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Local Solution of Three-Dimensional Axisymmetric Supersonic Flow in a Nozzle

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Abstract

In this paper, we construct a local supersonic flow in a 3-dimensional axis-symmetry nozzle when a uniform supersonic flow inserts the throat. We apply the local existence theory of boundary value problem for quasilinear hyperbolic system to solve this problem. The boundary value condition is set in particular to guarantee the character number condition. By this trick, the theory in quasilinear hyperbolic system can be employed to a large range of the boundary value problem.

Keywords

High-Dimensional Axisymmetric Hyperbolic Equations, Supersonic Flow in a Nozzle, Local Solutions to Boundary Value Problems of Quasilinear Hyperbolic Equations

1. Introduction

This paper considers a class of supersonic pipe flow problems. Supposed a three-dimensional axisymmetric pipe with steady uniform supersonic incoming flow at the throat, we consider the existence of local smooth supersonic flow field around there.

The pipe flow problem is the focus of the study for high dimensional conservation law, which is related to the theory and application of fluid mechanics and has important significance in the design of jet engine. This problem has been discussed in a large number of papers, and its research has a long history. In the past 20 years, it has been a hot topic in the study of high-dimensional conservation law.

In [1] [2], the authors discuss the problem of subsonic pipe flow, constructing the global subsonic flow solution by the theory of quasi conformal mapping. In [3] [4], the authors discuss the problem of for the two-dimensional convex pipe flow, constructing a global smooth solution and discussing the existence conditions of vacuum. In a series of papers like [5]-[17], the authors made a lot of work on 2-dimensional transonic flow in a nozzle. In these papers, via a detailed singularity analysis, a local solution is constructed for a class of flow fields with sound velocity at the throat, the subsonic flow accelerates and becomes supersonic across throat. A further problem is the transonic shock wave. We do not go into details here.

2. Statement of the Problem

In this paper, it is assumed that x is the axis of symmetry and y is the distance to x-axis, as shown in **Figure 1**. The gas motion is described by the following three-dimensional steady axisymmetric irrotational Euler equation.

$$\begin{cases} (\rho u)_x + (\rho v)_y + \dfrac{\rho v}{y} = 0, \\ u_y - v_x = 0, \end{cases} \tag{1}$$

where (u,v) is the velocity, ρ is the density function following Bernoulli's law

$$\frac{c^2}{\gamma-1} + \frac{u^2+v^2}{2} = \frac{\bar{q}^2}{2}, \tag{2}$$

where $\bar{q}$ is the limit speed.

We assume that the gas is a polytropic gas. That is, the gas pressure p satisfies $p = A\rho^\gamma$, where γ is the adiabatic exponent belong to $[1,3]$, and $A > 0$ is a constant. The sonic speed $c = \sqrt{\dfrac{\mathrm{d}p}{\mathrm{d}\rho}}$.

Assume the velocity of the incoming flow at the throat is

$$(u,v)\big|_{x=0} = (\underline{u}, \underline{v}) = (\underline{u}, 0). \tag{3}$$

The pipe wall is expressed by

$$\mathrm{W}: y = b(x), \tag{4}$$

where

$$b(0) = 1,\ b'(0) = 0 \ \text{ and } \ b''(x) > 0. \tag{5}$$

On the wall W, we have the solid boundary value condition

$$b' = \frac{v}{u} \ \text{ on } \mathrm{W}. \tag{6}$$

Let Ω be the domain bounded by the wall W and the axis of symmetry $y = 0$. That is

$$\Omega := \{(x,y) \,|\, 0 < y < b(x),\, x > 0\}. \tag{7}$$

We have the following

Basic Problem: For incoming flow (3), with the supersonic assumption as follows

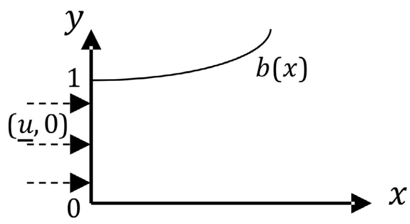

Figure 1. Supersonic axisymmetric flow in a nozzle.

$$\underline{u} > \underline{c}, \tag{8}$$

we intend to construct a continuous flow field (u,v) in $\Omega \cap \{0 < x < \delta\}$, satisfying

1) the wall condition(6);

2) the incoming flow conditions;

3) the Equation (1).

