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Entropy Formulation for Triply Nonlinear Degenerate Elliptic-Parabolic-Hyperbolic Equation with Zero-Flux Boundary Condition

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Abstract

In this note, we investigated existence and uniqueness of entropy solution for triply nonlinear degenerate parabolic problem with zero-flux boundary condition. Accordingly to the case of doubly nonlinear degenerate parabolic hyperbolic equation, we propose a generalization of entropy formulation and prove existence and uniqueness result without any structure condition.

Keywords

Degenerate Elliptic-Parabolic Hyerbolic Equation, Zero-Flux Boundary Condition, Structure Condition, Entropy Formulation, Multi-Step Approximation, Nonlinear Semigroup Theories, Integral and Mild Solution

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^{\ell}$ with a Lipschitz boundary $\partial\Omega$ and η the unit normal to $\partial\Omega$ outward to Ω. We consider the triply nonlinear degenerate parabolic-elliptic-hyperbolic problem with zero-flux boundary condition:

$$\begin{cases} b(u)_t + \operatorname{div} f(u) - \Delta\phi(u) + \psi(u) = 0 & \text{in } Q = (0,T)\times\Omega \\ b(u)(t=0,x) = b_0(x) & \text{in } \Omega \\ \big(f(u) - \nabla\phi(u)\big)\cdot\eta = 0 & \text{on } \Sigma = (0,T)\times\partial\Omega \end{cases} \tag{P}$$

The particularity of this problem is its strong degeneracy. For practical reason and physical consideration, we suppose that [0, 1] will be the invariant domain of solution of (P) and that there exist two particular values of the unknown u. We denote by u_s and u_c (with $0 < u_c \le u_s < 1$) such that $\phi(u)$ (resp $b(u)$) is strictly increasing only on $[u_c, 1]$ (resp $[0, u_c] \cup [u_s, 1]$) otherwise it has a flat region (see **Figure 1**). Then, problem (P) is of mixed elliptic parabolic hyperbolic type with absorption term, and thus combines the difficulties related to nonlinear conservation laws with those related to nonlinear degenerate diffusion equations. We refer to Kruzkov [1] for the case of hyperbolic problem ($\phi \equiv 0$) and Carrillo [2] for degenerate parabolic problem to inspiration. We need a notion of solution which is sufficient to deal with existence and uniqueness. One consequence is that the notion of weak solution generally leads to non-uniqueness, unless ϕ is strictly increasing. It is necessary to adopt an entropy formulation. The notion of entropy solution we use is adapted from the founding paper of [3] in the case where $u_s = u_c$ and $\phi = 0$. Several authors have studied the degenerate equation type we consider.

Some of these authors ([3] [4]) proved existence and uniqueness under the hypothesis that the convection flux *f* is a Lipschitz-continuous function and required that

$$f(0) = 0, \quad f(1) = 0. \tag{H}$$

This hypothesis is necessary to obtain the solution in [0, 1] if the initial datum u_0 belongs in [0, 1] in the sense that $b \circ u_0(x) = b_0(x)$ and the hypothesis (H) is below. The main idea in the paper is to keep this hypothesis but we impose that initial datum u_0 belongs to [0, 1]. We suppose that the function b satisfies:

$$\exists(\underline{\alpha}, \overline{\alpha}) \in \mathbb{R}_+^2 \text{ such that } b(u) \in [\underline{\alpha}, \overline{\alpha}]. \tag{1.1}$$

A simple choice is to take $b(0) = \underline{\alpha}$ and $b(1) = \overline{\alpha}$. Our assumptions on b and ϕ do not concern the case where the structure condition

$$b(u) = b(v) \Rightarrow \phi(u) = \phi(v). \tag{S.C}$$

holds. The presence of the absorption term in the equation requires us to assume it to be increasing. Further, we suppose:

$$\psi(r) = sign(r)\, g(r) \quad \text{where } g \text{ is a positif function.} \tag{1.2}$$

The case of triply nonlinear problems of the form (P) has been first addressed by Ouaro and Touré (see [5]) and the references therein) and Ouaro [6].

