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Incompressible Limit of the Oldroyd-B Model with Density-Dependent Viscosity

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Abstract

This paper studies the existence and uniqueness of local strong solutions to an Oldroyd-B model with density-dependent viscosity in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, via incompressible limit, in which the initial data is "well-prepared" and the velocity field enjoys the slip boundary conditions. The main idea is to derive the uniform energy estimates for nonlinear systems and corresponding incompressible limit.

Keywords

Incompressible Limit, Oldroyd-B Model, Slip Boundary Condition, Density-Dependent Viscosity

1. Introduction

The Oldroyd-B model is a fundamental set of equations in the field of fluid dynamics, which is used to characterize the motion of fluids that display complex viscoelastic behavior under the influence of strain. In this paper, we consider the Oldroyd-B model with density-dependent viscosity in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3. For the incompressible fluids of Oldroyd-B type, the governing equations are of the following form (see [1] [2], for instance):

$$\operatorname{div} u = 0, \tag{1}$$

$$u_t + u \cdot \nabla u + \nabla q = \mu \Delta u + \mu_1 \operatorname{div} \tau, \tag{2}$$

$$\tau_t + u \cdot \nabla \tau + a\tau + Q(\tau, \nabla u) = \mu_2 \Gamma(\nabla u), \tag{3}$$

where $u = (u_1, \cdots, u_d)$, q, τ are velocity, pressure and elastic part of the tangential stress tensor, respectively, and the density is usually set to be 1 without the loss of generality; ∇ denotes the gradient of a scalar field, and Δ is Laplace operator, a, μ_1, μ_2 and the viscosity coefficient μ are positive constants, and

$$\Gamma(\nabla u)=\frac{1}{2}\left(\nabla u+(\nabla u)^{\mathrm{T}}\right),$$

$$Q(\tau,\nabla u)=\tau W(u)-W(u)\tau-b\left(\Gamma(\nabla u)\tau+\tau\Gamma(\nabla u)\right),$$

where $W(u)=\frac{1}{2}\left(\nabla u-(\nabla u)^{\mathrm{T}}\right)$ and $b\in[-1,1]$.

The motion of compressible fluids is governed by the following nonlinear equations:

$$\rho_t^\lambda+u^\lambda\cdot\nabla\rho^\lambda+\rho^\lambda\operatorname{div}u^\lambda=0, \tag{4}$$

$$\begin{aligned} &u_t^\lambda+u^\lambda\cdot\nabla u^\lambda+\frac{1}{\rho^\lambda}\nabla P^\lambda \\ &=\frac{1}{\rho^\lambda}\left[\operatorname{div}\left(2\mu(\rho)D\left(u^\lambda\right)\right)+\nabla\left(\lambda(\rho)\operatorname{div}u^\lambda\right)\right]+\frac{\mu_1}{\rho^\lambda}div\tau^\lambda, \end{aligned} \tag{5}$$

$$\tau_t^\lambda+u^\lambda\cdot\nabla\tau^\lambda+a\tau^\lambda+\frac{1}{\rho^\lambda}Q\left(\tau^\lambda,\nabla u^\lambda\right)=\frac{\mu_2}{\rho^\lambda}\Gamma\left(\nabla u^\lambda\right), \tag{6}$$

where ρ^λ, u^λ, τ^λ are density, velocity and elastic stress tensor, respectively, $\lambda=1/\varepsilon$ is the non-dimensional constant, and ε is Mach number. Moreover, the pressure P^λ is given by the equation of states $P^\lambda(\rho)=\lambda^2p(\rho)$, where $p(\rho)$ satisfies $p'(\rho)>0$ for $\rho>0$. The constants $\mu(\rho)$ and $\lambda(\rho)$ are viscosity constants with $\mu(\rho)>0$, $\lambda(\rho)\geq 0$ and $\mu(\rho)+d\lambda(\rho)/2>0$. For the derivation of the non-dimensional system (4)-(6), one may refer to [3] for the details.

In the physical standpoint, as the Mach number tends to zero, the solutions of the compressible system (4)-(6) converge to the solutions of the incompressible system (1)-(3), it is known as the incompressible limit. However, rigorously proving this limit process mathematically is a challenging problem. Since Ebin [4] in the 1970s, researchers have made a lot of research achievements on the incompressible limit of hydrodynamic models. Klainerman and Majda [5] establish a general framework for studying the incompressible limits of locally smooth solutions. For further research on the incompressible limit problem of hydrodynamics, refer to [6]-[12].

Currently, significant progress has been made in the research outcomes of the Oldroyd-B model for the viscoelastic fluids. It is well known that long time existence of solutions to the viscoelastic equations depends on strong dispersive estimates (see [13] [14] for instance). The results by Klainerman [13] provided some answers for the wave equations based on the Lorentz invariance, and Sideris and Thomases [15] studied non-Lorentz invariant systems in three-dimensional space using weighted estimates method. However, none of these methods can be applied to Oldroyd-B model due to the existence of damping mechanisms. In H^s, the existence and uniqueness of local strong solutions of incompressible fluid satisfying the Oldroyd constitutive law are given by Guillopé and Saut [16]. In Besov spaces, Chemin and Masmoudi [2] studied the existence and uniqueness for local and global solutions.

In addition, Fang and Zi [17] investigated the incompressible limit of the Oldroyd-B model in the full space when the initial data and coupling constants are sufficiently small. They verify that when the Mach number tends to zero, the global solutions of compressible Oldroyd-B model converge to the solutions of the corresponding incompressible model. This demonstrates the existence of the solutions for the incompressible model, and the uniform estimates for the convergence rate are obtained. Lei [18] proved the incompressible limit of the Oldroyd-B model with small and "well-prepared" initial data in $\mathbb{T}^n$, as well as the local and global existence of classical solutions. For the Oldroyd-B model in bounded domain, Ren and Ou [19] proved the incompressible limit of local strong solutions. It is worth noting that the existing conclusions on the incompressible limit problem show that the viscosity coefficient is constant, while the case of density-dependent viscosity has not been studied.

In this paper, the incompressible limit of local strong solutions of compressible Oldroyd-B model in a bounded domain $\Omega\subset\mathbb{R}^2$ or $\mathbb{R}^3$ will be studied when the viscosity coefficient depends on the density, so as to prove the existence and uniqueness of local strong solutions of the compressible Oldroyd-B model. In a sense, it extends the result of constant viscosity coefficient in Ren and Ou [19] to the case where the viscosity coefficient depends on density. Moreover, compared with Ren and Ou [19], the density-dependent viscosity will bring more difficulties to energy estimate. This is due to the fact that the boundary effects produce more troubles in the estimates for high-order derivatives. The main idea is to derive a uniform spatial-time energy estimate for the linearized system of (4)-(6) which yields the uniform estimates for the nonlinear system (4)-(6) and the corresponding incompressible limit, provided that the initial data are well prepared and uniformly bounded with respect to the Mach number.

We impose the following initial condition

$$\left(\rho^\lambda,u^\lambda,\tau^\lambda\right)\Big|_{t=0}=\left(\rho_0^\lambda,u_0^\lambda,\tau_0^\lambda\right)(x),\quad x\in\Omega, \tag{7}$$

and the slip boundary condition

$$\begin{cases}u^\lambda\cdot n=0 \quad \text{on } \partial\Omega,\\ \text{curl}u^\lambda=0\,(d=2)\ \text{ or }\ n\times\text{curl}u^\lambda=0\,(d=3) \quad \text{on } \partial\Omega,\end{cases} \tag{8}$$

where $\Omega\in R^d\,(d=2,3)$ is the bounded domain with smooth boundary $\partial\Omega$, n is the unit outer normal, and the vorticity $\text{curl}u^\lambda=\partial_1u_2^\lambda-\partial_2u_1^\lambda$ for $d=2$ or $\text{curl}u^\lambda=\left(\partial_2u_3^\lambda-\partial_3u_2^\lambda,\partial_3u_1^\lambda-\partial_1u_3^\lambda,\partial_1u_2^\lambda-\partial_2u_1^\lambda\right)^{\mathrm{T}}$ for $d=3$. The boundary condition (8) is a particular case of Navier's slip boundary conditions which describe the interaction between a fluid and a wall,

$$u\cdot n=0,\quad s\cdot\mathbb{S}(u)\cdot n+\alpha u\cdot s=0 \quad \text{on } \partial\Omega, \tag{9}$$

where s is any unit tangential direction to $\partial\Omega$.

The main results of this paper are as follows.

Theorem 1.1 *Let* $\Omega \subset \mathbb{R}^d\ (d=2,3)$ *be a bounded domain with smooth boundary* $\partial\Omega$. *Suppose that the initial datum* $\left(\rho_0^\lambda(x), u_0^\lambda(x), \tau_0^\lambda(x)\right)$ *satisfies that for* $\lambda \ge \lambda_0$, *where* $\lambda_0 > 0$ *is a sufficiently large constant,*

$$\sum_{i=0}^{2}\left\|\left(\partial_i^t\left(\lambda\left(\rho^\lambda(0)-1\right)\right), \partial_i^t u^\lambda(0), \partial_i^t \tau^\lambda(0)\right)\right\|_{H^{2-i}(\Omega)} \le \tilde{C}. \tag{10}$$

Assuming that the following compatibility conditions are satisfied for $i=0,1$ *and* 2,

$$\begin{aligned}&\partial_t^i u^\lambda(0)\cdot n = 0 \quad \text{on } \partial\Omega,\\ &\text{curl}\,\partial_t^i u^\lambda(0) = 0\ (d=2) \text{ or } n\times \text{curl}\,\partial_t^i u^\lambda(0) = 0\ (d=3) \quad \text{on } \partial\Omega,\end{aligned} \tag{11}$$

There are positive constants $T_0 = T_0\left(d,\Omega,\tilde{C}\right)$ *and* $C_0 = C_0\left(d,\Omega,\tilde{C}\right)$ *independent of* $\lambda \ge \lambda_0$, *which make the initial-boundary problem* (4)-(8) *admit a unique solution* $\left(\rho^\lambda, u^\lambda, \tau^\lambda\right)$ *satisfying*

$$\sum_{i=0}^{2}\left\|\left(\partial_i^t\left(\lambda\left(\rho^\lambda-1\right)\right), \partial_i^t u^\lambda, \partial_i^t \tau^\lambda\right)(t)\right\|_{H^{2-i}(\Omega)} + \left\|\partial_i^t u^\lambda\right\|_{L^2\left(0,T_0;H^{3-i}(\Omega)\right)} \le C_0,\ 0 \le t \le T_0. \tag{12}$$

Remark 1.1 *The initial data of the time derivatives* $\rho_t^\lambda(0), u_t^\lambda(0), \tau_t^\lambda(0)$ *are determined by* (4)-(6) *and* $\rho_0^\lambda, u_0^\lambda, \tau_0^\lambda$, *e.g.,* $\rho_t^\lambda(0) = -\left(u_0^\lambda\cdot\nabla\rho_0^\lambda + \rho_0^\lambda \text{div} u_0^\lambda\right)$. *Similarly,* $\rho_{tt}^\lambda(0), u_{tt}^\lambda(0), \tau_{tt}^\lambda(0)$ *are determined by* (4)-(6) *and the initial data of the lower order time derivatives, e.g.,*

$$\rho_{tt}^\lambda(0) = -\left[u_t^\lambda(0)\cdot\nabla\rho_0^\lambda + u_0^\lambda\cdot\nabla\rho_t^\lambda(0) + \rho_t^\lambda(0)\text{div} u_0^\lambda + \rho_0^\lambda \text{div} u_t^\lambda(0)\right].$$

Theorem 1.2 *Let all the assumptions in Theorem* 1 *be satisfied. Then, the solution* $\left(\rho^\lambda, u^\lambda, \tau^\lambda\right)$ *to* (4)-(8) *satisfies that as* $\lambda \to \infty$,

$$\rho^\lambda \to 1 \text{ in } L^\infty\left(0,T_0;H^2(\Omega)\right)\cap \text{Lip}\left([0,T_0],H^1(\Omega)\right), \tag{13}$$

$$\left(u^\lambda,\tau^\lambda\right)\to(u,\tau) \text{ weakly}-* \text{ in } L^\infty\left(0,T_0;H^2(\Omega)\right)\cap \text{Lip}\left([0,T_0],H^1(\Omega)\right), \tag{14}$$

$$\left(u^\lambda,\tau^\lambda\right)\to(u,\tau) \text{ in } C\left([0,T_0],H^{2-\delta}(\Omega)\right),\ \forall 0<\delta<1, \tag{15}$$

where (u,τ) *is the unique solution to the following incompressible Oldroyd-B system*

$$\text{div} u = 0, \tag{16}$$

$$\Pi\left(u_t + u\cdot\nabla u - \gamma\Delta u - \mu_1 \text{div}\tau\right) = 0, \tag{17}$$

$$\tau_t + u\cdot\nabla\tau + a\tau + Q(\tau,\nabla u) = \mu_2\Gamma(\nabla u), \tag{18}$$

in $\Omega\times[0,T_0]$ *associated with the initial condition* $(u,\tau)|_{t=0} = \left(u_0(x), \tau_0(x)\right)$ *and the slip boundary conditions* (8), *where* Π *is the Leray-projection on the divergence free vector fields and* (u_0,τ_0) *is the weak limit of* $\left(u_0^\lambda, \tau_0^\lambda\right)$ *in* $H^2(\Omega)$.

2. Preliminaries and the Linearized Problem

During the subsequent proof, the following result is needed

$$\Delta u = \nabla\text{div} u - \overrightarrow{\text{curl}}\text{curl} u = 2\text{div}\left(D(u)\right) - \nabla\text{div} u,\ \forall u = \left(u_1,u_2,u_3\right)^{\text{T}}, \tag{19}$$

where $\overrightarrow{\text{curl}} = (\partial_2, -\partial_1)^{\mathrm{T}}$ for $d=2$, $\overrightarrow{\text{curl}} = \text{curl}$ for $d=3$.

Lemma 2.1 ([20]) *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outward normal n. There is a constant $C>0$ independent of u, such that*

$$\|u\|_{H^s(\Omega)} \le C\left(\|\text{div}u\|_{H^{s-1}(\Omega)} + \|\text{curl}u\|_{H^{s-1}(\Omega)} + \|u\cdot n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \|u\|_{H^{s-1}(\Omega)}\right), \tag{20}$$

for any $u \in H^s(\Omega)$.

