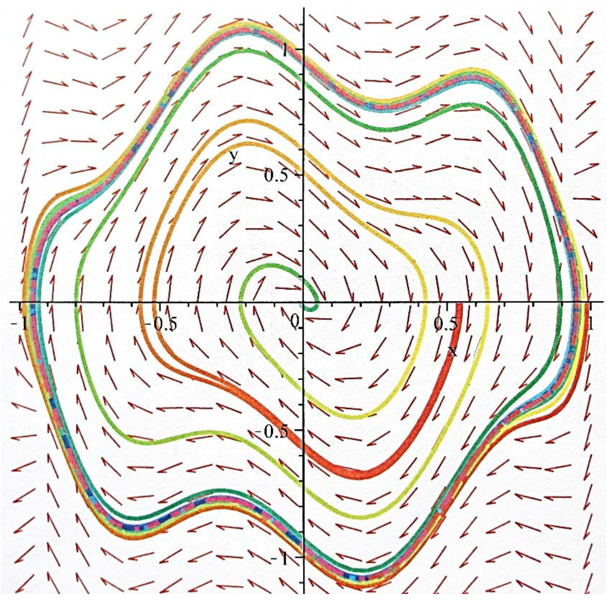

Figure 3. One stable LC containing an unstable LC.

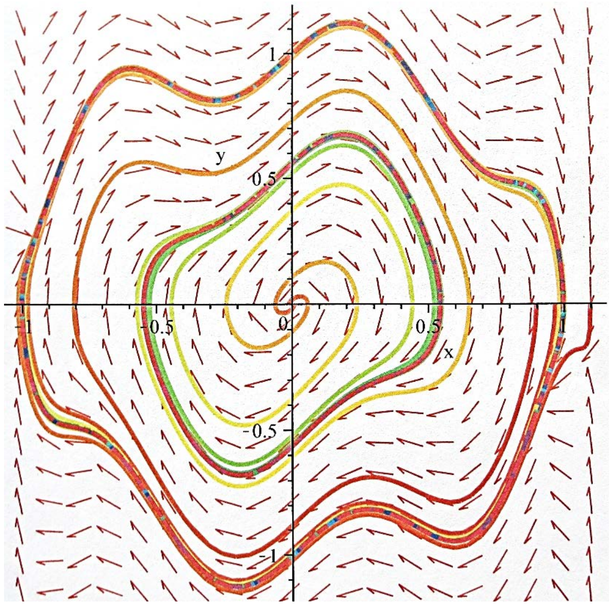

Figure 4. Two stable LC.

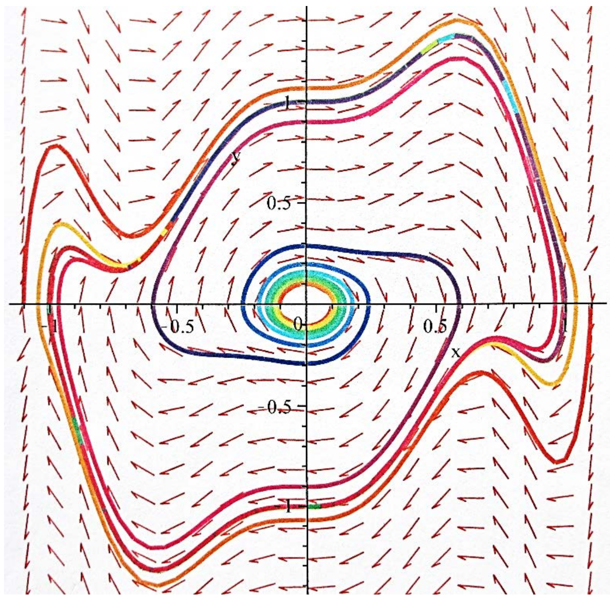

Figure 5. A LC with an unstable center as equilibrium point.

See **Figure 6**, where $k=-1$. This LC is almost identic to Van der Pol's LC. The parameter k has the same function as the parameter ε in Van der Pol's equation. When $h=0$, the integral (IF 56 B) is elementary. Equation (3425 C) has an elementary solution. This part of *The Jef-Family* gives a smooth passage from elementary functions to non-elementary functions.