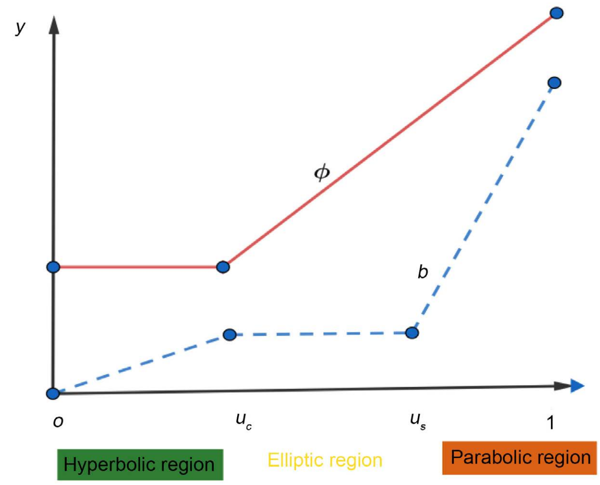

Figure 1. Convection and diffusion flow.

Well-posedness results are obtained in dimension $\ell = 1$, under very general coercivity conditions; see also the works of Bénilan and Touré ([7] and the references therein). Andreianov and Wittbold investigate in [8] about the continuous dependence of the solution of a degenerate elliptic-parabolic equation without structure condition related to b and f. They prove existence by passing to bi-monotonicity and penalization method as in [9]. Otherwise, in [10], Andreianov *et al.* obtain a general continuous dependence result on data for our kind of triply nonlinear problem with help of structure condition. They showed similar result for the degenerate elliptic problem, which corresponds to the case of $b \equiv 0$ and general non-decreasing surjective ψ. In our case the function ψ is bounded continuous and strictly increasing. If (S.C) fails, the convergence of approximate solutions to (P) is known for a particular monotone approximation method developed by Ammar and Wittbold [9]. This approach leads to an existence result which bypasses (S.C). Notice that some essential arguments of uniqueness result in this works are specific to the case $\ell = 1$. For Neumann boundary condition also called zero-flux boundary condition, it is easy to prove uniqueness of solutions such that the boundary condition is satisfied in the sense of strong boundary trace of the normal component of the flux $\left(f(u) - \nabla\phi(u)\right)$. Unfortunately, we are able to establish this additional solution regularity only for the stationary problem (S) associated to (P) and only in the case of one space dimension.

The paper is divided in three parts, in Section 2 we generalized the notion of entropy solution of paper [3] where $u_c < u_s$ and [4] in the pure hyperbolic case. In Section 3, we first prove existence and after uniqueness of entropy solution.

2. Formulation of Entropy Solution

2.1. Definition of Entropy Solution

We need the notion of weak solution for (P) with additional "entropy" conditions.

Definition 2.1. A measurable function u taking values on [0, 1] is called an entropy solution of the initial-boundary value problem (P) if satisfying the following conditions:

$$\phi(u) \in L^2\left(0,T;H^1(\Omega)\right) \text{ and } \forall \xi \in \mathcal{C}^\infty\left([0,T)\times\mathbb{R}^\ell\right), \text{ with } \xi \geq 0$$

$$\int_0^T \left\langle b(u)_t, \xi \right\rangle \mathrm{d}t - \int_0^T\int_\Omega \left(f(u) - \nabla\phi(u)\right)\cdot\nabla\xi \,\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \psi(u)\xi \,\mathrm{d}x\mathrm{d}t = 0. \tag{2.1}$$