Lemma 2.2 ([21]) *Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outward normal n. There is a constant $C>0$ independent of u, such that*

$$\|u\|_{H^s(\Omega)} \le C\left(\|\text{div}u\|_{H^{s-1}(\Omega)} + \|\text{curl}u\|_{H^{s-1}(\Omega)} + \|u\times n\|_{H^{s-\frac{1}{2}}(\partial\Omega)} + \|u\|_{H^{s-1}(\Omega)}\right), \tag{21}$$

for any $u \in H^s(\Omega)^N$.

Remark 2.1 *The general form of the conclusions of Lemma* 2.1 *and Lemma* 2.2 *in the range* [20] [21] *is: there is a constant $C>0$ independent of u, such that*

$$\|u\|_{W^{s,p}(\Omega)} \le C\left(\|\text{div}u\|_{W^{s-1,p}(\Omega)} + \|\text{curl}u\|_{W^{s-1,p}(\Omega)} + \|u\cdot n\|_{W^{s-\frac{1}{p},p}(\partial\Omega)} + \|u\|_{W^{s-1,p}(\Omega)}\right)$$

and

$$\|u\|_{W^{s,p}(\Omega)} \le C\left(\|\text{div}u\|_{W^{s-1,p}(\Omega)} + \|\text{curl}u\|_{W^{s-1,p}(\Omega)} + \|u\times n\|_{W^{s-\frac{1}{p},p}(\partial\Omega)} + \|u\|_{W^{s-1,p}(\Omega)}\right)$$

for any $u \in W^{s,p}(\Omega)$.

Lemma 2.3 ([3]). *If $f:\mathbb{R}^n \to \mathbb{R}$ is a smooth function with $f(0)=0$, then for any $k\in\mathbb{N}$, we have $f(u)\in H^k \cap L^\infty$ with*

$$\|f(u)\|_{H^k} \le C\|u\|_{H^k},$$

provided that $u \in H^k \cap L^\infty$, where C depends only on f, k and $\|u\|_{L^\infty}$.

It follows that for any smooth function $F(\cdot)$ and any $u\in H^k\cap L^\infty$, we can deduce that

$$\|f(u)\|_{L^\infty} \le C\|F(0) + (F(u)-F(0))\|_{H^k} \le C\left(1+\|u\|_{H^k}\right). \tag{22}$$

Lemma 2.4 ([22], Part 1, Theorem 10.1) *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with C^k-boundary, and let u be any function in $W^{k,r}(\Omega)\cap L^q(\Omega)$ with $1\le r,q\le\infty$. For any integer j with $0\le j<k$, and for any number a in the interval $[j/k,1]$, set*

$$\frac{1}{p} = \frac{j}{N} + a\left(\frac{1}{r} - \frac{k}{N}\right) + (1-a)\frac{1}{q}.$$

If $k-j-N/r$ is not a nonnegative integer, then

$$\|D^j u\|_{L^p(\Omega)} \le C\|u\|^a_{W^{k,r}(\Omega)}\|u\|^{1-a}_{L^q(\Omega)}. \tag{23}$$

If $k-j-N/r$ is a nonnegative integer, then (23) *only holds for $a=j/k$. The constant C depends only on Ω, r, q, k, j, a.*

In the following, we present some specific cases of Equation (23) in R^2 or R^3.

$$\|\nabla^2 u\|_{L^2} \le C\|u\|_{H^3}^{\frac{2}{3}}\|u\|_{L^2}^{\frac{1}{3}}, \quad \|\nabla^2 u\|_{L^2} \le C\|u\|_{H^3}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}},$$

$$\|u\|_{L^3} \le C\|u\|_{H^1}^{\frac{1}{2}}\|u\|_{L^2}^{\frac{1}{2}}, \quad \|u\|_{L^4} \le C\|u\|_{H^1}^{\frac{3}{4}}\|u\|_{L^2}^{\frac{1}{4}}.$$

Furthermore, according to the Sobolev embedding theorem, we have

$$\|u\|_{L^\infty} \le C\|u\|_{W^{1,4}} \le C\|u\|_{H^2}^{\frac{3}{4}}\|u\|_{H^1}^{\frac{1}{4}} \le C\|u\|_{H^2}^{\frac{7}{8}}\|u\|_{L^2}^{\frac{1}{8}}.$$

To simplify the calculations in this chapter, we will ignore the superscripts in Equations (4)-(6). Let's consider the linearization problem of Equations (4)-(6):

$$\rho_t + v\cdot\nabla\rho + \xi \operatorname{div} u = 0, \tag{24}$$

$$\begin{aligned} &u_t + v\cdot\nabla u + p'(\xi)\xi^{-1}\lambda^2\nabla\rho \\ &= \frac{1}{\xi}\left[\operatorname{div}\left(2\mu(\xi)D(u)\right) + \nabla\left(\lambda(\xi)\operatorname{div} u\right)\right] + \frac{\mu_1}{\xi}\operatorname{div}\tau, \end{aligned} \tag{25}$$

$$\tau_t + v\cdot\nabla\tau + a\tau + \frac{1}{\xi}Q(\tau,\nabla v) = \frac{\mu_2}{\xi}\Gamma(\nabla u), \tag{26}$$

where (ρ,u,τ) satisfies (7) and (8). When $\lambda \ge \lambda_0$ and $i = 0,1,2$, for any $T > 0$, (ξ,v) and (ξ,v) satisfy the following inequalities:

$$\left\|\left(\partial_i^t\left(\lambda(\xi-1)\right), \partial_i^t v\right)\right\|_{C\left([0,T],H^{2-i}(\Omega)\right)} \le M, \tag{27}$$

$$\left\|\xi^{-1}\right\|_{L^\infty\left(0,T;L^\infty(\Omega)\right)} \le M, \tag{28}$$

$$\left\|\partial_i^t v\right\|_{L^2\left(0,T;H^{3-i}(\Omega)\right)} \le M, \tag{29}$$

where ξ, v is a known function dependent on λ, $H^0(\Omega) = L^2(\Omega)$. Applying the method in [7] [15], we find that the existence of solutions (ρ,u,τ) to the linearized problem (24)-(26) satisfying the initial margin value conditions (7) and (8) in bounded regions $\Omega\times[0,T]$ can be proved, which is omitted here. In the following, we will derive the uniform estimate for the solution (ρ,u,τ) of the linearized system of equations with respect to λ.

Lemma 2.5 *Let* (ρ,u,τ) *be the solution to the linearized problem* (24)-(26) *with* (7) *and* (8) *in* $\Omega\times[0,T]$ *satisfying the initial conditions* (10), *which are defined recursively by* (24)-(26), *and the compatibility conditions* (11). *Then the solution* (ρ,u,τ) *to the linearized problem satisfies the uniform-on-*λ *estimates* (12).

Remark 2.2 *At the end of this section, we will introduce some notations for energy estimate. Norms* $W^{k,p}$ *and* H^k *usually denote the commonly used Sobolev spaces, and the positive constant C,* $C_i\,(i=0,1,\cdots)$ *does not depend on* λ. *In addition, we usually assume that the constant* $\eta > 0$, *and the constant* $C_\eta > 0$ *only depends on* η. *It is worth noting that C and* $C_\eta > 0$ *depend on*

M. ∂_i *and* ∂_{ij} *denote* $\frac{\partial}{\partial x_i}$ *and* $\frac{\partial}{\partial x_{ij}}$ *respectively.*

3. Uniform Estimates for the Linearized Problem

3.1. The Basic Estimate

In this section, we derive the uniform-on-λ estimates for (ρ,u,τ) to the linearized problem, which is stated in lemma 2.

Lemma 3.1 *The following inequality holds*:

$$\begin{aligned}&\left\|\left(\lambda(\rho-1),u,\tau\right)\right\|_{L^2}^2(t)+\int_0^t\left(\|\tau\|_{L^2}^2+\|\mathrm{div}u\|_{L^2}^2+\|\mathrm{curl}u\|_{L^2}^2\right)\mathrm{d}x\\&\le \mathrm{e}^{Ct}\left\|\left(\lambda(\rho_0-1),u_0,\tau_0\right)\right\|_{L^2}^2,\forall 0\le t\le T.\end{aligned}\tag{30}$$

Proof. Multiply both sides of Equations (24), (25) and (26) simultaneously by $\lambda^2(\rho-1)$, $\xi^2p'(\xi)^{-1}u$ and τ respectively, then summarizing the integrals of the resulting equations on Ω, and finally by integration by parts we obtain that:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\left(\lambda(\rho-1),\tau\right)\right\|_{L^2}^2+\left\|\xi\sqrt{p'(\xi)^{-1}}u\right\|_{L^2}^2\right]+a\|\tau\|_{L^2}^2\\&+\int_\Omega\xi p'(\xi)^{-1}\left[\left(2\mu(\xi)+\lambda(\xi)\right)|\mathrm{div}u|^2+\mu(\xi)|\mathrm{curl}u|^2\right]\mathrm{d}x=\sum_{i=1}^5 I_i,\end{aligned}\tag{31}$$

where

$$\begin{aligned}|I_1|&=\left|\frac{1}{2}\int_\Omega\partial_t\left(\xi^2p'(\xi)^{-1}\right)|u|^2\mathrm{d}x+\int_\Omega\lambda^2(\rho-1)u\cdot\nabla\xi\mathrm{d}x\right|\\&\le C\|\xi_t\|_{L^4}\|\xi\|_{L^\infty}\|u\|_{L^2}\|u\|_{L^4}+C\|\xi\|_{L^\infty}\|\xi\|_{L^\infty}\|\xi_t\|_{L^4}\|u\|_{L^2}\|u\|_{L^4}\\&\quad+C\|\lambda(\rho-1)\|_{L^2}\|u\|_{L^4}\|\nabla\xi\|_{L^4}\\&\le\eta\|u\|_{H^1}^2+C_\eta\left(\|\xi\|_{H^2}^2\|\xi_t\|_{H^1}^2\|u\|_{L^2}^2+\|\nabla\xi\|_{H^1}^2\|\lambda(\rho-1)\|_{L^2}^2\right)\\&\le\eta\|u\|_{H^1}^2+C_\eta\left(\|u\|_{L^2}^2+\|\lambda(\rho-1)\|_{L^2}^2\right),\end{aligned}$$

$$\begin{aligned}|I_2|&=\frac{1}{2}\left|\int_\Omega\left[\lambda^2\mathrm{div}v\left((\rho-1)^2+|\tau|^2\right)+\mathrm{div}\left(\xi^2p'(\xi)^{-1}v\right)|u|^2\right]\mathrm{d}x\right|\\&\le C\|\nabla v\|_{L^\infty}\left(\|\lambda(\rho-1)\|_{L^2}\|\lambda(\rho-1)\|_{L^2}+\|\tau\|_{L^2}\|\tau\|_{L^2}\right)\\&\quad+C\|\xi\|_{L^\infty}\|\nabla\xi\|_{L^4}\|v\|_{L^\infty}\|u\|_{L^2}\|u\|_{L^4}+C\|\xi\|_{L^\infty}\|\xi\|_{L^\infty}\|\nabla\xi\|_{L^4}\|v\|_{L^\infty}\|u\|_{L^2}\|u\|_{L^4}\\&\quad+C\|\xi\|_{L^\infty}\|\xi\|_{L^\infty}\|\nabla v\|_{L^\infty}\|u\|_{L^2}\|u\|_{L^2}\\&\le\eta\|u\|_{H^1}^2+C_\eta\|u\|_{L^2}^2+C\|v\|_{H^3}\left\|\left(\lambda(\rho-1),u,\tau\right)\right\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}|I_3|&=\left|\mu_1\int_\Omega\left[\partial_j\left(\xi p'(\xi)^{-1}\right)u_i\tau_{ij}+\xi p'(\xi)^{-1}\nabla u:\tau\right]\mathrm{d}x\right|\\&\le C\|\nabla\xi\|_{L^4}\|u\|_{L^4}\|u\|_{L^2}+C\|\xi\|_{L^\infty}\|\nabla\xi\|_{L^4}\|u\|_{L^4}\|u\|_{L^2}+C\|\xi\|_{L^\infty}\|\nabla u\|_{L^2}\|\tau\|_{L^2}\\&\le\eta\|u\|_{H^1}^2+C_\eta\|\tau\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}|I_4|&=\left|\int_\Omega\xi^{-1}\tau\cdot\left[\mu_2\Gamma(\nabla u)-Q(\tau,\nabla v)\right]\mathrm{d}x\right|\\&\le C\left\|\xi^{-1}\right\|_{L^\infty}\|\tau\|_{L^2}\left(\|\nabla u\|_{L^2}+\|\nabla v\|_{L^\infty}\|\tau\|_{L^2}\right)\\&\le\eta\|\nabla u\|_{L^2}^2+C_\eta\|\tau\|_{L^2}^2+C\|v\|_{H^3}\|\tau\|_{L^2}^2,\end{aligned}$$

$$
\begin{aligned}
|I_5| &= \left|-\int_\Omega u\left[\nabla\left(\xi p'(\xi)^{-1}\left(2\mu(\xi)+\lambda(\xi)\right)\right)\mathrm{div}u+\nabla\left(\xi p'(\xi)^{-1}\mu(\xi)\right)\times\mathrm{curl}u\right]\mathrm{d}x\right| \\
&\quad+\left|\int_\Omega \xi p'(\xi)^{-1}u\left[2\nabla\left(\mu(\xi)\right)D(u)+\nabla\left(\lambda(\xi)\right)\mathrm{div}u\right]\mathrm{d}x\right| \\
&\le C\|\nabla\xi\|_{L^6}\|\nabla u\|_{L^2}\|u\|_{L^3}+C\|\xi\|_{L^\infty}\|u\|_{L^3}\|\nabla\xi\|_{L^6}\left(\|D(u)\|_{L^2}+\|\nabla u\|_{L^2}\right) \\
&\le \eta\|u\|_{H^1}^2+C_\eta\|u\|_{H^1}\|u\|_{L^2}\|\xi\|_{H^2}^2 \\
&\le \eta\|u\|_{H^1}^2+C_\eta\|u\|_{L^2}^2.
\end{aligned}
$$