When $h=0$ is

$$u=u(\varphi)=\frac{2\arctan\left(\frac{2\tan\varphi+k}{\sqrt{4-k^2}}\right)}{\sqrt{4-k^2}}+C \tag{85}$$

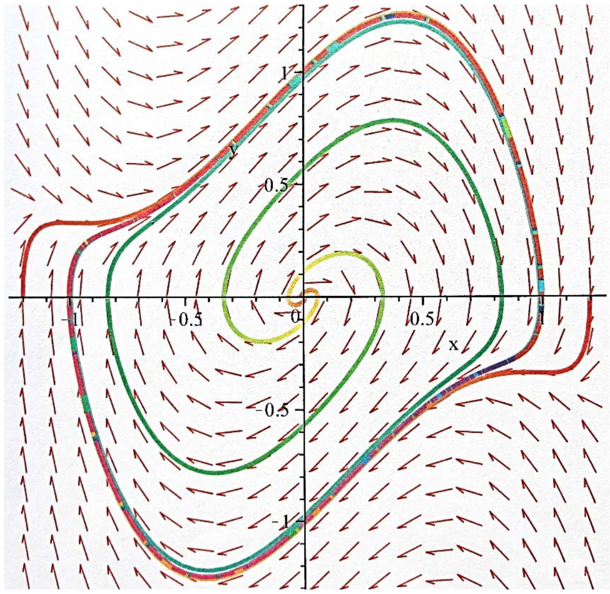

Figure 6. A LC almost identic to Van der Pol's LC.

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi}=\frac{1}{1+k\sin\varphi\cos\varphi} \tag{86}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u}=1+k\sin\varphi\cos\varphi \tag{87}$$

Choose $x(t)=\cos\varphi$, and after twice differentiating we obtain Equation (3425 C).

Notice also that the solutions to the differential equations above contain only the functions cos and sine to the amplitude φ. No exponential functions, as we could expect when systems of ODEs are exhibiting limit cycles.

4.2. The Functions *bsn*, *bcn* and *bdn*

Define an integral function u (IF 56 C):

$$u=u(\varphi)=\int_0^{\varphi}\frac{\mathrm{d}\theta}{1+k\sin\theta\cos\theta+f\cos\theta-h^2\cos^2\theta} \tag{88}$$

$$-1<f,k<1,\quad 0\le h<1$$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi}=\frac{1}{1+k\sin\varphi\cos\varphi+f\cos\varphi-h^2\cos^2\varphi} \tag{89}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u}=1+k\sin\varphi\cos\varphi+f\cos\varphi-h^2\cos^2\varphi \tag{90}$$

Define a set of three functions *bsn*, *bcn* and *bdn*, so that

$$bsn(u)=\sin\varphi \tag{91}$$

$$bcn(u)=\cos\varphi \tag{92}$$

$$\begin{aligned} bdn(u)&=1+k\sin\varphi\cos\cos\varphi+f\cos\varphi-h^2\cos^2\varphi \\ &=1+k\,bsn(u)bcn(u)+f\,bcn(u)-h^2bcn^2(u) \end{aligned} \tag{93}$$

The connection between the functions *bsn*, *bcn* and *bdn*:

$$bsn^2(u)+bcn^2(u)=1 \tag{94}$$

$$bdn(u)-k\,bsn(u)bcn(u)-f\,bcn(u)+h^2bcn^2(u)=1 \tag{95}$$

The derivative to these functions:

$$\frac{\mathrm{d}}{\mathrm{d}u}bsn(u)=bcn(u)bdn(u) \tag{96}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}bcn(u)=-bsn(u)bdn(u) \tag{97}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}bdn(u)=&\left(k\,bcn^2(u)-kbsn^2(u)-f\,bsn(u)+2h^2bsn(u)bcn(u)\right)\\&\times\left(1+k\,bsn(u)bcn(u)+f\,bcn(u)-h^2bcn^2(u)\right)\end{aligned} \tag{98}$$

Define a solution $x(t)=bcn(t)=\cos\varphi$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-bsn(t)\left(1+k\,bsn(t)bcn(t)+f\,bcn(t)-h^2bcn^2(t)\right) \tag{99}$$

Then we become Equation (3426):

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=&k\frac{\mathrm{d}x}{\mathrm{d}t}\left(4x^2-1\right)+k^2x^3\left(1-x^2\right)-x\left(1+fx-h^2x^2\right)^2\\&+\frac{\left(\frac{\mathrm{d}x}{\mathrm{d}t}+kx-kx^3\right)\left(f\frac{\mathrm{d}x}{\mathrm{d}t}-2h^2x\frac{\mathrm{d}x}{\mathrm{d}t}\right)}{1+fx-h^2x^2}\end{aligned} \tag{100}$$

$$\begin{aligned}&\left(1+fx-h^2x^2\right)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\left(k\frac{\mathrm{d}x}{\mathrm{d}t}\left(4x^2-1\right)+k^2x^3\left(1-x^2\right)\right)\left(1+fx-h^2x^2\right)\\&+x\left(1+fx-h^2x^2\right)^3-\frac{\mathrm{d}x}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}+kx-kx^3\right)\left(f-2h^2x\right)=0\end{aligned} \tag{101}$$

This, a bit complicated equation appears through two times differentiating the "simple" function *bcn*(*t*). This is showing some of the strength of these functions.