$\phi(u) \in L^2\left(0,T;H^1(\Omega)\right)$ and $\forall k \in [0,u_c] \cup [u_s,1]$, $\forall \xi \in \mathcal{C}^\infty\left([0,T)\times\mathbb{R}^\ell\right)$, with $\xi \geq 0$,

$$\begin{aligned}
&\int_0^T\int_\Omega \left\{ \left|b(u) - b(k)\right|\xi_t + sign(u-k)\left(f(u) - f(k) - \nabla\phi(u)\right)\cdot\nabla\xi \right\} \mathrm{d}x\mathrm{d}t \\
&+ \int_0^T\int_{\partial\Omega} \left|f(k)\cdot\eta(x)\right|\xi(t,x)\,\mathrm{d}\mathcal{H}^{\ell-1}(x)\,\mathrm{d}t - \int_0^T\int_\Omega sign(u-k)\psi(u)\xi\,\mathrm{d}x\mathrm{d}t \\
&+ \int_\Omega \left|b_0 - b(k)\right|\xi(0,x)\,\mathrm{d}x \geq 0.
\end{aligned} \tag{2.2}$$

Here $\mathcal{H}$ represents the $(\ell - 1)$-dimensional Hausdorff measure and $\langle .,. \rangle$ is the duality pairing between $L^2\left(0,T,\left(H^1(\Omega)\right)^\star\right) + L^1(Q)$ and

$L^2\left(0,T,H^1(\Omega)\right)\cap L^\infty(Q)$

It is well known that the distributional derivative $b(u)_t$ of $b(u)$ can be identified with an element of the space $L^2\left(0,T,\left(H^1\right)^*(\Omega)\right)+L^1(Q)$. More exactly, we have

$$\int_0^T \left\langle b(u)_t,\xi\right\rangle \mathrm{d}t=\int_Q b(u)\xi_t-\int_\Omega b_0\xi(0,x)\mathrm{d}x \tag{2.3}$$

for all $\xi\in L^2\left(0,T;H^1(\Omega)\right)$ with $\xi_t\in L^\infty(Q)$ and $\xi(T,x)=0$.

We obtain notions of entropy sub-solution and entropy super-solution respectively if we replace (2.2) by one of the following inequalities

$$\begin{aligned}&\int_0^T\int_\Omega\left\{\left(b(u)-b(k)\right)^+\xi_t+sign^+(u-k)\left(f(u)-f(k)-\nabla\phi(u)\right)\cdot\nabla\xi\right\}\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega\left(f(k)\cdot\eta(x)\right)^+\xi(t,x)\mathrm{d}\mathcal{H}^{\ell-1}(x)\mathrm{d}t-\int_0^T\int_\Omega sign^+(u-k)\psi(u)\xi\mathrm{d}x\mathrm{d}t\\&+\int_\Omega\left(b_0-b(k)\right)^+\xi(0,x)\mathrm{d}x\ge 0\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\int_0^T\int_\Omega\left\{\left(b(u)-b(k)\right)^-\xi_t+sign^-(u-k)\left(f(u)-f(k)-\nabla\phi(u)\right)\cdot\nabla\xi\right\}\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega\left(f(k)\cdot\eta(x)\right)^-\xi(t,x)\mathrm{d}\mathcal{H}^{\ell-1}(x)\mathrm{d}t-\int_0^T\int_\Omega sign^-(u-k)\psi(u)\xi\mathrm{d}x\mathrm{d}t\\&+\int_\Omega\left(b_0-b(k)\right)^-\xi(0,x)\mathrm{d}x\ge 0.\end{aligned} \tag{2.5}$$

Remark 2.2.

Obviously, a function u is an entropy solution if and only if u is entropy sub-solution and entropy super-solution simultaneously.

Our notion of entropy solution coincide with the Definition of [3] in the case $u_s=u_c$, $\psi\equiv 0$ and assumption (S.C) is trivially satisfied.

Notice that if u satisfy (2.2), then use (1.1) and (H), we have also u verify (2.1).