Note that lemma 2 is used in the above estimation, and the symmetry of τ is used for estimating I_3. Thus, the lemma is proved by Grönwall's inequality [23] [24] and lemma 2. □

3.2. The Estimates of Low-Order Derivatives

Lemma 3.2 *The following inequality holds*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\mathrm{div}u\right\|_{L^2}^2+\left\|\sqrt{\mu(\xi)}\mathrm{curl}u\right\|_{L^2}^2\right] \\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \xi u_t\cdot u\mathrm{d}x+\left\|\sqrt{P'(\xi)\xi^{-1}}\lambda\rho_t\right\|_{L^2}^2 \\
&\le \eta\left(\|u\|_{H^1}^2+\|u_t\|_{H^1}^2+\|\nabla u\|_{H^1}^2\right)+C_\eta\left(\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{L^2}^2+\left\|(\lambda(\rho-1),u)\right\|_{H^1}^2\right).
\end{aligned}
\tag{32}
$$

Proof. Integrating $u\cdot\partial_t(\xi(24))$ on Ω, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\mathrm{div}u\right\|_{L^2}^2+\left\|\sqrt{\mu(\xi)}\mathrm{curl}u\right\|_{L^2}^2\right] \\
&+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \xi u_t\cdot u\mathrm{d}x+\left\|\sqrt{P'(\xi)\xi^{-1}}\lambda\rho_t\right\|_{L^2}^2 \\
&=\int_\Omega \lambda^2p''(\xi)\nabla\xi\rho_t\cdot u\mathrm{d}x+\int_\Omega \xi u_t\cdot u_t\mathrm{d}x-\int_\Omega\left[(\xi v)_t\cdot\nabla u+\lambda^2p''(\xi)\xi_t\nabla\rho\right]\cdot u\mathrm{d}x \\
&-\int_\Omega \xi(v\cdot u)u_t\cdot u\mathrm{d}x+\mu_1\int_\Omega \mathrm{div}\tau_t\cdot u\mathrm{d}x-\int_\Omega\nabla\left(2\mu(\xi)+\lambda(\xi)\right)\mathrm{div}u_t\cdot u\mathrm{d}x \\
&-\int_\Omega\nabla\left(\mu(\xi)\right)\times\mathrm{curl}u_t\cdot u\mathrm{d}x+\frac{1}{2}\int_\Omega\partial_t\left(2\mu(\xi)+\lambda(\xi)\right)|\mathrm{div}u|^2\mathrm{d}x \\
&+\frac{1}{2}\int_\Omega\partial_t\left(\mu(\xi)\right)|\mathrm{curl}u|^2\mathrm{d}x+\int_\Omega\left[2\nabla\left(\mu(\xi)\right)D(u_t)+\nabla\left(\lambda(\xi)\right)\mathrm{div}u_t\right]\cdot u\mathrm{d}x \\
&+\int_\Omega\left[\mathrm{div}\left(2\partial_t\left(\mu(\xi)\right)D(u)\right)+\nabla\left(\partial_t\left(\lambda(\xi)\right)\mathrm{div}u\right)\right]\cdot u\mathrm{d}x \\
&=\sum_{i=1}^{6}J_i,
\end{aligned}
$$

According to (24), it follows that

$$
\begin{aligned}
|J_1| &= C\left(\|\lambda\rho_t\|_{L^2}^2+\left(1+\|\xi\|_{H^2}^2\right)\|\lambda\nabla\xi\|_{H^1}^2\|u\|_{H^1}^2\right) \\
&\le C\left(\|\lambda\rho_t\|_{L^2}^2+\|u\|_{H^1}^2\right),
\end{aligned}
$$

$$
|J_2| = C\|\xi\|_{L^\infty}\|u_t\|_{L^2}^2\le C\|u_t\|_{L^2}^2,
$$

$$
\begin{aligned}
|J_3| &= C\left[\left(\|\xi\|_{L^\infty}^2\|v_t\|_{H^1}^2+\|v\|_{L^\infty}^2\|\xi_t\|_{H^1}^2\right)\|\nabla u\|_{L^2}^2\right. \\
&\quad\left.+\|p''(\xi)\|_{L^\infty}^2\|\lambda\xi_t\|_{H^1}^2\|\lambda\nabla\rho\|_{L^2}^2+\|u\|_{H^1}^2\right] \\
&\le C\left(\|\lambda\nabla\rho\|_{L^2}^2+\|u\|_{H^1}^2\right),
\end{aligned}
$$

$$\begin{aligned}|J_4| &\le \left|\int_\Omega \operatorname{div}(\xi v)u_t\cdot u\mathrm{d}x\right| + \left|\int_\Omega \xi v\cdot u_t \operatorname{div}u\mathrm{d}x\right| \\ &\le C\left(\|u\|_{H^1}^2 + \left(\|\xi\|_{L^\infty}^2\|\operatorname{div}v\|_{H^1}^2 + \|v\|_{L^\infty}^2\|\nabla\xi\|_{H^1}^2\right)\|u_t\|_{L^2}^2 + \|\xi\|_{L^\infty}^2\|v\|_{L^\infty}^2\|u_t\|_{L^2}^2\right) \\ &\le C\left(\|u\|_{H^1}^2 + \|u_t\|_{L^2}^2\right),\end{aligned}$$

In addition, based on the symmetry of τ and the boundary condition $u\cdot n|_{\partial\Omega}=0$, it can be deduced that

$$|J_5| = \left|\int_\Omega \tau_t : \nabla u\mathrm{d}x\right| \le C\left(\|\tau_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right).$$

$$\begin{aligned}|J_6| &= \left|-\int_\Omega \nabla\left(2\mu(\xi)+\lambda(\xi)\right)\operatorname{div}u_t\cdot u\mathrm{d}x - \int_\Omega \nabla\left(\mu(\xi)\right)\times\operatorname{curl}u_t\cdot u\mathrm{d}x\right| \\ &\quad + \left|\frac{1}{2}\int_\Omega \partial_t\left(2\mu(\xi)+\lambda(\xi)\right)|\operatorname{div}u|^2 dx + \frac{1}{2}\int_\Omega \partial_t\left(\mu(\xi)\right)|\operatorname{curl}u|^2\mathrm{d}x\right| \\ &\quad + \left|\int_\Omega\left[2\nabla\left(\mu(\xi)\right)D(u_t) + \nabla\left(\lambda(\xi)\right)\operatorname{div}u_t + \operatorname{div}\left(2\partial_t\left(\mu(\xi)\right)D(u)\right)\right]\cdot u\mathrm{d}x \right.\\ &\quad \left. + \int_\Omega \nabla\left(\partial_t\left(\lambda(\xi)\right)\operatorname{div}u\right)\cdot u\mathrm{d}x\right| \\ &\le \eta\left(\|u\|_{H^1}^2 + \|u_t\|_{H^1}^2 + \|\nabla u\|_{H^1}^2\right) + C_\eta\left(\left\|(\lambda\rho_t, u_t, \tau_t)\right\|_{L^2}^2 + \left\|(\lambda(\rho-1), u)\right\|_{H^1}^2\right).\end{aligned}$$

A direct calculation shows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\operatorname{div}u\right\|_{L^2}^2 + \left\|\sqrt{\mu(\xi)}\operatorname{curl}u\right\|_{L^2}^2\right] \\ &+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \xi u_t\cdot u\mathrm{d}x - \int_\Omega \lambda^2 P'(\xi)\rho_t\operatorname{div}u\mathrm{d}x \\ &\le \eta\left(\|u\|_{H^1}^2 + \|u_t\|_{H^1}^2 + \|\nabla u\|_{H^1}^2\right) + C_\eta\left(\left\|(\lambda\rho_t, u_t, \tau_t)\right\|_{L^2}^2 + \left\|(\lambda(\rho-1), u)\right\|_{H^1}^2\right).\end{aligned} \tag{33}$$

On the other hand, integrating the product of (24) with $\lambda^2 p'(\xi)\xi^{-1}\rho_t$ yields

$$\left\|\sqrt{p'(\xi)\xi^{-1}}\lambda\rho_t\right\|_{L^2}^2 + \int_\Omega \lambda^2 p'(\xi)\rho_t\operatorname{div}u\mathrm{d}x \le \eta\|\lambda\rho_t\|_{L^2}^2 + C_\eta\|\lambda(\rho-1)\|_{H^1}^2. \tag{34}$$

Summarizing (33) and (34), the lemma is proved. □

Lemma 3.3 *The following inequality holds*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\lambda\nabla\rho\|_{L^2}^2 + \left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\sqrt{P'(\xi)^{-1}\xi}\nabla\operatorname{div}u\right\|_{L^2}^2 \\ &\le \eta\|\operatorname{div}u\|_{H^1}^2 + C_\eta\left(\left\|(\lambda\nabla\rho, \nabla u, \nabla\tau)\right\|_{L^2}^2 + \|u_t\|_{L^2}^2\right) + C\|v\|_{H^3}\|\lambda\nabla\rho\|_{L^2}^2.\end{aligned} \tag{35}$$

Proof. Multiply $\nabla(24)$ by $\lambda^2\nabla\rho$ and then integrate the resulting equation on Ω, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\lambda\nabla\rho\|_{L^2}^2 + \int_\Omega \xi\lambda^2\nabla\rho\nabla\operatorname{div}u\mathrm{d}x \\ &\le C\|\nabla v\|_{L^\infty}\|\lambda\nabla\rho\|_{L^2}^2 + \eta\|\operatorname{div}u\|_{H^1}^2 + C_\eta\|\lambda\nabla\xi\|_{H^1}^2\|\lambda\nabla\rho\|_{L^2}^2 \\ &\le \|\operatorname{div}u\|_{H^1}^2 + C_\eta\|\lambda\nabla\rho\|_{L^2}^2 + C\|v\|_{H^3}\|\lambda\nabla\rho\|_{L^2}^2.\end{aligned} \tag{36}$$

Based on the relation $\overrightarrow{\operatorname{curl}}\nabla = 0$ and the boundary condition (8), we can obtain

$$\int_\Omega \nabla\operatorname{div}u\cdot\overrightarrow{\operatorname{curl}}\operatorname{curl}u\mathrm{d}x = 0.$$

Multiply both sides of (25) by $p'(\xi)^{-1}\xi^2\nabla\mathrm{div}u$ and then integrate the resulting equation on Ω, it follows that

$$
\begin{aligned}
&\left\|\sqrt{p'(\xi)^{-1}\xi}\sqrt{2\mu(\xi)+\lambda(\xi)}\nabla\mathrm{div}u\right\|_{L^2}^2-\int_\Omega \xi\nabla\mathrm{div}u\lambda^2\nabla\rho\mathrm{d}x\\
&=\int_\Omega p'(\xi)^{-1}\xi^2\nabla\mathrm{div}u\left(u_t+(v\cdot\nabla)u-\xi^{-1}\mu_1\mathrm{div}\tau\right)\mathrm{d}x\\
&\quad-\int_\Omega p'(\xi)^{-1}\xi\nabla\mathrm{div}u\left[2\nabla(\mu(\xi))D(u)+\nabla(\lambda(\xi))\mathrm{div}u\right]\mathrm{d}x\\
&\quad+\int_\Omega p'(\xi)^{-1}\xi\mu(\xi)\overrightarrow{\mathrm{curl}}\mathrm{curl}u\cdot\nabla\mathrm{div}u\mathrm{d}x\\
&\le\eta\|\nabla\mathrm{div}u\|_{L^2}^2+C_\eta\left(\|u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla\tau\|_{L^2}^2\right)+C\|\xi\|_{L^\infty}\|\nabla\mathrm{div}u\|_{L^2}\|\nabla\xi\|_{L^6}\|\nabla u\|_{L^3}\\
&\le\eta\|\nabla\mathrm{div}u\|_{L^2}^2+C_\eta\left(\|u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\nabla\tau\|_{L^2}^2\right).
\end{aligned}
\tag{37}
$$

Therefore, with the above inequalities (36) and (37), the lemma is proved. □

Lemma 3.4 *The following inequality holds*

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tau\|_{L^2}^2+a\|\nabla\tau\|_{L^2}^2\le\eta\|\nabla u\|_{H^1}^2+C_\eta\|\tau\|_{H^1}^2+C\|v\|_{H^3}\|\tau\|_{H^1}^2. \tag{38}
$$

Proof. Applying the operator ∇ to Equation (26), multiplying the result by $\nabla\tau$ and then integrating it, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\tau\|_{L^2}^2+a\|\nabla\tau\|_{L^2}^2\\
&=\frac{1}{2}\int_\Omega \mathrm{div}v|\nabla\tau|^2\mathrm{d}x-\int_\Omega\nabla\tau\cdot\nabla\left(\xi^{-1}\right)Q(\tau,\nabla v)\mathrm{d}x\\
&\quad-\int_\Omega\nabla\tau\cdot\xi^{-1}\nabla\left(Q(\tau,\nabla v)\right)\mathrm{d}x+\int_\Omega\nabla\tau\cdot\nabla\left(\mu_2\xi^{-1}\right)\Gamma(\nabla u)\mathrm{d}x\\
&\quad+\int_\Omega\nabla\tau\cdot\mu_2\xi^{-1}\nabla\left(\Gamma(\nabla u)\right)\mathrm{d}x-\int_\Omega(\nabla\tau\cdot\nabla)v\cdot\nabla\tau\mathrm{d}x\\
&=\sum_{i=1}^6 I_i,
\end{aligned}
$$

where

$$
|I_1|\le C\|\mathrm{div}v\|_{L^\infty}\|\nabla\tau\|_{L^2}^2\le C\|v\|_{H^3}\|\nabla\tau\|_{L^2}^2,
$$

$$
|I_2|\le C\left(\|\nabla\tau\|_{L^2}^2+\left\|\xi^{-2}\right\|_{L^\infty}^2\|\nabla\xi\|_{H^1}^2\|\nabla v\|_{H^1}^2\|\tau\|_{H^1}^2\right)\le C\|\tau\|_{H^1}^2,
$$

$$
|I_3|\le C\|\nabla v\|_{H^2}\left(\|\nabla\tau\|_{L^2}^2+\left\|\xi^{-1}\right\|_{L^\infty}^2\|\tau\|_{H^1}^2\right)\le C\|v\|_{H^3}\|\tau\|_{H^1}^2,
$$

$$
|I_4|\le\eta\|\nabla v\|_{H^1}^2+C_\eta\left\|\xi^{-2}\right\|_{L^\infty}^2\|\nabla\xi\|_{H^1}^2\|\nabla\tau\|_{L^2}^2\le\eta\|\nabla u\|_{H^1}^2+C_\eta\|\nabla\tau\|_{L^2}^2,
$$

$$
|I_5|\le\eta\left\|\nabla^2u\right\|_{L^2}^2+C_\eta\left\|\xi^{-1}\right\|_{L^\infty}^2\|\nabla\tau\|_{L^2}^2\le\eta\left\|\nabla^2u\right\|_{L^2}^2+C_\eta\|\nabla\tau\|_{L^2}^2,
$$

$$
|I_6|\le C\|\nabla v\|_{L^\infty}\|\nabla\tau\|_{L^2}^2\le C\|v\|_{H^3}\|\nabla\tau\|_{L^2}^2.
$$