In this paper, we prove

Theorem 1 If incoming flow $\underline{u}$ satisfies (8), then there exists a sufficiently small constant $\delta > 0$, such that there exists a solution for **Basic Problem** in $x \in [0, \delta]$.

In this paper, we construct the local solution of three-dimensional axisymmetric supersonic pipe flow, using [18] theory of local solutions of quasilinear hyperbolic equations. There are two difficulties in solving the problem here.

One is a relatively trivial difficulty, that is, the equation has singularity along the axis of symmetry. As a result, the theory of local solutions to the problem cannot be applied directly. However the incoming flow is uniform, the singularity of the equations on the axis of symmetry can be directly verified by taking a constant states. This difficulty is not complicated when we only consider the local existence of the solution.

The second difficulty is that the characteristic number condition, which is required to prove the existence of the local solution in [18], is not valid. This condition is not easy to guaranteed, it is only a sufficient condition. These limited the application of the results in [18] to a range.

In this paper, we use a deformation to extend the application of characteristic number to a more general range and show the existence of local solution for supersonic flows. Our proof is expected to generalize the results of [18] in further studies.

3. Characteristic Form of Hyperbolic Equations

For the smooth flow field (u,v) in Ω, (1) can be written as

$$\begin{cases} v_x - u_y = 0, \\ \left(1 - \dfrac{u^2}{c^2}\right) u_x - \dfrac{2uv}{c^2} v_x + \left(1 - \dfrac{v^2}{c^2}\right) v_y + \dfrac{v}{y} = 0. \end{cases} \tag{9}$$

Then we rewrite (9) in a matrix form

$$\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} \frac{-2uv}{c^2-u^2} & \frac{c^2-v^2}{c^2-u^2} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_y + \begin{pmatrix} 0 & \frac{c^2}{(c^2-u^2)y} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{10}$$

Under the supersonic assumption $u^2+v^2>c^2$, the above equations have eigenvalues

$$\lambda_\pm := \frac{uv \pm c\sqrt{u^2+v^2-c^2}}{u^2-c^2}. \tag{11}$$

Multiplying the left-hand side of (10) by the eigenvector $l_\pm = (1, \lambda_\mp)$, we obtain

$$\partial^+ u + \lambda_- \partial^+ v + \frac{c^2 v}{(c^2-u^2)y} = 0, \text{ and } \partial^- u + \lambda_+ \partial^- v + \frac{c^2 v}{(c^2-u^2)y} = 0, \tag{12}$$

where derivative operator $\partial^\pm$ is defined as

$$\partial^\pm := \partial_x + \lambda_\pm \partial_y. \tag{13}$$

4. The Proof of Theorem 1

Proof. We are going to divide this boundary value problem into two parts, as shown in Figure 2.

First: calculate the eigenvalue of the incoming flow field

$$\lambda_+(\underline{u},0) = -\lambda_-(\underline{u},0) = \frac{\underline{c}}{\sqrt{\underline{u}^2-\underline{c}^2}} > 0. \tag{14}$$

Therefore, region

$$\Omega_0 := \{(x,y) \mid y \le 1 + \lambda_-(\underline{u},0)x, x \ge 0, y \ge 0\}, \tag{15}$$

is the hyperbolic determining region of the throat flow field. We verify directly that

$$(u,v) = (\underline{u},0) \tag{16}$$

is a C^1 solution of the Equation (12) satisfying incoming flow conditions (3) in Ω_0.

Second: we intend to obtain a C^1 solution of the Equation (9) in region

$$\Omega_1 := \{(x,y) \mid 1 + \lambda_-(\underline{u},0)x \le y \le b(x), 0 \le x \le \delta\} \tag{17}$$

which satisfies:

1) the wall condition (6);

2) the boundary value conditions

$$(u,v) = (\underline{u},0), \text{ and } 1 + \lambda_-(\underline{u},0)x = y. \tag{18}$$

We consider applying Theorem 3.3.1 in [18] to this part. Following the notations in [18], we assume

$$\begin{cases} x = t; \\ y = 1 - x. \end{cases}$$

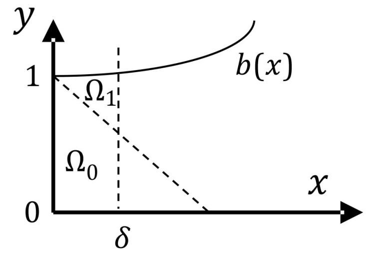

Figure 2. Solve the problem by two regions.