Let us stress that, if (H) is satisfy, in particular, the zero-flux boundary condition $\left(f(u)-\nabla\phi(u)\right)\cdot\eta=0$ is verified literally in the weak sense (see for exemple [3] and [11]). A forthcoming work is to envisage envisage (P) if assumption (H) is dropped. We expect that the boundary condition should be relaxed.

2.2. Dissipative Property

We propose here an essential property of entropy solutions, based on the idea of J. Carrillo [2].

Proposition 2.3. Let $\xi\in\mathcal{C}^\infty\left([0,T[\times\mathbb{R}^\ell\right)$, $\xi\ge 0$. Then for all $k\in[u_s,1]$, for all $D\in\mathbb{R}^\ell$ for all entropy solution u of (P), we have

$$\begin{aligned}&\int_0^T\int_\Omega\left\{\left|b(u)-b(k)\right|\xi_t+sign(u-k)\left(f(u)-f(k)-\nabla\phi(u)-D\right)\cdot\nabla\xi\right\}\\&+\int_0^T\int_\Omega\left|\left(f(k)-D\right)\cdot\eta(x)\right|\xi(t,x)\mathrm{d}\mathcal{H}^{\ell-1}(x)\mathrm{d}t-\int_0^T\int_\Omega sign(u-k)\psi(u)\xi\mathrm{d}x\mathrm{d}t\\&+\int_\Omega\left|b_0-b(k)\right|\xi(0,x)\mathrm{d}x\ge\overline{\lim_{\sigma\to 0}}\frac{1}{\sigma}\int\int_{Q\cap A_\phi^\sigma}\nabla\phi(u)\cdot\left(\nabla\phi(u)-D\right)\xi.\end{aligned} \tag{2.6}$$

where $A_\phi^\sigma = \{(t,x) \in Q \text{ such that } -\sigma < \phi(u) - \phi(k) < \sigma\}$.

Proof. The ingredient of the proof of Proposition 2.7 can from firstly, for all $u \in [0,u_c] \cup [u_s,1]$ and for all $k \in [u_s,1]$, one has:

$$sign(u-k) = sign(b(u)-b(k)) = sign(\phi(u)-\phi(k)).$$

Secondly, $\nabla\phi(u) = 0$ $\mathcal{L}^N$ a.e. on the set $\{(t,x) \in Q \text{ such that } u(t,x) \in [0,u_c]\}$.

Taking as test function in (2.1) $sign_\sigma(\phi(u)-\phi(k))\xi$ with $\xi \in C^\infty([0,T[\times\mathbb{R}^\ell)$ and $sign_\sigma$ is the approximation of $sign$ function, using Chain rule (see [12]) and passing to the limit $\sigma \to 0$. For more details see [3]. ■

3. Existence and Uniqueness Result

3.1. Existence of Entropy Solution

The main result of this subsection is the following theorem:

Theorem 3.1. Assume that (1.1), (1.2), (H) holds. Then there exists an entropy solution u for (P).

3.1.1. Bi-Monotonicity Approach

Because of we are not in the case where (S.C) holds, we use the particular multi-step approximation approach of Ammar and Wittbold (see [9]).

Theorem 3.2. Let $(b_l, \phi_l, \psi_{m,n}, u_0^n)_{(l,m,n)\in\mathbb{N}^{*3}}$, a sequences converging to (b,ϕ,ψ,u_0) in the following sense:

$$b_l, \phi_l, \psi_{m,n} \text{ converge pointwise to } b,\phi,\psi \text{ as } l,m,n \to +\infty \tag{3.1}$$

$$u_0^n \in [0,1] \text{ converge a.e to } u_0 \tag{3.2}$$

$$b_l(u_0^n) \text{ converge to } b(u_0) \text{ in } L^1(\Omega). \tag{3.3}$$

There exists a weak solution $u_{m,n}^l$ of $(\mathrm{P})_{m,n}^l$ the analogue of (P) with corresponding data $(b_l, \phi_l, \psi_{m,n}, u_0^n)$.