Thus, the lemma is proved. □

Lemma 3.5 *Let* $w:=\mathrm{curl}u$. *We have*

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\xi}w\right\|_{L^2}^2+\mu(\xi)\|w\|_{H^1}^2\le\eta\|\nabla\mathrm{div}u\|_{L^2}^2+C_\eta\|(\lambda\nabla\rho,\nabla u,\nabla\tau)\|_{L^2}^2. \tag{39}
$$

Proof. Applying the operator curl to Equation (26), multiplying the result by

ξ and then integrating it, we deduce that

$$\xi w_t + \xi(v\cdot\nabla)w - \mu(\xi)\Delta w = g + \text{curldiv}\,\tau, \tag{40}$$

where

$$\begin{aligned} g = &-\frac{1}{\xi}\nabla\xi\times\left[\text{div}\left(2\mu(\xi)D(u)\right)+\nabla\left(\lambda(\xi)\text{div}u\right)+\mu_1\text{div}\,\tau\right] \\ &+\nabla\left(\mu(\xi)\right)\times\Delta u+\nabla\left(\mu(\xi)\right)\times\nabla\text{div}u+\nabla\left(\lambda(\xi)\right)\times\nabla\text{div}u \\ &-\xi\nabla\left(p'(\xi)\xi^{-1}\right)\times\left(\lambda^2\nabla\rho\right)+\xi\nabla v\times\nabla u. \end{aligned}$$

Multiply (40) by w and integrate it on $L^2(\Omega)$, one has

$$\begin{aligned} &\frac{1}{2}\frac{\text{d}}{\text{d}t}\left\|\sqrt{\xi}w\right\|_{L^2}^2-\int_\Omega\mu(\xi)\Delta w\cdot w\text{d}x \\ &=\frac{1}{2}\int_\Omega\left(\xi_t+\text{div}(\xi v)\right)|w|^2\,\text{d}x+\int_\Omega g\cdot w\text{d}x+\int_\Omega\text{curldiv}\,\tau\cdot w\text{d}x \\ &=\sum_{i=1}^{4}J_i, \end{aligned}$$

where

$$|J_1|\le\eta\|w\|_{H^1}^2+C_\eta\|w\|_{L^2}^2,$$

$$|J_3|\le\eta\|w\|_{H^1}^2+C_\eta\|\text{div}\,\tau\|_{L^2}^2,$$

Based on boundary condition (8), it follows that

$$\begin{aligned} |J_2| &= \left|\int_\Omega g\cdot w\text{d}x\right| \\ &\le C\|w\|_{L^3}\left[\left\|\xi^{-1}\right\|_{L^\infty}\|\nabla\xi\|_{L^6}\left(\|\nabla\text{div}u\|_{L^2}+\|\text{curlcurl}u\|_{L^2}\right)\right] \\ &\quad+C\|w\|_{L^6}\left\|\xi^{-1}\right\|_{L^\infty}\|\nabla\xi\|_{L^6}\|\nabla\xi\|_{L^6}\left(\|\nabla u\|_{L^2}+\|\nabla\tau\|_{L^2}\right) \\ &\quad+C\|w\|_{L^3}\|\nabla\xi\|_{L^6}\left(\|\Delta u\|_{L^2}+\|\nabla\text{div}u\|_{L^2}\right) \\ &\quad+C\|w\|_{L^3}\|\nabla\xi\|_{L^6}\|\lambda\nabla\rho\|_{L^2}+C\|w\|_{L^4}\|\xi\|_{L^\infty}\|\nabla v\|_{L^4}\|\nabla u\|_{L^2} \\ &\le\eta\|\nabla\text{div}u\|_{L^2}^2+C_\eta\left\|(\lambda\nabla\rho,\nabla u,\nabla\tau)\right\|_{L^2}^2. \end{aligned}$$

Thus, the lemma is proved by Lemma 2. □

Lemma 3.6 *The following inequality holds*:

$$\begin{aligned} &\frac{1}{2}\frac{\text{d}}{\text{d}t}\int_\Omega\left[\mu_2\lambda^2P'(\xi)\xi^{-1}\rho_t^2+\xi\left(\mu_2|u_t|^2+\mu_1|\tau_t|^2\right)\right]\text{d}x+a\mu_1\int_\Omega\xi|\tau_t|^2\,\text{d}x \\ &+\mu_2\left(\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\text{div}u_t\right\|_{L^2}^2+\left\|\sqrt{\mu(\xi)}\text{curl}u_t\right\|_{L^2}^2\right) \\ &\le\eta\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{H^1}^2+\eta\|\nabla u\|_{H^1}^2 \\ &+C_\eta\left(\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{L^2}^2+\left\|(\lambda(\rho-1),u,\tau)\right\|_{H^1}^2+\|\tau\|_{H^1}^2\right). \end{aligned} \tag{41}$$

Proof. Taking the derivatives of Equations (24) and (25) with respect to *t*, respectively, one has

$$\rho_{tt}+v\cdot\nabla\rho_t+\xi\text{div}u_t=-v_t\cdot\nabla\rho-\xi_t\text{div}u, \tag{42}$$

$$\begin{aligned}&\xi\left(u_{tt}+(v\cdot\nabla)u_t\right)+\lambda^2P'(\xi)\nabla\rho_t\\&=\partial_t\left[\operatorname{div}\left(2\mu(\xi)D(u)\right)+\nabla\left(\lambda(\xi)\operatorname{div}u\right)\right]+\mu_1\operatorname{div}\tau_t-\xi_t\left(u_t+(v\cdot\nabla)u\right)\\&-\xi v_t\cdot\nabla u-\lambda^2P''(\xi)\xi_t\nabla\rho.\end{aligned}\tag{43}$$

Then, multiply (26) by ξ and take the derivative of the result with respect to t, we have

$$\begin{aligned}&\xi\left(\tau_{tt}+(v\cdot\nabla)\tau_t\right)+a\xi\tau_t\\&=-Q(\tau_t,\nabla v)-Q(\tau,\nabla v_t)+\mu_2\Gamma(\nabla u_t)-\xi_t\left(\tau_t+(v\cdot\nabla)\tau+a\tau\right)-\xi v_t\cdot\nabla\tau.\end{aligned}\tag{44}$$

Multiply both sides of Equations (42), (43) and (44) simultaneously by $\mu_2\lambda^2p'(\xi)\xi^{-1}\rho_t$, μ_2u_t and $\mu_1\tau_t$ respectively, then summarizing the integrals of the resulting equations on Ω, and finally by integration by parts we obtain that:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left[\mu_2\lambda^2P'(\xi)\xi^{-1}\rho_t^2+\xi\left(\mu_2|u_t|^2+\mu_1|\tau_t|^2\right)\right]\mathrm{d}x+a\mu_1\int_\Omega\xi|\tau_t|^2\,\mathrm{d}x\\&+\mu_2\left(\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\operatorname{div}u_t\right\|_{L^2}^2+\left\|\sqrt{\mu(\xi)}\operatorname{curl}u_t\right\|_{L^2}^2\right)\\&=\sum_{i=1}^{8}J_i,\end{aligned}$$

where

$$\begin{aligned}|J_1|&\le\eta\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{H^1}^2+C_\eta\left(\left\|\frac{p''(\xi)}{\xi}\right\|_{L^\infty}^2+\left\|\frac{p'(\xi)}{\xi^2}\right\|_{L^\infty}^2\right)\|\xi_t\|_{H^1}^2\|\lambda\rho_t\|_{L^2}^2\\&\quad+C_\eta\|\xi_t\|_{H^1}^2\left\|(u_t,\tau_t)\right\|_{L^2}^2\\&\le\eta\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{H^1}^2+C_\eta\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{L^2}^2,\end{aligned}$$

$$\begin{aligned}|J_2|&\le\eta\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{H^1}^2+C_\eta\left\{\left(\|\operatorname{div}v\|_{H^1}^2+\|\nabla\xi\|_{H^1}^2\|v\|_{L^\infty}^2\right)\|\lambda\rho_t\|_{L^2}^2\right.\\&\quad\left.+\left(\|\xi\|_{L^\infty}^2\|\operatorname{div}v\|_{H^1}^2+\|\nabla\xi\|_{H^1}^2\|v\|_{L^\infty}^2\right)\left\|(u_t,\tau_t)\right\|_{L^2}^2\right\}\\&\le\eta\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{H^1}^2+C_\eta\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{L^2}^2\end{aligned}$$

$$|J_3|\le\eta\|u_t\|_{H^1}^2+C_\eta\|\lambda\nabla\xi\|_{H^1}^2\|\lambda\rho_t\|_{L^2}^2\le\eta\|u_t\|_{H^1}^2+C_\eta\|\lambda\rho_t\|_{L^2}^2,$$

$$\begin{aligned}|J_4|&\le\eta\|\lambda\rho_t\|_{H^1}^2+C_\eta\left(\|v_t\|_{H^1}^2\|\lambda\nabla\rho\|_{L^2}^2+\|\lambda\xi_t\|_{H^1}^2\|\operatorname{div}u\|_{L^2}^2\right)\\&\le\eta\|\lambda\rho_t\|_{H^1}^2+C_\eta\left(\|\lambda\nabla\rho\|_{L^2}^2+\|\operatorname{div}u\|_{L^2}^2\right),\end{aligned}$$

$$\begin{aligned}|J_6|&\le\eta\|u_t\|_{H^1}^2+C_\eta\left[\|\xi_t\|_{H^1}^2\left(\|u_t\|_{L^2}^2+\|v\|_{L^\infty}^2\right)+\|v_t\|_{H^1}^2\|\nabla u\|_{L^2}^2+\|\lambda\xi_t\|_{H^1}^2\|\lambda\nabla\rho\|_{L^2}^2\right]\\&\le\eta\|u_t\|_{H^1}^2+C_\eta\left(\|u_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|\lambda\nabla\rho\|_{L^2}^2\right),\end{aligned}$$

$$\begin{aligned}|J_7|&\le\eta\|\tau_t\|_{H^1}^2+C_\eta\left[\|\nabla v\|_{H^1}^2\|\tau_t\|_{L^2}^2+\|\nabla v_t\|_{L^2}^2\|\tau\|_{H^1}^2\right.\\&\quad\left.+\|\xi_t\|_{H^1}^2\left(\|\tau_t\|_{L^2}^2+\|v\|_{L^\infty}^2\|\nabla\tau\|_{L^2}^2+\|\tau\|_{L^2}^2\right)+\|\xi\|_{L^\infty}^2\|v_t\|_{H^1}^2\|\nabla u\|_{L^2}^2\right]\\&\le\eta\|\tau_t\|_{H^1}^2+C_\eta\left(\|\tau_t\|_{L^2}^2+\|\tau\|_{H^1}^2\right),\end{aligned}$$

$$\begin{aligned}|J_8| &= \left|-\int_\Omega \nabla\left(2\mu(\xi)+\lambda(\xi)\right)\mathrm{div}u_t\cdot\mu_2u_t\mathrm{d}x-\int_\Omega\nabla\left(\mu(\xi)\right)\times\mathrm{curl}u_t\cdot\mu_2u_t\mathrm{d}x\right|\\&+\left|\int_\Omega\mu_2u_t\left[2\nabla\left(\mu(\xi)\right)D(u_t)+\nabla\left(\lambda(\xi)\right)\mathrm{div}u_t\right.\right.\\&\left.\left.+\mathrm{div}\left(2\mu(\xi)_t\,D(u)\right)+\nabla\left(\lambda(\xi)_t\,\mathrm{div}u\right)\right]\mathrm{d}x\right|\\&\le C\|\nabla\xi\|_{L^6}\|\nabla u_t\|_{L^2}\|u_t\|_{L^3}+C\|u_t\|_{L^3}\|\xi_t\|_{L^6}\|\nabla^2u\|_{L^2}+C\|\nabla\xi_t\|_{L^2}\|\nabla u\|_{L^6}\|u_t\|_{L^3}\\&\le\eta\left(\|u_t\|_{H^1}^2+\|\nabla u\|_{H^1}^2\right)+C_\eta\|u_t\|_{L^2}^2\\&\le\eta\|(u_t,\nabla u)\|_{H^1}^2+C_\eta\|u_t\|_{L^2}^2.\end{aligned}$$

By applying the symmetry of τ and integration by parts, one can derive that $J_5=0$. Therefore, the lemma is proved □

According to Lemmas 3.1-3.2 and Lemmas 2-2, it follows that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|(\lambda(\rho-1),u,\tau)\|_{H^1}^2+\|(\lambda\rho_t,u_t,\tau_t)\|_{L^2}^2\right)\\&+\|(\tau,\mathrm{div}u,\mathrm{curl}u)\|_{H^1}^2+\|(\tau_t,\mathrm{div}u_t,\mathrm{curl}u_t)\|_{L^2}^2\\&\le\eta\left(\|\lambda\nabla\rho\|_{H^1}^2+\|\nabla u\|_{H^1}^2+\|\mathrm{curl}u\|_{H^1}^2+\|(\lambda\rho_t,u_t,\tau_t)\|_{H^1}^2\right)\\&+C_\eta\left(\|(\lambda(\rho-1),u,\tau)\|_{H^1}^2+\|(\lambda\rho_t,u_t,\tau_t)\|_{L^2}^2\right)\\&+C\left(\|v\|_{H^3}\|\lambda\nabla\rho\|_{L^2}^2+\|v\|_{H^3}\|\tau\|_{H^1}^2\right).\end{aligned}\tag{45}$$