Then, we have a differential relation:

$$\partial_x + \lambda_{\pm}\partial_y = \partial_t - \lambda_{\pm}\partial_x,$$

Therefore

$$u_1 = u,\ \ u_2 = v;$$

$$\lambda_1 = -\lambda_+,\ \ \lambda_2 = -\lambda_-.$$

From the characteristic form (12), we write the corresponding coefficient expression of the boundary value problem in [18]

$$\begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix} = \begin{pmatrix} 1 & \lambda_- \\ 1 & \lambda_+ \end{pmatrix}. \tag{19}$$

Moreover, we have

$$\begin{pmatrix} \zeta^{11} & \zeta^{12} \\ \zeta^{21} & \zeta^{22} \end{pmatrix} = \frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} \lambda_+ & -\lambda_- \\ -1 & 1 \end{pmatrix}. \tag{20}$$

The corresponding boundary value conditions are denoted as

1) The characteristic boundary value condition (18) is set as follows

$$u_1 + \lambda_-^0 u_2 = u + \lambda_-^0 v = \underline{u} =: G_1,\ \ x = -\lambda_-^0 t; \tag{21}$$

2) The wall boundary value condition (6) is set as follows

$$u_1 + \lambda_+^0 u_2 = u + \lambda_+^0 v = u + \lambda_+^0 v + \lambda_+^0\left(b'u - v\right) =: G_2,\ \ x = -b(t) + 1. \tag{22}$$

Then, we have

$$\left.\begin{pmatrix} \dfrac{\partial G_1}{\partial u_1} & \dfrac{\partial G_1}{\partial u_2} \\ \dfrac{\partial G_2}{\partial u_1} & \dfrac{\partial G_2}{\partial u_2} \end{pmatrix}\right|_{x=t=u_2=0,u_1=\underline{u}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Therefore,

$$\left.\begin{pmatrix} \dfrac{\partial G_1}{\partial u_1} & \dfrac{\partial G_1}{\partial u_2} \\ \dfrac{\partial G_2}{\partial u_1} & \dfrac{\partial G_2}{\partial u_2} \end{pmatrix}\begin{pmatrix} \zeta^{11} & \zeta^{12} \\ \zeta^{21} & \zeta^{22} \end{pmatrix}\right|_{x=t=u_2=0,u_1=\underline{u}}$$

$$= \frac{1}{\lambda_+^0 - \lambda_-^0}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \lambda_+^0 & -\lambda_-^0 \\ -1 & 1 \end{pmatrix} = \frac{1}{\lambda_+^0 - \lambda_-^0}\begin{pmatrix} 0 & 0 \\ \lambda_+^0 & -\lambda_-^0 \end{pmatrix}.$$

Further, using the definition of [18], we calculate the minimal characteristic number

$$\theta_{\min} = \inf_{\gamma_1,\gamma_2 \neq 0} \left(\left| \frac{\gamma_2}{\gamma_1} \frac{\lambda_+^0}{\lambda_+^0 - \lambda_-^0} \right| + \left| \frac{-\lambda_-^0}{\lambda_+^0 - \lambda_-^0} \right| \right) = \left| \frac{-\lambda_-^0}{\lambda_+^0 - \lambda_-^0} \right|. \tag{23}$$

Because

$$\lambda_+^0 = -\lambda_-^0 = \lambda_+(\underline{u}, 0) = \frac{\underline{c}}{\sqrt{\underline{u}^2 - \underline{c}^2}}.$$

(23) yields

$$\theta_{\min} = \frac{1}{2} < 1.$$

The proof is complete.

Note The most important step is the construction of the right side of the equation (22). We multiply the wall relation $b'u - v$ by λ_+^0, that ensures the final characteristic number is less than 1. Unless, there's no way to guarantee this. The result inspires us to further optimize and generalize the conclusions in [18].

5. Conclusions and Suggestions

In this paper, we use a new method of boundary value problems for quasilinear hyperbolic systems to successfully solve the local solution of 3D axisymmetric supersonic flow in a nozzle with a singularity on its symmetrical axis. Especially, we make an equivalent linear transformation (22) of its boundary condition to satisfy the sufficient condition of the existence theory of local solution in [18]'s Theorem 3.3.1, which shall give others new ideas to solve this type of problem.

On the other hand, we can generalize the formula (22) to $u + \lambda_+^0 = u + \lambda_+^0 + K(b'u - v) := G_2$, which inspires us to improve the existence theorem in [18], in order to be applied to more typical boundary value problems.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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