Proof. We consider the particular multi-step approximation approach of Ammar and Witt bold [9]. We replace b by $b_l := b + \frac{1}{l} Id$, ϕ by $\phi_l := \phi + \frac{1}{l} Id$ and ψ, by $\psi_{m,n} := \psi + \frac{1}{n} Id^+ - \frac{1}{m} Id^-$. Hence $\phi_l, \phi_l^{-1}, b_l, b_l^{-1}$ are Lipschitz continuous on $\mathbb{R}$.

We obtain the following equation:

$$b_l(u_{m,n}^l)_t + \mathrm{div} f(u_{m,n}^l) - \Delta\phi_l(u_{m,n}^l) + \psi_{m,n}(u_{m,n}^l) = 0 \text{ in } Q = (0,T)\times\Omega$$

Take $v_{m,n}^l = b_l(u_{m,n}^l)$, hence b_l is invertible, one puts the problem into the doubly non-linear framework then we obtain the following problem

$$(v_{m,n}^l)_t + \mathrm{div}\tilde{f}(v_{m,n}^l) - \Delta\tilde{\phi}_l(v_{m,n}^l) + \tilde{\psi}_{m,n}(v_{m,n}^l) = 0 \text{ in } Q$$

where $\tilde{f} = f \circ b_l^{-1}$; $\tilde{\phi}_l = \phi \circ b_l^{-1}$ and $\tilde{\psi}_{m,n} = \psi_{m,n} \circ b_l^{-1}$. Using classical methods (cf. Andreianov and Gazibo [3]), one shows that there exists a weak solution $u_{m,n}^l$

for the corresponding problem $(\mathrm{P})_{m,n}^{l}$. ■

Theorem 3.3. Let $u_{m,n}^{l}$ be the weak solution of $(\mathrm{P})_{m,n}^{l}$ the analogue of (P) with corresponding data $\left(b_l,\phi_l,\psi_{m,n},u_0^n\right)$. Then $u_{m,n}^{l}$ is also entropy solution of $(\mathrm{P})_{m,n}^{l}$ in the sense of Definition 2.1 and converge to u entropy solution of (P) in L^∞ weakly star up to a subsequence. Furthermore:

$$\phi_l\left(u_{m,n}^{l}\right)\to\phi(u)\text{ in }L^1(Q) \tag{3.4}$$

$$b_l\left(u_{m,n}^{l}\right)\to b(u)\text{ in }L^1(Q) \tag{3.5}$$

$$\psi_{m,n}\left(u_{m,n}^{l}\right)\to\psi(u)\text{ in }L^1(Q). \tag{3.6}$$

3.1.2. A priori Estimates

Lemma 3.4. Let $\left(b_l,\phi_l,\psi_{m,n},u_0^n\right)_{(l,m,n)\in\mathbb{N}^{*3}}$, be a sequence of data satisfying the assumption of theorem 3.2. Assume that the corresponding data $\left(b,\phi,\psi,u_0\right)$ verifies (1.1), (1.2), (H). Let $u_{m,n}^{l}$ be an entropy solution of $(\mathrm{P})_{m,n}^{l}$ then there exist $L>0$ such that:

$$0\le u_{m,n}^{l}\le 1 \tag{3.7}$$

$$\left\|\phi_l\left(u_{m,n}^{l}\right)\right\|_{L^2\left(0,T;H^1(\Omega)\right)}\le L \tag{3.8}$$

$$\left\|\phi_l\left(u_{m,n}^{l}\right)\right\|_{L^2(Q)}\le L \tag{3.9}$$

$$\left\|\phi_l\left(u_{m,n}^{l}\right)\psi_{m,n}\left(u_{m,n}^{l}\right)\right\|_{L^1(Q)}\le L. \tag{3.10}$$