By Grönwall's inequality [25], we obtain

Lemma 3.7 *The following inequality holds*

$$\begin{aligned}&\left(\|(\lambda(\rho-1),u,\tau)\|_{H^1}^2+\|(\lambda\rho_t,u_t,\tau_t)\|_{L^2}^2\right)(t)\\&+\int_0^t\left(\|(\tau,\mathrm{div}u,\mathrm{curl}u)\|_{H^1}^2+\|(\tau_t,\mathrm{div}u_t,\mathrm{curl}u_t)\|_{L^2}^2\right)\mathrm{d}s\\&\le\eta\int_0^t\|(\lambda\rho_t,\tau_t)\|_{H^1}^2\,\mathrm{d}s+\mathrm{e}^{C_\eta t}\sum_{i=0}^{1}\left\|\left(\partial_i^t\left(\lambda(\rho-1)\right),\partial_i^tu,\partial_i^t\tau\right)(0)\right\|_{H^{2-i}}^2.\end{aligned}\tag{46}$$

3.3. The Estimates of High-Order Derivatives

Lemma 3.8 *The following inequality holds*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\nabla\mathrm{div}u\right\|_{L^2}^2-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\xi u_t\cdot\nabla\mathrm{div}u\mathrm{d}x+\left\|\sqrt{P'(\xi)\xi^{-1}}\lambda\nabla\rho_t\right\|_{L^2}^2\\&\le C\left(\|\nabla\mathrm{div}u\|_{H^1}^2+\|u_t\|_{H^1}^2+\|\nabla u\|_{H^1}^2+\|\lambda\nabla\rho\|_{H^1}^2+\|\nabla\tau_t\|_{L^2}^2\right).\end{aligned}\tag{47}$$

Proof. Multiply $\partial_t(\xi)$ by $\nabla\mathrm{div}u$ and then integrate it on $L^2(\Omega)$, one has

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\nabla\mathrm{div}u\right\|_{L^2}^2-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\xi u_t\cdot\nabla\mathrm{div}u\mathrm{d}x-\int_\Omega\lambda^2P'(\xi)\nabla\rho_t\cdot\nabla\mathrm{div}u\mathrm{d}x\\&=\int_\Omega\left[\left(\xi_t(v\cdot\nabla)u+\xi(v_t\cdot\nabla)u+\xi(v\cdot\nabla)u_t\right)+\lambda^2P''(\xi)\xi_t\nabla\rho-\mu_1\mathrm{div}\tau_t\right]\cdot\nabla\mathrm{div}u\mathrm{d}x\\&+\int_\Omega\left[-\frac{1}{2}\left(2\mu(\xi)+\lambda(\xi)\right)_t\nabla\mathrm{div}u+\mu(\xi)\mathrm{curlcurl}u_t+\mu(\xi)_t\,\mathrm{curlcurl}u\right.\\&-2\nabla\left(\mu(\xi)\right)D(u_t)+\nabla\left(\lambda(\xi)\right)\mathrm{div}u_t+2\nabla\left(\mu(\xi)_t\right)D(u)\\&\left.+\nabla\left(\lambda(\xi)_t\right)\mathrm{div}u\right]\cdot\nabla\mathrm{div}u\mathrm{d}x-\int_\Omega\xi u_t\cdot\nabla\mathrm{div}u_t\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\|\nabla\tau_t\|_{L^2}^2+\|u_t\|_{H^1}^2+\|\nabla u\|_{H^1}^2+\|\lambda\nabla\rho\|_{H^1}^2\right)+C\|\xi_t\|_{L^4}\|\nabla \mathrm{div}u\|_{L^2}\|\nabla \mathrm{div}u\|_{L^4}\\
&\quad+C\|\nabla\xi\|_{L^4}\|\mathrm{curl}u_t\|_{L^2}\|\nabla \mathrm{div}u\|_{L^4}+C\|\nabla\xi_t\|_{L^4}\|\mathrm{curl}u\|_{L^2}\|\nabla \mathrm{div}u\|_{L^4}\\
&\quad+C\|\nabla\xi\|_{L^4}\|\nabla u_t\|_{L^2}\|\nabla \mathrm{div}u\|_{L^4}+C\|\nabla\xi_t\|_{L^2}\|\nabla u\|_{L^4}\|\nabla \mathrm{div}u\|_{L^4}\\
&\le C\left(\|\nabla \mathrm{div}u\|_{H^1}^2+\|u_t\|_{H^1}^2+\|\nabla u\|_{H^1}^2+\|\lambda\nabla\rho\|_{H^1}^2+\|\nabla\tau_t\|_{L^2}^2\right).
\end{aligned}
\tag{48}
$$

Then, integrate $\nabla(30)\times p'(\xi)\xi^{-1}\lambda^2\nabla\rho_t$ on Ω, this implies

$$
\begin{aligned}
&\left\|\sqrt{p'(\xi)\xi^{-1}}\lambda\nabla\rho_t\right\|_{L^2}^2+\int_\Omega\lambda^2 p'(\xi)\nabla\rho_t\cdot\nabla \mathrm{div}u\mathrm{d}x\\
&=\int_\Omega p'(\xi)\xi^{-1}\lambda\nabla\rho_t\cdot\left(v\cdot\nabla^2\rho+\nabla v\cdot\nabla\rho+\nabla\xi \mathrm{div}u\right)\mathrm{d}x\\
&\le C\left(\|\lambda\nabla\rho_t\|_{L^2}^2+\|\nabla u\|_{H^1}^2+\|\lambda\nabla\rho\|_{H^1}^2\right).
\end{aligned}
\tag{49}
$$

Therefore, the lemma is proved by (48) and (49). □

Lemma 3.9 *The following inequality holds*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\lambda\nabla^2\rho\|_{L^2}^2+\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\sqrt{\xi P'(\xi)^{-1}}\nabla^2\mathrm{div}u\right\|_{L^2}^2\\
&\le\eta\|u\|_{H^3}^2+C\|v\|_{H^3}\|\lambda\nabla\rho\|_{H^1}^2\\
&\quad+C_\eta\left(\left\|\nabla\overrightarrow{\mathrm{curl}}\mathrm{curl}u\right\|_{L^2}^2+\|\nabla \mathrm{div}\tau\|_{L^2}^2+\|\lambda\nabla\rho\|_{H^1}^2+\|u_t\|_{H^1}^2+\|\nabla u\|_{H^1}^2+\|u\|_{H^1}^2\right).
\end{aligned}
\tag{50}
$$

Proof. By applying the operator ∇^2 to Equation (24), multiplying the result by $\lambda^2\nabla^2\rho$ and then integrating it, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\lambda\nabla^2\rho\|_{L^2}^2+\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\sqrt{\xi P'(\xi)^{-1}}\nabla^2\mathrm{div}u\right\|_{L^2}^2\\
&=\frac{1}{2}\int_\Omega\lambda^2\mathrm{div}v\left|\nabla^2\rho\right|^2dx-\int_\Omega\lambda^2\nabla^2\rho\left[\nabla^2v\nabla\rho+2\nabla v\nabla^2\rho\right]\mathrm{d}x\\
&\quad-\int_\Omega\lambda^2\nabla^2\rho\left[\nabla^2\xi \mathrm{div}u+2\nabla\xi\nabla \mathrm{div}u\right]\mathrm{d}x\\
&\le\eta\|u\|_{H^3}^2+C_\eta\|\lambda\nabla^2\rho\|_{L^2}^2+C\|v\|_{H^3}\|\lambda\nabla\rho\|_{H^1}^2.
\end{aligned}
\tag{51}
$$

Applying ∇ to (25) and then integrate the resulting equation with $\xi^2p'(\xi)^{-1}\nabla^2\mathrm{div}u$ leads to

$$
\begin{aligned}
&\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\sqrt{\xi p'(\xi)^{-1}}\nabla^2\mathrm{div}u\right\|_{L^2}^2-\int_\Omega\xi\lambda^2\nabla^2\rho\nabla^2\mathrm{div}u\mathrm{d}x\\
&=\int_\Omega\xi^2p'(\xi)^{-1}\nabla^2\mathrm{div}u\left(\nabla u_t+\nabla v\cdot\nabla u+v\cdot\nabla^2u+p''(\xi)\nabla\xi\xi^{-1}\lambda^2\nabla\rho\right.\\
&\quad\left.-p'(\xi)\xi^{-2}\nabla\xi\lambda^2\nabla\rho-\xi^{-1}\mu_1\nabla \mathrm{div}\tau-p'(\xi)\xi^{-2}\nabla\xi\lambda^2\nabla\rho-\xi^{-1}\mu_1\nabla \mathrm{div}\tau\right)\mathrm{d}x\\
&\quad+\int_\Omega\xi p'(\xi)^{-1}\nabla^2\mathrm{div}u\left[-\nabla\left(2\mu(\xi)+\lambda(\xi)\right)\nabla \mathrm{div}u+\nabla\left(\mu(\xi)\right)\overrightarrow{\mathrm{curl}}\mathrm{curl}u\right.\\
&\quad\left.+\mu(\xi)\overrightarrow{\mathrm{curl}}\mathrm{curl}u-\nabla\left(2\nabla\left(\mu(\xi)\right)D(u)+\nabla\left(\lambda(\xi)\right)\mathrm{div}u\right)\right]\mathrm{d}x\\
&\quad+\int_\Omega p'(\xi)^{-1}\nabla\xi\nabla^2\mathrm{div}u\left[\mathrm{div}\left(2\mu(\xi)D(u)\right)+\nabla\left(\lambda(\xi)\mathrm{div}u\right)\right]\mathrm{d}x\\
&\le\|\nabla^2\mathrm{div}u\|_{L^2}^2+C_\eta\left(\left\|\left(\nabla\overrightarrow{\mathrm{curl}}\mathrm{curl}u,\nabla \mathrm{div}\tau\right)\right\|_{L^2}^2+\left\|\left(u_t,\nabla u,\lambda\nabla\rho\right)\right\|_{H^1}^2\right)\\
&\quad+C\|\nabla\xi\|_{L^6}\|\nabla \mathrm{div}u\|_{L^2}\left(\|\nabla^2u\|_{L^3}+\|\nabla\xi\|_{L^6}\|\nabla u\|_{L^6}\right)+C\|\nabla^2\mathrm{div}u\|_{L^2}\|\nabla^2\xi\|_{L^2}\|\nabla u\|_{L^\infty}
\end{aligned}
$$

$$\le \eta\left\|\nabla^2 \mathrm{div}u\right\|_{L^2}^2 + C_\eta\left(\left\|\left(\nabla\overrightarrow{\mathrm{curl}}\mathrm{curl}u, \nabla \mathrm{div}\,\tau\right)\right\|_{L^2}^2 + \left\|\left(u_t, \nabla u, \lambda\nabla\rho\right)\right\|_{H^1}^2\right)$$
$$+\eta\|u\|_{H^3}^2 + C_\eta\|u\|_{H^3}^2. \tag{52}$$

Thus, this lemma is proved by (51) and (52) □

Lemma 3.10 *The following inequality holds*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\xi^{-1}\sqrt{P'(\xi)}\lambda\nabla\rho_t\right\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathrm{div}u_t\right\|_{L^2}^2 + \frac{1}{2}\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\sqrt{\xi^{-1}}\nabla \mathrm{div}u_t\right\|_{L^2}^2$$
$$\le \eta\|u\|_{H^3}^2 + \left\|\left(\lambda\nabla\rho, \nabla u, \nabla\tau, u_t\right)\right\|_{H^1}^2 + \left\|\left(\mathrm{div}\,\tau_t, \lambda\nabla\rho_t\right)\right\|_{L^2}^2 + \|u\|_{H^1}^2 \tag{53}$$
$$+C\left[\|v\|_{H^3}\|\lambda\nabla\rho_t\|_{L^2}^2 + \|v_t\|_{H^2}\left(\|\lambda\nabla\rho_t\|_{L^2}^2 + \|\lambda\nabla\rho\|_{H^1}^2\right)\right].$$

Proof. According to (8), integrating (43) with $\xi^{-1}\nabla\mathrm{div}u_t$, this implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\mathrm{div}u_t\right\|_{L^2}^2 + \frac{1}{2}\left\|\sqrt{2\mu(\xi)+\lambda(\xi)}\sqrt{\xi^{-1}}\nabla\mathrm{div}u_t\right\|_{L^2}^2 - \int_\Omega \xi^{-1}p'(\xi)\lambda^2\nabla\rho_t\nabla\mathrm{div}u_t\mathrm{d}x$$
$$= \int_\Omega -\xi^{-1}\nabla\mathrm{div}u_t\left[\mu_1\mathrm{div}\,\tau_t - \xi_t\left(u_t + v\cdot\nabla u\right) - \xi v_t\cdot\nabla u - \lambda^2 p''(\xi)\xi_t\nabla\rho - \xi v\cdot\nabla u_t\right]\mathrm{d}x$$
$$+\int_\Omega \xi^{-1}\nabla\mathrm{div}u_t\left[\mu(\xi)\mathrm{curlcurl}u_t - \left(2\mu(\xi)+\lambda(\xi)\right)_t\nabla\mathrm{div}u + \left(\mu(\xi)\right)_t\mathrm{curlcurl}u\right]\mathrm{d}x$$
$$+\int_\Omega \xi^{-1}\nabla\mathrm{div}u_t\left[-2\nabla\left(\mu(\xi)\right)D(u_t) - \nabla\left(\lambda(\xi)\right)\mathrm{div}u_t\right]\mathrm{d}x$$
$$+\int_\Omega \xi^{-1}\nabla\mathrm{div}u_t\left[-2\nabla\left(\mu(\xi)\right)_t D(u) - \nabla\left(\lambda(\xi)\right)_t\mathrm{div}u\right]\mathrm{d}x$$
$$\le \eta\left\|\nabla\mathrm{div}u_t\right\|_{L^2}^2 + C_\eta\left(\left\|\left(\lambda\nabla\rho, \nabla u, \nabla\tau\right)\right\|_{H^1}^2 + \|u_t\|_{H^1}^2 + \|\mathrm{div}\,\tau_t\|_{L^2}^2\right)$$
$$+C\left\|\xi^{-1}\right\|_{L^\infty}\left\|\nabla\mathrm{div}u_t\right\|_{L^2}\left(\left\|\nabla^2 u_t\right\|_{L^2} + \|\xi_t\|_{L^6}\left\|\nabla^2 u\right\|_{L^3} + \|\nabla\xi\|_{L^6}\|\nabla u_t\|_{L^3} + \|\nabla\xi_t\|_{L^2}\|\nabla u\|_{L^\infty}\right) \tag{54}$$
$$\le \eta\|u\|_{H^3}^2 + C_\eta\left(\left\|\left(\lambda\nabla\rho, \nabla u, \nabla\tau\right)\right\|_{H^1}^2 + \|u_t\|_{H^1}^2 + \|\mathrm{div}\,\tau_t\|_{L^2}^2 + \|u\|_{H^1}^2\right).$$