We can see that Equation (3426) has two asymptotes for $h\neq 0$. And there is one hole in each of them where trajectories can pass. They will pass the asymptotes in the points that are making the fraction term in Equation (3426), $\frac{0}{0}$. In these points $\frac{\mathrm{d}y}{\mathrm{d}t}$ exist and is not zero.

We will study one special case:

4.2.1. The Parameter $f=0$

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=&k\frac{\mathrm{d}x}{\mathrm{d}t}\left(4x^2-1\right)+k^2x^3\left(1-x^2\right)-x\left(1-h^2x^2\right)^2\\&-\frac{\left(\frac{\mathrm{d}x}{\mathrm{d}t}+kx-kx^3\right)2h^2x\frac{\mathrm{d}x}{\mathrm{d}t}}{1-h^2x^2}\end{aligned} \tag{102}$$

This equation has two asymptotes: $x=\pm\frac{1}{h}$, with two holes: $\left(\frac{1}{h},\frac{k\left(1-h^2\right)}{h^3}\right)$ and $\left(-\frac{1}{h},-\frac{k\left(1-h^2\right)}{h^3}\right)$

Through these two holes in the asymptotes can the trajectories pass, and they also intersect when passing these holes. See **Figure 7**, where $k=-1, h^2=\frac{1}{2}$. The trajectories pass the asymptotes in the points $\left(\sqrt{2},-\sqrt{2}\right)$ and $\left(-\sqrt{2},\sqrt{2}\right)$. In these two points $\frac{\mathrm{d}y}{\mathrm{d}t}$ exist, and is not zero, but nowhere else along the asymptotes. When putting in the two points in the fraction term, it becomes $\frac{0}{0}$, and this exist and is 0. These two holes in asymptotes are examples of when the Uniqueness Theorem fails. All derivatives don't exist in the neighborhood of these two points. They are attracting the solution curves from outside the asymptotes. The holes act as sink on one side and as source on the other side.

Let us now try to give **Figure 7** an extra dimension, so that the holes in the asymptotes becomes cracks in an asymptotic wall. The easiest way is to add an equation $\frac{\mathrm{d}z}{\mathrm{d}t}=sz$, having the solution $z(t)=\mathrm{e}^{st}$. Then we become a system of three ODEs:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=y \tag{103}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=ky\left(4x^2-1\right)+k^2x^3\left(1-x^2\right)-x\left(1-h^2x^2\right)^2-\frac{2h^2xy\left(y+kx-kx^3\right)}{1-h^2x^2} \tag{104}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=sz \tag{105}$$

Choosing $s<0$, the *xy*-plane will act as a sink and prevent the trajectories going to infinity parallel with the *z*-axis. Now the asymptotes $x=\pm\frac{1}{h}$ are vertical planes, parallel with the z-axis. The trajectories can pass through the asymptotes where the plane $x=\frac{1}{h}$ intersect the plane $y=\frac{k\left(1-h^2\right)}{h^3}$, and where the plane $x=-\frac{1}{h}$ intersect the plane $y=-\frac{k\left(1-h^2\right)}{h^3}$. Along these two vertical lines $\frac{\mathrm{d}y}{\mathrm{d}t}$ exist and is not zero. $\left(\frac{1}{h},\frac{k\left(1-h^2\right)}{h^3},z\right),\left(-\frac{1}{h},-\frac{k\left(1-h^2\right)}{h^3},z\right)$.

See **Figure 8**, where $k=-1, h^2=\frac{1}{2}, s=-1$. We have a crack in the asymptotes where the plane $x=\sqrt{2}$ intersect the plane $y=-\sqrt{2}$, and also where the plane $x=-\sqrt{2}$ intersect the plane $y=\sqrt{2}$ I have sketched two vertical lines where these cracks are.