Proof. Since $u_{m,n}^{l}$ is entropy solution, it is also entropy subsolution and entropy super solution of $(\mathrm{P})_{m,n}^{l}$. Take $k=0$ in (2.5) and $\xi(t,x)=\xi(t)$ and (H) and (1.2) then

$$\int_0^T\left(\int_\Omega\left(\left(b_l\left(u_{m,n}^{l}\right)-b_l(0)\right)^- -\left(b_l\left(u_n^0\right)-b_l(0)\right)^-\right)\mathrm{d}x\right)\xi_t\mathrm{d}t\ge 0 \tag{3.11}$$

Let us introduce the function

$$H(t)=\int_\Omega\left(\left(b_l\left(u_{m,n}^{l}\right)-b_l(0)\right)^- -\left(b_l\left(u_n^0\right)-b_l(0)\right)^-\right)\mathrm{d}x\text{ if }t\in(0,T)$$

Since (1.1), we have $u_{m,n}^{l}\ge 0$. In the same way, $u_{m,n}^{l}$ satisfy (2.4), take now $k=1$, we prove that $u_{m,n}^{l}\le 1$ *i.e.*

$$0\le u_{m,n}^{l}\le 1 \tag{3.12}$$

We use the test function $\phi_l\left(u_{m,n}^{l}\right)$ in the weak formulation of $(\mathrm{P})_{m,n}^{l}$. The duality product between $\phi_l\left(u_{m,n}^{l}\right)\in L^2\left(0,T;H^1(\Omega)\right)\cap L^\infty(Q)$ and $\partial_t b_l\left(u_{m,n}^{l}\right)\in L^2\left(0,T,\left(H^1\right)^*(\Omega)\right)+L^1(Q)$ is treated via the standard chain rule argument

$$\begin{aligned}&\int_0^t\left\langle b_l\left(u_{m,n}^{l}\right)_t,\phi_l\left(u_{m,n}^{l}\right)\right\rangle\mathrm{d}t-\int_0^t\int_\Omega\left(f\left(u_{m,n}^{l}\right)-\nabla\phi_l\left(u_{m,n}^{l}\right)\right)\cdot\nabla\phi_l\left(u_{m,n}^{l}\right)\mathrm{d}x\mathrm{d}t\\&+\int_0^t\int_\Omega\psi_{m,n}\left(u_{m,n}^{l}\right)\phi_l\left(u_{m,n}^{l}\right)\mathrm{d}x\mathrm{d}t=0.\end{aligned} \tag{3.13}$$

Take $B_l(z)=\int_0^z \phi_l(s)\,\mathrm{d}b_l(s)$. Since $B_l(z)\le b_l(z)\phi_l(z)$, we obtain the inequality

$$\int_\Omega B_l\left(u_{m,n}^l(t,.)\right)+\int_0^T\int_\Omega \psi_{m,n}\left(u_{m,n}^l\right)\phi_l\left(u_{m,n}^l\right)+C\left|\nabla\phi_l\left(u_{m,n}^l\right)\right|^2\le\int_\Omega b_l\left(u_n^0\right)\phi_l\left(u_n^0\right).$$

with some $C>0$ independent of n. Note that the functions b_l,ϕ_l are locally uniformly bounded because they are monotone and converge pointwise to b,ϕ respectively.

Therefore the right-hand side of the above inequality is bounded uniformly in l, thanks to (3.12) and the uniform bounds on the data u_n^0 in $L^\infty(\Omega)$. The uniform estimate of the left-hand side follows. We then estimate $\left\|\phi_l\left(u_{m,n}^l\right)\right\|$ in $L^2(Q)$ by the Poincaré inequality and $\left\|\phi_l\left(u_{m,n}^l\right)\psi\left(u_{m,n}^l\right)\right\|_{L^1}$ follows. ■