Applying $\partial_t\nabla$ to (24) and then integrate the resulting equation with $\lambda^2\xi^{-2}p'(\xi)\nabla\rho_t$, it can be deduced that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\xi^{-1}\sqrt{p'(\xi)}\lambda\nabla\rho_t\right\|_{L^2}^2 + \int_\Omega \xi^{-1}p'(\xi)\lambda^2\nabla\rho_t\nabla\mathrm{div}u_t\mathrm{d}x$$
$$\le \eta\left(\|u\|_{H^3}^2 + \|\mathrm{div}u_t\|_{H^1}^2\right) + C_\eta\|\lambda\nabla\rho_t\|_{L^2}^2 \tag{55}$$
$$+C\left(\|v\|_{H^3}\|\lambda\nabla\rho_t\|_{L^2}^2 + \|v_t\|_{H^2}\left(\|\lambda\nabla\rho_t\|_{L^2}^2 + \|\lambda\nabla\rho\|_{H^1}^2\right)\right).$$

Therefore, the lemma is proved. □

Lemma 3.11 *Let* $w := \mathrm{curl}u$. *Then we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\xi}w_t\right\|_{L^2}^2 + \left\|\sqrt{\mu(\xi)}w_t\right\|_{H^1}^2$$
$$\le \eta\left(\|\nabla\mathrm{div}u_t\|_{L^2}^2 + \left\|\left(\nabla\mathrm{div}u\overrightarrow{\mathrm{curl}}w\right)\right\|_{H^1}^2\right) \tag{56}$$
$$+C_\eta\left(\left\|\left(\nabla u_t, \lambda\nabla\rho_t, \mathrm{div}\,\tau_t\right)\right\|_{L^2}^2 + \left\|\left(\nabla u, \lambda\nabla\rho, \mathrm{div}\,\tau\right)\right\|_{H^1}^2\right).$$

Proof. Differentiate (40) with respect to t to obtain

$$\xi\left(w_{tt} + v\cdot\nabla w_t\right) - \mu\Delta w_t = h + \mathrm{curldiv}\,\tau_t, \tag{57}$$

where $-h = \xi_t w_t + (\xi v)_t\cdot\nabla w - \partial_t\left(\mu(\xi)\right)\Delta w + g_t$. Base on (22) and $\|\xi\|_{H^2}\le C$, it follows that

$$\begin{aligned}|g_t| \le C\Big\{&|\xi_t|\Big[|\nabla\xi|\big(|\nabla \mathrm{div} u|+|\overrightarrow{\mathrm{curl}} w|+|\mathrm{div}\,\tau|+|\nabla\xi||\nabla u|\big)+|\lambda\nabla\xi||\lambda\nabla\rho|+|\nabla v||\nabla u|\Big]\\&+|\nabla\xi_t|\big(|\nabla \mathrm{div} u|+|\overrightarrow{\mathrm{curl}} w|+|\mathrm{div}\,\tau|+|\nabla\xi||\nabla u|\big)\\&+|\nabla\xi|\big(|\nabla \mathrm{div} u_t|+|\overrightarrow{\mathrm{curl}} w_t|+|\mathrm{div}\,\tau_t|+|\nabla\xi||\nabla u_t|\big)\\&+|\lambda\nabla\xi||\lambda\nabla\rho_t|+|\lambda\nabla\xi_t||\lambda\nabla\rho|+|\nabla v_t||\nabla u|+|\nabla v||\nabla u_t|\Big\}.\end{aligned}$$

Multiply (57) by w_t and integrate it on $L^2(\Omega)$, one has

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\xi}w_t\right\|_{L^2}^2+\left\|\sqrt{\mu(\xi)}\mathrm{curl} w_t\right\|_{L^2}^2\\&=\frac{1}{2}\int_\Omega\big(\xi_t+\mathrm{div}(\xi v)\big)|w_t|^2\,\mathrm{d}x+\int_\Omega h w_t\mathrm{d}x\\&\quad+\int_\Omega \mathrm{curl}\mathrm{div}\,\tau_t\cdot w_t\mathrm{d}x-\int_\Omega\nabla\big(\mu(\xi)\big)\times\mathrm{curl} w_t\cdot w_t\mathrm{d}x\\&=\sum_{i=1}^4 I_i,\end{aligned}$$

where

$$|I_1|\le C\left(\|\xi_t\|_{H^1}+\|\xi\|_{H^2}\|v\|_{H^2}\right)\|w_t\|_{H^1}\|w_t\|_{L^2}\le\eta\|w_t\|_{H^1}^2+C_\eta\|w_t\|_{L^2}^2,$$

$$\begin{aligned}|I_2|&\le C\|w_t\|_{L^4}\Big[\|\xi_t\|_{L^4}\left(\|w_t\|_{L^2}+\|v\|_{L^\infty}\|\nabla w\|_{L^2}\right)+\|v_t\|_{L^4}\|\nabla w\|_{L^2}\Big]\\&\quad+C\|\xi_t\|_{L^6}\left\|\overrightarrow{\mathrm{curl}}\mathrm{curl} w\right\|_{L^2}\|w_t\|_{L^3}+\int_\Omega|w_t||g_t|\mathrm{d}x\\&\le C\|w_t\|_{H^1}\Big[\|\xi_t\|_{H^1}\|w_t\|_{L^2}+\|(\xi_t,v_t)\|_{H^1}\|\nabla w\|_{L^2}\Big]+\eta\|(w_t,\mathrm{curl} w)\|_{H^1}^2\\&\quad+C_\eta\|\nabla u_t\|_{L^2}^2+C\|\xi_t\|_{L^6}\|\nabla\xi\|_{L^6}\left(\|\nabla\mathrm{div} u\|_{L^2}+\|\mathrm{curl} w\|_{L^2}+\|\mathrm{div}\,\tau\|_{L^2}\right)\|w_t\|_{L^6}\\&\quad+C\|\xi_t\|_{L^6}\|w_t\|_{L^6}\left(\|\lambda\nabla\xi\|_{L^6}+\|\lambda\nabla\rho\|_{L^2}+\|\nabla v\|_{L^6}\|\nabla u\|_{L^2}\right)\\&\qquad+C\|\xi_t\|_{L^8}\|\nabla\xi\|_{L^8}\|\nabla\xi\|_{L^8}\|\nabla u\|_{L^2}\|w_t\|_{L^8}\\&\qquad+C\|\nabla\xi_t\|_{L^2}\|w_t\|_{L^3}\left(\|\nabla\mathrm{div} u\|_{L^6}+\|\mathrm{curl} w\|_{L^6}+\|\mathrm{div}\,\tau\|_{L^6}\right)\\&\qquad+C\|\nabla\xi_t\|_{L^2}\|w_t\|_{L^3}\|\nabla\xi\|_{L^6}\|\nabla u\|_{L^6}\|w_t\|_{L^6}\\&\qquad+C\|w_t\|_{L^3}\left(\|\nabla\mathrm{div} u_t\|_{L^2}+\|\mathrm{curl} w_t\|_{L^2}+\|\mathrm{div}\,\tau_t\|_{L^2}\right)\|\nabla\xi\|_{L^6}\\&\qquad+C\|w_t\|_{H^1}\|\nabla u_t\|_{L^2}\|\nabla\xi\|_{L^4}+C\|w_t\|_{L^3}\left(\|\lambda\nabla\xi\|_{L^6}+\|\lambda\nabla\rho_t\|_{L^2}\right.\\&\quad\left.+\|\lambda\nabla\xi_t\|_{L^2}\|\lambda\nabla\rho\|_{L^6}\right)+C\|w_t\|_{L^3}\left(\|\nabla v_t\|_{L^2}\|\nabla u\|_{L^6}+\|\nabla v\|_{L^6}\|\nabla u_t\|_{L^2}\right)\\&\le\eta\left(\|\nabla\mathrm{div} u_t\|_{L^2}+\|(w_t,\nabla\mathrm{div} u,\mathrm{curl} w)\|_{H^1}^2\right)\\&\quad+C_\eta\left(\|(\nabla u_t,\lambda\nabla\rho_t,\mathrm{div}\,\tau_t)\|_{L^2}^2+\|(\nabla u,\lambda\nabla\rho,\mathrm{div}\,\tau)\|_{H^1}^2\right),\end{aligned}$$

In addition, according to the boundary condition $\partial_t(8)$, it can be deduced that

$$|I_3|=\left|\int_\Omega\mathrm{div}\,\tau_t\,\overrightarrow{\mathrm{curl}} w_t\mathrm{d}x\right|\le\eta\|w_t\|_{H^1}^2+C_\eta\|\mathrm{div}\,\tau_t\|_{L^2}^2,$$

$$|I_4|=\left|\int_\Omega\nabla\big(\mu(\xi)\big)\times\mathrm{curl} w_t\cdot w_t\mathrm{d}x\right|\le\eta\|w_t\|_{H^1}^2+C_\eta\|w_t\|_{L^2}^2.$$

Thus, the lemma is proved. □

Lemma 3.12 *Let* $w:=\mathrm{curl} u$, *We have*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\xi}\,\overrightarrow{\mathrm{curl}}w\right\|_{L^2}^2+\left\|\sqrt{\mu(\xi)}\Delta w\right\|_{L^2}^2\\&\le\eta\left(\|\nabla\mathrm{div}u\|_{H^1}^2+\left\|\overrightarrow{\mathrm{curl}}w\right\|_{H^1}^2\right)+C_\eta\left(\left\|(\lambda\nabla\rho,\nabla u,\nabla\tau)\right\|_{H^1}^2+\|w_t\|_{L^2}^2\right).\end{aligned}\tag{58}$$

Proof. By integrating $\Delta w\cdot(40)$ on Ω to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\xi}\,\overrightarrow{\mathrm{curl}}w\right\|_{L^2}^2+\alpha\|\Delta w\|_{L^2}^2\\&=\frac{1}{2}\int_\Omega\xi_t\left|\overrightarrow{\mathrm{curl}}w\right|^2\mathrm{d}x+\int_\Omega w_t\cdot\left(\overrightarrow{\mathrm{curl}}w\times\nabla\xi\right)\mathrm{d}x\\&\quad+\int_\Omega\Delta w\left(-g-\mathrm{curl}\mathrm{div}\tau+\xi v\cdot\nabla w\right)\mathrm{d}x\\&\equiv\sum_{i=1}^{3}J_i,\end{aligned}$$

where

$$|J_1|\le\|\xi_t\|_{H^1}\left\|\overrightarrow{\mathrm{curl}}w\right\|_{L^2}\left\|\overrightarrow{\mathrm{curl}}w\right\|_{L^3}\le\eta\left\|\overrightarrow{\mathrm{curl}}w\right\|_{H^1}^2+C_\eta\left\|\overrightarrow{\mathrm{curl}}w\right\|_{L^2}^2,$$

$$|J_2|\le\|\nabla\xi\|_{H^1}\|w_t\|_{L^2}\left\|\overrightarrow{\mathrm{curl}}w\right\|_{L^3}\le\eta\left\|\overrightarrow{\mathrm{curl}}w\right\|_{H^1}^2+C_\eta\left\|\left(\overrightarrow{\mathrm{curl}}w,w_t\right)\right\|_{L^2}^2,$$

$$\begin{aligned}|J_3|\le&\|\Delta w\|_{L^2}\Big\{\|\mathrm{curl}\mathrm{div}\tau\|_{L^2}^2+\|\xi\|_{L^\infty}\|v\|_{L^\infty}\|\nabla w\|_{L^2}\\&+\|\nabla\xi\|_{L^6}\left(\|\nabla\mathrm{div}u\|_{L^3}+\|\mathrm{curl}w\|_{L^3}+\|\nabla\xi\|_{L^6}\|\nabla u\|_{L^6}+\|\mathrm{div}\tau\|_{L^3}\right)\\&+\|\lambda\nabla\xi\|_{L^6}\|\lambda\nabla\rho\|_{L^3}+\|\nabla v\|_{L^4}\|\nabla u\|_{L^4}\Big\}\\\le&\,\eta\left(\|\nabla\mathrm{div}u\|_{H^1}^2+\|\mathrm{curl}w\|_{H^1}^2\right)+C_\eta\|(\lambda\nabla\rho,\nabla u,\nabla\tau)\|_{H^1}^2.\end{aligned}$$

Therefore, the lemma is proved. □

We now star to estimate the second order derivative of τ.