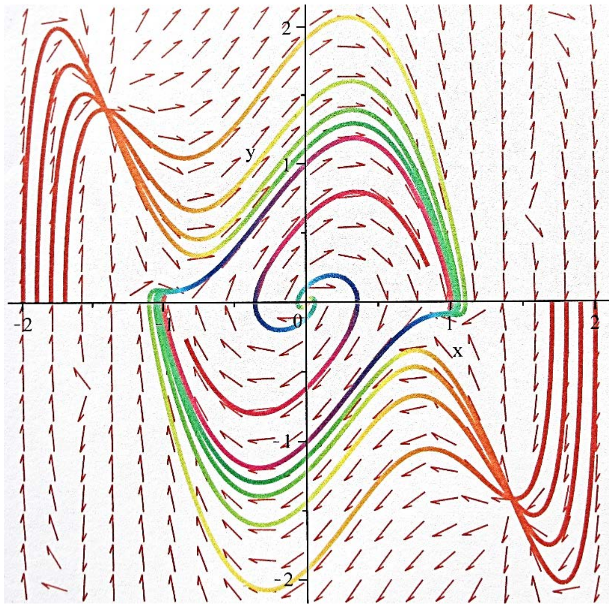

Figure 7. Two holes in two asymptotes with a LC in the middle.

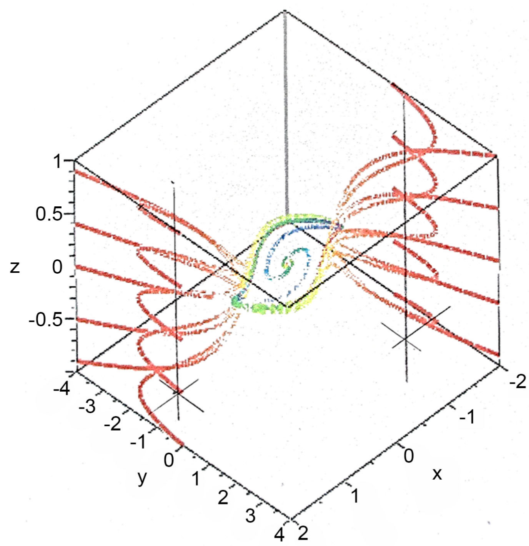

Figure 8. Two cracks in two asymptotic walls.

Subgroup B2. $\frac{\mathrm{d}\varphi}{\mathrm{d}u} = f\left(\sin\left(\sin\varphi, \cos\varphi\right), \mathrm{e}^{\sin\varphi, \cos\varphi}\right)$

4.2.2. The Functions *hsn*, *hcn* and *hdn*

Define an integral function u (IF 40):

$$u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{1 + k\sin\theta\sin\left(p\cos\theta\right) + h\sin\theta\cos\left(v\cos\theta\right)} \tag{106}$$

$$-1 < k, h < 1, \quad -\infty < p, v < \infty$$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{1}{1 + k\sin\varphi\sin\left(p\cos\varphi\right) + h\sin\varphi\cos\left(v\cos\varphi\right)} \tag{107}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u}=1+k\sin\varphi\sin(p\cos\varphi)+h\sin\varphi\cos(v\cos\varphi) \tag{108}$$

Define a set of three functions: *hsn*, *hcn* and *hdn*, so that $hsn(u)=\sin\varphi$, (109)

$$hcn(u)=\cos\varphi, \tag{110}$$

$$\begin{aligned}hdn(u)&=1+k\sin\varphi\sin(p\cos\varphi)+h\sin\varphi\cos(v\cos\varphi)\\&=1+k\,hsn(u)\sin(p\,hcn(u))+h\,hsn(u)\cos(v\,hcn(u))\end{aligned} \tag{111}$$

The connection between these functions:

$$hsn^2(u)+hcn^2(u)=1 \tag{112}$$

$$hdn(u)-k\,hsn(u)\sin(p\,hcn(u))-h\,hsn(u)\cos(v\,hcn(u))=1 \tag{113}$$

The derivative to the functions *hsn*, *hcn* and *hdn*:

$$\frac{\mathrm{d}}{\mathrm{d}u}hsn(u)=hcn(u)hdn(u) \tag{114}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}hcn(u)=-hsn(u)hdn(u) \tag{115}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}hdn(u)&=\left(1+k\,hsn(u)\sin(p\,hcn(u))+h\,hsn(u)\cos(v\,hcn(u))\right)\\&\quad\times\left[k\,hcn(u)\sin(p\,hcn(u))-kp\,hsn^2(u)\cos(p\,hcn(u))\right.\\&\quad\left.+h\,hcn(u)\cos(v\,hcn(u))+hv\,hsn^2(u)\sin(v\,hcn(u))\right]\end{aligned} \tag{116}$$