Lemma 3.5. Let $u_{m,n}^l$ be the weak solution of $(\mathrm{P})_{m,n}^l$ the analogue of (P) with corresponding data $\left(b_l,\phi_l,\psi_{m,n},u_0^n\right)$. For $\tau>0$, we have

$$\int_0^{T-\tau}\int_\Omega\left|\phi_l\left(u_{m,n}^l\right)(t+\tau,x)-\phi_l\left(u_{m,n}^l\right)(t,x)\right|\le\omega(\tau) \tag{3.14}$$

Proof. Let $\tau>0$, Multiplying the first equation of $(\mathrm{P})_{m,n}^l$ by $\xi\in H^1(\Omega)\cap L^\infty(\Omega)$ and integer in $(t,t+\tau)\subset(0,T)$ we get:

$$\int_\Omega\left(b_l\left(u_{m,n}^l(x,t+\tau)\right)-b_l\left(u_{m,n}^l(x,t)\right)\right)\xi+\int_t^{t+\tau}\int_\Omega\left(f\left(u_{m,n}^l\right)-\nabla\phi_l\left(u_{m,n}^l\right)\right)\cdot\nabla\xi$$
$$+\int_t^{t+\tau}\int_\Omega\psi\left(u_{m,n}^l\right)\xi=0.$$

Take now $\xi=\phi_l\left(u_{m,n}^l\right)(t+\tau,x)-\phi_l\left(u_{m,n}^l\right)(t,x)$ and integrate in t. By Fubini theorem and estimates of Lemma 3.4, it appear a factor τ in the right hand side and we get

$$\int\int_{Q_\tau=(0,T-\tau)\times\Omega}\left|h_\tau\left(b_l\left(u_{m,n}^l\right)(t,x)\right)\right|\left|h_\tau\left(\phi_l\left(u_{m,n}^l\right)(t,x)\right)\right|\le C\tau \tag{3.15}$$

with $h_\tau(g)(t,x)g(t+\tau,x)-g(t,x)$. Here C is a constant independent of l,m,n. Take $\bar\phi=\phi_l\circ b_l^{-1}$ which is continuous function, let $\bar w$ the modulus of continuity of $\bar\phi$ on $[0,1]$ and $\bar W$ be its inverse and set $W(r)=r\bar W(r)$ and w be the inverse of W, notice that $w(0)=0$. Denote by $s=b_l\left(u_{m,n}^l\right)(t+\tau,x)$ and $r=b_l\left(u_{m,n}^l\right)(t,x)$. Then

$$\int\int_{Q_\tau}\left|\bar\phi(s)-\bar\phi(r)\right|=\int\int_{Q_\tau}w\left(W\left(\left|\bar\phi(s)-\bar\phi(r)\right|\right)\right)$$
$$\le\left|Q_\tau\right|w\left(\frac{1}{\left|Q_\tau\right|}\int\int_{Q_\tau}W\left(\left|\bar\phi(s)-\bar\phi(r)\right|\right)\right)$$

Since

$$\left|\bar\phi(s)-\bar\phi(r)\right|\le\bar w\left(\left|s-r\right|\right)$$

we have:

$$\bar W\left(\left|\bar\phi(s)-\bar\phi(r)\right|\right)\le\left|s-r\right|$$

$$W\left(\left|\bar{\phi}(s)-\bar{\phi}(r)\right|\right)=\bar{W}\left(\left|\bar{\phi}(s)-\bar{\phi}(r)\right|\right)\left|\bar{\phi}(s)-\bar{\phi}(r)\right|$$
$$\leq |s-r|\left|\bar{\phi}(s)-\bar{\phi}(r)\right|$$
$$=\left|h_\tau\left(b_l\left(u_{m,n}^l\right)\right)(t,x)\right|\left|h_\tau\left(\phi_l\left(u_{m,n}^l\right)(t,x)\right)\right|.$$