Lemma 3.13 *The following inequality holds*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^2\tau\right\|_{L^2}^2+a\left\|\nabla^2\tau\right\|_{L^2}^2\le\eta\|u\|_{H^3}^2+C_\eta\left\|\nabla^2\tau\right\|_{L^2}^2+C\|v\|_{H^3}\|\tau\|_{H^2}^2.\tag{59}$$

Proof. Multiply $\nabla^2(26)$ by $\nabla^2\tau$ and then integrate on Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_{ij}\tau\right\|_{L^2}^2+a\left\|\partial_{ij}\tau\right\|_{L^2}^2=\frac{1}{2}\int_\Omega\mathrm{div}\left|\partial_{ij}\tau\right|^2\mathrm{d}x+\sum_{i=1}^{4}K_i,$$

where

$$|K_1|=\left|\int_\Omega\partial_{ij}\tau\left(\partial_i v\cdot\nabla\partial_j\tau+\partial_j v\cdot\nabla\partial_i\tau+\partial_{ij}v\cdot\nabla\tau\right)\mathrm{d}x\right|\le C\|\nabla v\|_{H^2}\|\nabla\tau\|_{H^1}^2,$$

$$\begin{aligned}|K_2|&=\left|\int_\Omega\xi^{-1}\partial_{ij}\tau\left(-\partial_{ij}Q(\tau,\nabla v)+\mu_2\partial_{ij}\Gamma(\nabla u)\right)\mathrm{d}x\right|\\&\le C\left\|\partial_{ij}\tau\right\|_{L^2}\left(\|\tau\|_{L^2}\|v\|_{H^3}+\|\nabla\tau\|_{H^1}\|\nabla v\|_{H^1}+\left\|\nabla^2\tau\right\|_{L^2}\|\nabla v\|_{L^\infty}+\|u\|_{H^3}\right)\\&\le\eta\|u\|_{H^3}^2+C_\eta\left\|\partial_{ij}\tau\right\|_{L^2}^2+C\|v\|_{H^3}\|\tau\|_{H^2}^2,\end{aligned}$$

$$\begin{aligned}|K_3|&=\left|\int_\Omega\partial_{ij}\tau\left(\partial_{ij}\left(\xi^{-1}\right)\left(-Q(\tau,\nabla v)+\mu_2\Gamma(\nabla u)\right)\right)\mathrm{d}x\right|\\&\le C\left\|\partial_{ij}\tau\right\|_{L^2}\left(\|\tau\|_{L^\infty}\|\nabla v\|_{L^\infty}+\|\nabla u\|_{L^\infty}\right)\\&\le\eta\|u\|_{H^3}^2+C_\eta\left\|\partial_{ij}\tau\right\|_{L^2}^2+C\|v\|_{H^3}\|\tau\|_{H^2}^2,\end{aligned}$$

By the symmetry of τ, one has

$$\begin{aligned}|K_4| &= \left|2\int_\Omega \partial_{ij}\tau\left(\partial_i\left(\xi^{-1}\right)\left(-\partial_j Q(\tau,\nabla v)+\mu_2\partial_j\Gamma(\nabla u)\right)\right)\mathrm{d}x\right| \\ &\le C\left\|\partial_{ij}\tau\right\|_{L^2}\left(\|\nabla\tau\|_{H^1}\|\nabla v\|_{H^2}+\|\tau\|_{H^2}\left\|\nabla^2 v\right\|_{L^2}+\|u\|_{H^2}\right) \\ &\le \eta\|u\|_{H^3}^2+C_\eta\left\|\partial_{ij}\tau\right\|_{L^2}^2+C\|v\|_{H^3}\|\tau\|_{H^2}^2.\end{aligned}$$

Thus, the lemma is proved. □

According to 3.1-3.3, Lemmas 2-2 and Grönwall's inequality, we obtain

$$\begin{aligned}&\left(\left\|(\lambda(\rho-1),u,\tau)\right\|_{H^2}^2+\left\|(\lambda\rho_t,u_t,\tau_t)\right\|_{H^1}^2\right) \\ &+\int_0^t\left(\left\|(\tau,\mathrm{div}u,\mathrm{curl}u)\right\|_{H^2}^2+\left\|(\tau_t,\mathrm{div}u_t,\mathrm{curl}u_t)\right\|_{H^1}^2\right)\mathrm{d}t \\ &\le \mathrm{e}^{Ct}\left[\left\|(\lambda(\rho_0-1),u_0,\tau_0)\right\|_{H^2}^2+\left\|(\lambda\rho_t(0),u_t(0),\tau_t(0))\right\|_{H^1}^2\right].\end{aligned} \tag{60}$$

Using the Grönwall's inequality and invoking the constraint conditions of the initial data, the following lemma can be obtained.

Lemma 3.14 *There is a positive constant* C_0 *such that*

$$\sum_{i=0}^{1}\left\|\left(\partial_i^t(\lambda(\rho-1)),\partial_i^t u,\partial_i^t\tau\right)\right\|_{H^{2-i}(\Omega)}(t)+\left\|\partial_i^t u\right\|_{L^2\left(0,T;H^{3-i}(\Omega)\right)}\le C_0,\quad 0\le t\le T, \tag{61}$$

here T is sufficiently small.

Note that by the above estimates, we obtain

$$\|\rho-1\|_{H^2(t)}\le C_0\lambda^{-1}.$$

According to the Sobolev imbedding $H^2\hookrightarrow L^\infty$, one has

$$|\rho(x,t)-1|\le C'\lambda^{-1}.$$

Hence, there is a sufficiently large constant λ_0 such that $\lambda\ge\lambda_0$, it can be deduced that

$$C_1^{-1}\le\left|\rho^{-1}(x,t)\right|\le C_1,\quad (x,t)\in\Omega\times[0,T]$$

for some constant $C>0$.

Then by estimating the second-order time, we can complete the energy estimates of the solutions to the linearized system.

Lemma 3.15 *The following inequality holds*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\sqrt{P'(\xi)\xi^{-1}}\lambda\rho_{tt}\right\|_{L^2}^2+\left\|\sqrt{\xi}u_{tt}\right\|_{L^2}^2\right)+K\|u_{tt}\|_{H^1}^2 \\ &\le\eta\left(\|\mathrm{div}u_t\|_{H^1}^2+\|u\|_{H^3}^2\right)+C_\eta\|\lambda\rho_{tt}\|_{L^2}^2+C\left(1+\|v\|_{H^3}+\|v_t\|_{H^2}+\|v_{tt}\|_{H^1}\right) \\ &\quad\times\left(\left\|(\lambda\rho_{tt},\lambda\nabla\rho_t,u_{tt},\tau_{tt})\right\|_{L^2}^2+\left\|(\lambda\nabla\rho,\nabla u,u_t)\right\|_{H^1}^2+\|u\|_{H^1}^2\right).\end{aligned} \tag{62}$$

where K is a positive constant.

Proof. Multiplying $\partial_{tt}(24)$ by $p'(\xi)\xi^{-1}\lambda^2\rho_{tt}$ and then integrating on Ω to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{p'(\xi)\xi^{-1}}\lambda\rho_{tt}\right\|_{L^2}^2-\int_\Omega p'(\xi)\lambda^2\nabla\rho_{tt}\cdot u_{tt}\mathrm{d}x\\
&=\frac{1}{2}\int_\Omega\left[\left(p'(\xi)\xi^{-1}\right)_t+\operatorname{div}\left(p'(\xi)\xi^{-1}v\right)\right]|\lambda\rho_{tt}|^2\mathrm{d}x+\int_\Omega\lambda^2p''(\xi)\nabla\xi\rho_{tt}u_{tt}\mathrm{d}x\\
&\quad-\int_\Omega p'(\xi)\xi^{-1}\lambda^2\rho_{tt}\left[v_{tt}\cdot\nabla\rho+2v_t\cdot\nabla\rho_t+\xi_{tt}\operatorname{div}u+2\xi_t\operatorname{div}u_t\right]\mathrm{d}x.\\
&=\sum_{i=1}^3L_i.
\end{aligned}\tag{63}
$$

Let $G_2(\xi):=p'(\xi)\xi^{-1}$. It turns out by the (24) that

$$\left(p'(\xi)\xi^{-1}\right)_t+\operatorname{div}\left(p'(\xi)\xi^{-1}v\right)=\left[G_2(\xi)-G_2'(\xi)\xi\right]\operatorname{div}v,$$

This implies

$$|L_1|\le C\left(\|G_2(\xi)\|_{L^\infty}+\|G_2'(\xi)\|_{L^\infty}\|\xi\|_{L^\infty}\right)\|\operatorname{div}v\|_{L^\infty}\|\lambda\rho_{tt}\|_{L^2}^2\le C\|v\|_{H^3}\|\lambda\rho_{tt}\|_{L^2}^2,$$

$$|L_2|\le C\|p''(\xi)\|_{L^\infty}\|\lambda\nabla\xi\|_{H^1}\|\lambda\rho_{tt}\|_{L^2}\|u_{tt}\|_{H^1}\le\eta\|u_{tt}\|_{H^1}^2+C_\eta\|\lambda\rho_{tt}\|_{L^2}^2,$$

$$
\begin{aligned}
|L_3|&\le C\|\lambda\rho_{tt}\|_{L^2}\left(\|v_{tt}\|_{L^3}\|\lambda\nabla\rho\|_{H^1}+\|v_t\|_{L^\infty}\|\lambda\nabla\rho_t\|_{L^2}\right.\\
&\quad\left.+\|\lambda\xi_{tt}\|_{L^2}\|\operatorname{div}u\|_{L^\infty}+\|\lambda\xi_t\|_{H^1}\|\operatorname{div}u_t\|_{L^3}\right.\\
&\le C\left(\|v_{tt}\|_{H^1}+\|v_t\|_{H^2}\right)\left(\|(\lambda\rho_{tt},\lambda\nabla\rho_t)\|_{L^2}^2+\|\lambda\nabla\rho\|_{H^1}^2\right)\\
&\quad+\eta\left(\|u\|_{H^3}^2+\|\operatorname{div}u_t\|_{H^1}^2\right)+C_\eta\|\lambda\rho_{tt}\|_{L^2}.
\end{aligned}\tag{64}
$$

Clearly,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{P'(\xi)\xi^{-1}}\lambda\rho_{tt}\right\|_{L^2}^2-\int_\Omega P'(\xi)\lambda^2\nabla\rho_{tt}\cdot u_{tt}\mathrm{d}x\\
&\le\eta\left(\|u_{tt}\|_{H^1}^2+\|\operatorname{div}u_t\|_{H^1}^2+\|u\|_{H^3}^2\right)+C_\eta\|\lambda\rho_{tt}\|_{L^2}^2\\
&\quad+C\left(\|v\|_{H^3}+\|v_t\|_{H^2}+\|v_{tt}\|_{H^1}\right)\left(\|(\lambda\rho_{tt},\lambda\nabla\rho_t)\|_{L^2}^2+\|\lambda\nabla\rho\|_{H^1}^2\right).
\end{aligned}\tag{65}
$$

Multiply $\partial_{tt}\xi(25)$ by u_{tt} and then integrate the result on Ω, it follows that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\xi}u_{tt}\right\|_{L^2}^2+\left(2\mu(\xi)+\lambda(\xi)\right)\|\operatorname{div}u_{tt}\|_{L^2}^2+\mu(\xi)\|\operatorname{curl}u_{tt}\|_{L^2}^2+\int_\Omega p'(\xi)\lambda^2\nabla\rho_{tt}\cdot u_{tt}\mathrm{d}x\\
&=\frac{1}{2}\int_\Omega\xi_t|u_{tt}|^2\mathrm{d}x-\int_\Omega u_{tt}\{\xi_{tt}u_t+2\xi_tu_{tt}+\xi_{tt}v\cdot\nabla u+2\xi_tv_t\cdot\nabla u+2\xi_tv\cdot\nabla u+\xi v_{tt}\cdot\nabla u\\
&\quad+2\xi v_t\cdot\nabla u_t+\xi v\cdot\nabla u_{tt}+\int_\Omega p'''(\xi)\xi_t^2\lambda^2\nabla\rho+p''(\xi)\xi_{tt}\lambda^2\nabla\rho+2p''(\xi)\xi_t\lambda^2\nabla\rho_t\\
&\quad-\nabla\left(2\mu(\xi)+\lambda(\xi)\right)\operatorname{div}u_{tt}-\nabla\left(\mu(\xi)\right)\times\operatorname{curl}u_{tt}+2\nabla\left(\mu(\xi)\right)D(u_{tt})\\
&\quad+\nabla\left(\lambda(\xi)\right)\operatorname{div}u_{tt}-2\operatorname{div}\left(\partial_t\left(\mu(\xi)\right)D(u_t)\right)-2\nabla\left(\partial_t\left(\lambda(\xi)\right)\operatorname{div}u_t\right)\\
&\quad-\operatorname{div}\left(2\partial_{tt}\left(\mu(\xi)\right)D(u)\right)-\nabla\left(\partial_{tt}\left(\lambda(\xi)\right)\operatorname{div}u\right)-\mu_1\operatorname{div}\tau_{tt}\Big\}\mathrm{d}x\\
&\le\eta\|u_{tt}\|_{H^1}^2+C_\eta\|(\lambda\nabla\rho,\lambda\nabla\rho_t,u_{tt},\tau_{tt})\|_{L^2}^2+\|(\nabla u,u_t)\|_{H^1}^2+C\|\nabla u_{tt}\|_{L^2}\|\xi_{tt}\|_{L^2}\|u\|_{H^3}.
\end{aligned}\tag{66}
$$

Therefore, the lemma is proved by (65), (66) and lemma 2. □

Finally, multiplying $\partial_{tt}(26)$ by τ_{tt} in $L^2(\Omega)$ to obatin

Lemma 3.16 *The following inequality holds*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau_{tt}\|_{L^2}^2+a\|\tau_{tt}\|_{L^2}^2\\&\le\eta\left(\|\nabla u_{tt}\|_{L^2}^2+\|\nabla u_t\|_{H^1}^2+\|\nabla u\|_{H^2}^2\right)+C_\eta\|\tau_{tt}\|_{L^2}^2\\&\quad+C\left(\|v\|_{H^3}+\|v_t\|_{H^2}+\|v_{tt}\|_{H^1}\right)\left(\|\tau_{tt}\|_{L^2}^2+\|\tau_t\|_{H^1}^2+\|\tau\|_{H^2}^2\right).\end{aligned}\tag{67}$$

The calculations are quite similar as the above lemmas, here we omit the details. Summarizing (62) and (67) and using Grönwall's inequality, we deduce that

Lemma 3.17 *The following inequality holds*

$$\left\|\left(\sqrt{p'(\xi)\xi^{-1}}\lambda\rho_{tt},\sqrt{\xi}u_{tt},\tau_{tt}\right)\right\|_{L^2}^2+\int_0^t\left(K\|u_{tt}\|_{H^1}^2+2a\|\tau_{tt}\|_{L^2}^2\right)\mathrm{d}s\le C_0,\quad 0\le t\le T,\tag{68}$$

where T is sufficiently small.