Define a solution $x(t)=hcn(t)=\cos\varphi$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-\sin\varphi-k\left(1-x^2\right)\sin(px)-h\left(1-x^2\right)\cos(vx) \tag{117}$$

Then we become Equation (3438):

$$\begin{aligned}&\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-3x\frac{\mathrm{d}x}{\mathrm{d}t}\left(k\sin(px)+h\cos(px)\right)-\frac{\mathrm{d}x}{\mathrm{d}t}\left(1-x^2\right)\left(hv\sin(vx)-kp\cos(px)\right)\\&-x\left(1-x^2\right)\left(k\sin(px)+h\cos(vx)\right)^2+x=0\end{aligned} \tag{118}$$

See **Figure 9**, where $k=1, p=\frac{1}{2}, v=2, h=-\frac{1}{2}$

The special case $h=0$ gives Equation (3437):

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=3kx\frac{\mathrm{d}x}{\mathrm{d}t}\sin(px)-kp\left(1-x^2\right)\frac{\mathrm{d}x}{\mathrm{d}t}\cos(px)+k^2x\left(1-x^2\right)\sin^2(px)-x \tag{119}$$

See **Figure 10**, where $k=-1, p=\frac{1}{4}$.

5. The Complex Expo-Elliptic Function

The solutions defined in this paper are trigonometric functions of the amplitude φ, that is the upper limit of integration in a non-elementary integral. These functions are special cases of the complex expo-elliptic function. This function may also be named the complex μ-function:

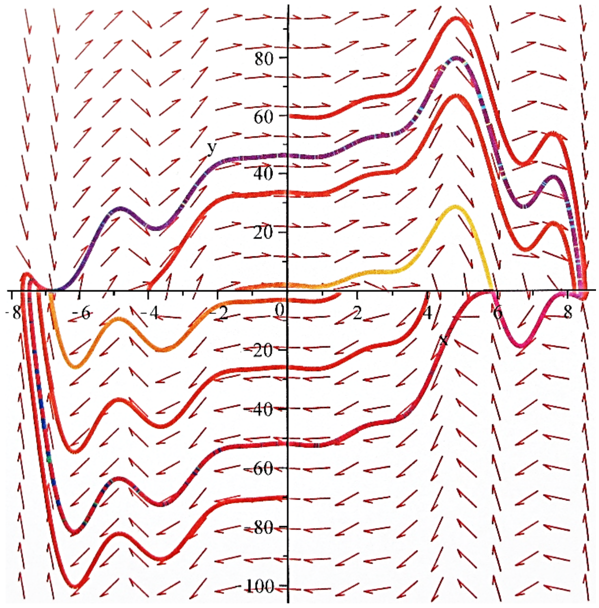

Figure 9. A large irregular LC.

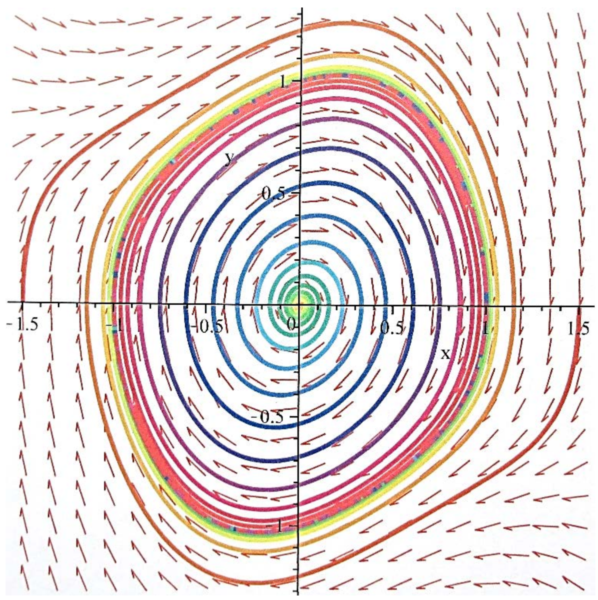

Figure 10. A regular stable LC.

$$M = M(u) = e^{\lambda\varphi},\ \varphi = \varphi(u),\ \lambda = a + ib(\text{a complex number}),\ i = \sqrt{-1},\ a,b \in R \quad (120)$$

$$M = M(u) = e^{\lambda\varphi} = e^{a\varphi}\left(\cos(b\varphi) + i\sin(b\varphi)\right) \quad (121)$$

In Equation (121) we find the expo-elliptic function $e^{a\varphi}$, the solutions in *The Jef-Family*: $\sin\varphi$ and $\cos\varphi$, and the amplitude function φ. Then we have combined three groups of non-elementary functions in one single function: $e^{\lambda\varphi}$.