Collecting all the lemmas proved in this section, Lemma 2 is obtained.

4. The Proof of Main Theorems

Proof of Theorem 1.1. The proof of this theorem is based on the use of the method of successive approximations and uniform-on-λ estimates obtained in Lemma 2. Set $(\rho^0,u^0,\tau^0)=(\rho_0,u_0,\tau_0)$. For any fixed $\lambda(\ge\lambda_0)$, a sequence $\left\{(\rho^{n+1},u^{n+1},\tau^{n+1})\right\}_{n\ge0}$ is generated by the standard Picard iteration [26] that satisfies the following equations:

$$\rho_t^{n+1}+u^n\cdot\nabla\rho^{n+1}+\rho^n\mathrm{div}u^{n+1}=0,\tag{69}$$

$$\begin{aligned}&u_t^{n+1}+u^n\cdot\nabla u^{n+1}+\frac{p'(\rho^n)}{\rho^n}\lambda^2\nabla p^{n+1}\\&=\frac{1}{\rho^n}\left[\mathrm{div}\left(2\mu(\xi)D(u^{n+1})\right)+\nabla\left(\lambda(\xi)\mathrm{div}u^{n+1}\right)\right]+\frac{\mu_1}{\rho^n}\mathrm{div}\tau^{n+1},\end{aligned}\tag{70}$$

$$\tau_t^{n+1}+u^n\cdot\nabla\tau^{n+1}+a\tau^{n+1}+\frac{1}{\rho^n}Q(\tau^{n+1},\nabla u^n)=\frac{\mu_2}{\rho^n}\Gamma(\nabla u^{n+1})\tag{71}$$

with (7), (8), and

$$\sup_{0\le t\le T}\sum_{i=0}^{2}\left\|\partial_i^t\left(\lambda(\rho^n-1)\right),\partial_i^tu^n,\partial_i^t\tau^n\right\|_{H^{2-i}}+\left\|\partial_i^tu^n\right\|_{L^2(0,T_0;H^{3-i})}\le C_0.\tag{72}$$

Denote $(\bar\rho^n,\bar u^n,\bar\tau^n):=(\rho^{n+1}-\rho^n,u^{n+1}-u^n,\tau^{n+1}-\tau^n)$. Base on the above equations, it can be deduced that

$$\bar\rho_t^{n+1}+u^{n+1}\cdot\nabla\bar\rho^{n+1}+\rho^{n+1}\mathrm{div}\bar u^{n+1}+\bar u^n\cdot\nabla\rho^{n+1}+\bar\rho^n\mathrm{div}u^{n+1}=0,\tag{73}$$

$$\begin{aligned}&\bar u_t^{n+1}+u^{n+1}\cdot\nabla\bar u^{n+1}+\frac{p'(\rho^{n+1})}{\rho^{n+1}}\lambda^2\nabla\bar\rho^{n+1}+\bar u^n\cdot\nabla u^{n+1}+\lambda^2\left(\frac{p'(\rho^{n+1})}{\rho^{n+1}}-\frac{p'(\rho^n)}{\rho^n}\right)\nabla\rho^{n+1}\\&=\frac{1}{\rho^{n+1}}\left[\mathrm{div}\left(2\mu(\xi)D(u^\lambda)\right)+\nabla\left(\lambda(\xi)\mathrm{div}u^\lambda\right)\right]+\frac{\mu_1}{\rho^{n+1}}\mathrm{div}\tau^{n+1}\\&\quad+\left(\frac{1}{\rho^{n+1}}-\frac{1}{\rho^n}\right)\left(\mathrm{div}\left(2\mu(\xi)D(u^\lambda)\right)+\nabla\left(\lambda(\xi)\mathrm{div}u^\lambda\right)\right)+\mu_1\left(\frac{1}{\rho^{n+1}}-\frac{1}{\rho^n}\right)\mathrm{div}\tau^{n+1},\end{aligned}\tag{74}$$

$$\bar{\tau}_t^{n+1} + u^{n+1} \cdot \nabla \bar{\tau}^{n+1} + a\bar{\tau}^{n+1} + \left[\frac{1}{\rho^{n+1}} Q\left(\tau^{n+2}, \nabla u^{n+1}\right) - \frac{1}{\rho^n} Q\left(\tau^{n+1}, \nabla u^n\right) \right] + \bar{u}^n \cdot \nabla \tau^{n+1}$$
$$= \frac{\mu_2}{\rho^{n+1}} \Gamma\left(\nabla \bar{u}^{n+1}\right) + \mu_2 \left(\frac{1}{\rho^{n+1}} - \frac{1}{\rho^n} \right) \Gamma\left(\nabla u^{n+1}\right). \quad (75)$$

Estimating as before, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\| \left(\lambda\bar{\rho}^{n+1}, \bar{u}^{n+1}, \bar{\tau}^{n+1}\right) \right\|_{L^2}^2 + \left\| \bar{\tau}^{n+1} \right\|_{L^2}^2 + \left\| \mathrm{div}\bar{u}^{n+1} \right\|_{L^2}^2 + \left\| \mathrm{curl}\bar{u}^{n+1} \right\|_{L^2}^2 \\ &\le \eta \left\| \bar{u}^n \right\|_{H^1}^2 + C_\eta \left\| \left(\lambda\bar{\rho}^{n+1}, \bar{\tau}^{n+1}\right) \right\|_{L^2}^2 + C\left(\left\| \bar{u}^{n+1} \right\|_{L^2}^2 + \left\| \left(\lambda\bar{\rho}^n, \bar{u}^n\right) \right\|_{L^2}^2 \right) \\ &\quad + C\left\| u^{n+1} \right\|_{H^3} \left(\left\| \left(\lambda\bar{\rho}^{n+1}, \bar{u}^{n+1}, \bar{\tau}^{n+1}\right) \right\|_{L^2}^2 + \left\| \lambda\bar{\rho}^n \right\|_{L^2}^2 \right) \end{aligned} \quad (76)$$

According to Grönwall's inequality, we obtain that

$$\begin{aligned} &\left\| \left(\lambda\bar{\rho}^{n+1}, \bar{u}^{n+1}, \bar{\tau}^{n+1}\right) \right\|_{L^2}^2 + \int_0^t \left(\left\| \bar{\tau}^{n+1} \right\|_{L^2}^2 + \left\| \mathrm{div}\bar{u}^{n+1} \right\|_{L^2}^2 + \left\| \mathrm{curl}\bar{u}^{n+1} \right\|_{L^2}^2 \right) \mathrm{d}s \\ &\le \exp\left[\int_0^t \left(C_\eta + \left\| u^{n+1} \right\|_{H^3} \right) \mathrm{d}s \right] \int_0^t \left[\eta \left\| \bar{u}^n \right\|_{H^1}^2 + C\left\| \left(\lambda\bar{\rho}^n, \bar{u}^n\right) \right\|_{L^2}^2 + C\left\| u^{n+1} \right\|_{H^3} \left\| \lambda\bar{\rho}^n \right\|_{L^2}^2 \right] \mathrm{d}s \quad (77) \\ &\le \frac{1}{2}\left(\int_0^t \left\| \bar{u}^n \right\|_{H^1}^2 + \sup_{0\le t\le T} \left\| \left(\lambda\bar{\rho}^n, \bar{u}^n\right) \right\|_{L^2}^2 \right), \end{aligned}$$

where η and T are small enough.

Therefore, there is a constant $T_0 > 0$ such that

$$\left(\rho^n, u^n, \tau^n\right) \to \left(\rho^\lambda, u^\lambda, \tau^\lambda\right) \text{ in } L^\infty\left(0, T_0; L^2\right), \text{ and } u^n \to u^\lambda \text{ in } L^2\left(0, T_0; H^1\right). \quad (78)$$

Base on the estimate (72), $\left(\rho^\lambda, u^\lambda, \tau^\lambda\right)$ satisfies (12). Therefore, Theorem 1 is proved. □

Proof of Theorem 1.2. Due to the similarity of the proof process with the previous article [19], it is not repeated here.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Chemin, J.-Y. and Masmoudi, N. (2001) About Lifespan of Regular Solutions of Equations Related to Viscoelastic Fluids. *SIAM Journal on Mathematical Analysis*, **33**, 84-112. https://doi.org/10.1137/S0036141099359317

[2] Lions, P. and Masmoudi, N. (2000) Global Solutions for Some Oldroyd Models of Non-Newtonian Flows. *Chinese Annals of Mathematics*, **21**, 131-146. https://doi.org/10.1142/S0252959900000170

[3] Majda, A. (2012) Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables. Springer Science & Business Media, Berlin, 53.

[4] Ebin, D.G. (1975) Motion of a Slightly Compressible Fluid. *Proceedings of the National Academy of Sciences*, **72**, 539-542. https://doi.org/10.1073/pnas.72.2.539

[5] Klainerman, S. and Majda, A. (1981) Singular Limits of Quasilinear Hyperbolic Systems with Large Parameters and the Incompressible Limit of Compressible Fluids. *Communications on Pure and Applied Mathematics*, **34**, 481-524. https://doi.org/10.1002/cpa.3160340405

[6] Jiang, S., Ju, Q. and Li, F. (2010) Incompressible Limit of the Compressible Magnetohydrodynamic Equations with Periodic Boundary Conditions. *Communications in Mathematical Physics*, **297**, 371-400. https://doi.org/10.1007/s00220-010-0992-0

[7] Ou, Y. (2011) Low Mach Number Limit of Viscous Polytropic Fluid Flows. *Journal of Differential Equations*, **251**, 2037-2065. https://doi.org/10.1016/j.jde.2011.07.009

[8] Feireisl, E. and Novotny, A. (2013) Inviscid Incompressible Limits of the Full Navier-Stokes-Fourier System. *Communications in Mathematical Physics*, **321**, 605-628. https://doi.org/10.1007/s00220-013-1691-4

[9] Fan, J., Gao, H. and Guo, B. (2011) Low Mach Number Limit of the Compressible Magnetohydrodynamic Equations with Zero Thermal Conductivity Coefficient. *Mathematical Methods in the Applied Sciences*, **34**, 2181-2188. https://doi.org/10.1002/mma.1515

[10] Zhang, J.Y. (2022) Local Well-Posedness and Incompressible Limit of the Free-Boundary Problem in Compressible Elastodynamics. *Archive for Rational Mechanics and Analysis*, **244**, 599-697. https://doi.org/10.1007/s00205-022-01774-4

[11] Wang, Y.-X. (2023) Incompressible Limit of the Compressible Q-Tensor System of Liquid Crystals. *Acta Mathematicae Applicatae Sinica*, *English Series*, **39**, 179-201. https://doi.org/10.1007/s10255-023-1033-z

[12] Chen, Z.-M. and Zhai, X.P. (2019) Global Large Solutions and Incompressible Limit for the Compressible Navier-Stokes Equations. *Journal of Mathematical Fluid Mechanics*, **21**, 1-23. https://doi.org/10.1007/s00021-019-0428-3

[13] Klainerman, S. (1985) Uniform Decay Estimates and the Lorentz Invariance of the Classical Wave Equation. *Communications on Pure and Applied Mathematics*, **38**, 321-332. https://doi.org/10.1002/cpa.3160380305

[14] Sideris, T.C. (2000) Nonresonance and Global Existence of Prestressed Nonlinear Elastic Waves. *Annals of Mathematics*, **151**, 849-874. https://doi.org/10.2307/121050

[15] Sideris, T.C. and Thomases, B. (2005) Global Existence for 3d Incompressible Isotropic Elastodynamics. *Communications on Pure and Applied Mathematics*.

[16] Guillopé, C. and Saut, J.-C. (1989) Existence Results for the Flow of Viscoelastic Fluids with a Differential Constitutive Law.

[17] Fang, D. and Zi, R. (2014) Incompressible Limit of Oldroyd-B Fluids in the Whole Space. *Journal of Differential Equations*, **256**, 2559-2602. https://doi.org/10.1016/j.jde.2014.01.017

[18] Lei, Z. (2006) Global Existence of Classical Solutions for Some Oldroyd-B Model via the Incompressible Limit. *Chinese Annals of Mathematics*, *Series B*, **27**, 565-580. https://doi.org/10.1007/s11401-005-0041-z

[19] Ren, D. and Ou, Y. (2015) Strong Solutions to an Oldroyd-B Model with Slip Boundary Conditions via Incompressible Limit. *Mathematical Methods in the Applied Sciences*, **38**, 330-348. https://doi.org/10.1002/mma.3071

[20] Bourguignon, J.-P. and Brezis, H. (1974) Remarks on the Euler Equation. *Journal of Functional Analysis*, **15**, 341-363. https://doi.org/10.1016/0022-1236(74)90027-5

[21] Xiao, Y. and Xin, Z. (2007) On the Vanishing Viscosity Limit for the 3d Navier-Stokes Equations with a Slip Boundary Condition. *Communications on Pure and Applied Mathematics*, **60**, 1027-1055. https://doi.org/10.1002/cpa.20187

[22] Friedman, A. (2008) Partial Differential Equations of Parabolic Type. Courier Dover Publications, Mineola.

[23] Song, S.M. (1990) A Note on Gronwall's Inequality. *Journal of the Chungcheong Mathematical Society*, **3**, 115-120.

[24] Clark, D.S. (1987) Short Proof of a Discrete Gronwall Inequality. *Discrete Applied Mathematics*, **16**, 279-281. https://doi.org/10.1016/0166-218X(87)90064-3

[25] Wang, X.Q. (2022) Several Inequalities of Gronwall and Their Proofs. *Insight-Information*, **4**, 58-63.

[26] Andrade, F., Figueiredo, M. and Xavier, J. (2023) Distributed Banach-Picard Iteration: Application to Distributed Parameter Estimation and PCA. *IEEE Transactions on Signal Processing*, **71**, 17-30. https://doi.org/10.1109/TSP.2023.3239806