6. Conclusions

What is new in this paper are the 6 sets of non-elementary functions:

1) *dsn*, *dcn*, *ddn*
2) *ksn*, *kcn*, *kdn*
3) *lsn*, *lcn*, *ldn*
4) *rsn*, *rcn*, *rdn*
5) *bsn*, *bcn*, *bdn*
6) *hsn*, *hcn*, *hdn*

These 6 sets of functions are giving solution to these second-order nonlinear ODEs:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\frac{1}{2}hp\left(1-x^2\right)\cos\left(px\right)+hx\sin\left(px\right)+x=0 \tag{122}$$

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}&=-x\mathrm{e}^{ax}\left(1-k^2\sin^2\left(px\right)\right)\\&\quad+\left(1-x^2\right)\mathrm{e}^{ax}\left(\frac{a}{2}\left(1-k^2\sin^2\left(px\right)\right)-pk^2\sin\left(px\right)\cos\left(px\right)\right)\end{aligned} \tag{123}$$

$$\begin{aligned}&\left(1+v+px-vx^2\right)^3\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\left(1+v+px-vx^2\right)\Bigg[x\left(n-h^2x-k^2+k^2x^2\right)\\&+\left(1-x^2\right)\left(\frac{1}{2}h^2-k^2x\right)\Bigg]-\left(1-x^2\right)\left(2vx-p\right)\left(n-h^2x-k^2+k^2x^2\right)=0\end{aligned} \tag{124}$$

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}&=k\frac{\mathrm{d}x}{\mathrm{d}t}\left(4x^2-1\right)+hx^2\left(1-x^2\right)\left(8x^2-3\right)\frac{\mathrm{d}x}{\mathrm{d}t}+k^2x^3\left(1-x^2\right)\\&\quad+2hkx^5\left(1-x^2\right)^2+h^2x^7\left(1-x^2\right)^3-x\end{aligned} \tag{125}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-hx^2\frac{\mathrm{d}x}{\mathrm{d}t}\left(1-x^2\right)\left(8x^2-3\right)-h^2x^7\left(1-x^2\right)^3+x=0 \tag{126}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-k\frac{\mathrm{d}x}{\mathrm{d}t}\left(4x^2-1\right)-k^2x^3\left(1-x^2\right)+x=0 \tag{127}$$

$$\begin{aligned}&\left(1+fx-h^2x^2\right)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-\left(k\frac{\mathrm{d}x}{\mathrm{d}t}\left(4x^2-1\right)+k^2x^3\left(1-x^2\right)\right)\left(1+fx-h^2x^2\right)\\&+x\left(1+fx-h^2x^2\right)^3-\frac{\mathrm{d}x}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}+kx-kx^3\right)\left(f-2h^2x\right)=0\end{aligned} \tag{128}$$

$$\begin{aligned}&\frac{\mathrm{d}^2x}{\mathrm{d}t^2}-3x\frac{\mathrm{d}x}{\mathrm{d}t}\left(k\sin\left(px\right)+h\cos\left(px\right)\right)-\frac{\mathrm{d}x}{\mathrm{d}t}\left(1-x^2\right)\left(hv\sin\left(vx\right)-kp\cos\left(px\right)\right)\\&-x\left(1-x^2\right)\left(k\sin\left(px\right)+h\cos\left(vx\right)\right)^2+x=0\end{aligned} \tag{129}$$

These second-order nonlinear ODEs are showing some of the strength of the non-elementary functions defined in this paper. They all appear from twice differentiating the solution $x(t)=\cos\varphi$, where $\varphi=\varphi(t)$ is the upper limit of integration in a non-elementary integral, which differ in the different functions.

It is possible to make many non-elementary functions using the Abel's methods described by Armitage and Eberlein, by how they define the Jacobi elliptic functions. In the same way as Jacobi's functions *sn*, *cn*, *dn* and *am* are giving solutions to a few ODEs, the non-elementary functions described in this paper and a lot more, are giving solutions to many different ODEs. I don't see any limit for this subject. The only limit is our imagination.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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