Therefore (3.15) implies

$$\begin{aligned}\int\int_{Q_\tau}\left|h_\tau\left(\phi_l\left(u_{m,n}^l\right)\right)(t,x)\right| &\leq |Q_\tau|\, w\left(\frac{1}{|Q_\tau|}\int\int_{Q_\tau}\left|h_\tau\left(u_{m,n}^l\right)(t,x)\right|\left|h_\tau\left(\phi_l\left(u_{m,n}^l\right)(t,x)\right)\right|\right)\\ &= |Q_\tau|\, w\left(\frac{1}{|Q_\tau|}\tau\right)\leq Cw(C\tau)\end{aligned} \tag{3.16}$$

the left-hand side of (3.16) tends to zero as $\tau\to 0$, we deduce (3.14). ■

The proof of Theorem 3.1 is a direct consequence of Theorem 3.2 and Theorem 3.3.

Since, we have establish the proof of theorem 3.2, let us demonstrate Theorem 3.3.

Proof of Theorem 3.3. There exists a function $u_{m,n}^k$ constructed by means of the nonlinear semigroup theory (see, e.g., [3] [13]), such that $v_k^{m,n}=b^k\left(u_{m,n}^k\right)$ is the unique integral solution to the abstract evolution problem associated with $(P)_{m,n}^k$ (here and below, we refer to Andreianov and Gazibo [3], Ammar and Wittbold [9], Ammar and Redwane [14] for details). One then shows that $u_{m,n}^k$ coincides with the unique entropy solution of $(P)_{m,n}^k$, the existence of this entropy solution being already shown. Further, the whole set $\left(u_{m,n}^k\right)_{k,m,n}$ verifies the uniform a priori estimates of Lemma 3.4 and 3.5. We then pass to the limit in $u_{m,n}^k$ in the following order: first $k\to+\infty$, then $n\to+\infty, m\to+\infty$.

While letting $k\to+\infty$, we use the fact that $\psi_{m,n}^{-1}$ is Lipschitz continuous. The fundamental estimates for the semigroup solutions permit to show that $\psi_{m,n}\left(u_{m,n}\right)$ are uniformly continuous on $(0,T)$ with values in $L^1(\Omega)$; thus we get the strong precompactness of $\left(u_{m,n}^k\right)_k\in L^1(Q)$. Thus, up to a subsequence, $u_{m,n}^k$ converge to $u_{m,n}$ which is an entropy solution of problem $(\mathrm{P})_{m,n}^l$ corresponding to the data $\left(b,\psi_{m,n},\phi,u_0\right)$. Finally, we use the inequalities $u_{m+1,n}\leq u_{m,n}\leq u_{m,n+1}$ which follow readily form the comparison principle (this come from uniqueness of $(P)_{m,n}^k$). The monotonicity argument yields the strong convergence of $u_{m,n}$. Passing to the limit in $u_{m,n}$ we conclude that the limit u is an entropy solution of the original problem (P) (one can use Lemma 3.6 and 3.7 of [3]). ■

3.2. Uniqueness of Entropy Solution in One Space Dimension

3.2.1. Stationary Problem

Let us stress that to our knowledge the problem of uniqueness is still open in multiple space dimensions. The definition of strong traces of the solution with respect to the lateral boundary of the domain Ω is possible if for example the diffusion term $\phi(u)$ is such that $f(u)-\nabla\phi(u)$ is continuous up to the boundary $\partial\Omega$. If there existed "sufficiently many" solutions (in the sense of [3], [4], see Definition 3.9 below) having this regularity, uniqueness would follow. Unfortunate, we could obtain this regularity for this moment only for the stationary problem associate to (P) and in one space dimension.

Now, we consider the stationary problem associated to (P)

$$\begin{cases} b(u)_t + \operatorname{div} f(u) - \Delta\phi(u) + \psi(u) = s & \text{in } \Omega \\ \left(f(u) - \nabla\phi(u)\right)\cdot\eta = 0 & \text{on } \partial\Omega \end{cases} \tag{S}$$

where $s \in L^\infty(\Omega)$.

Remark 3.6.

1) If, we suppose that $(b+\psi)(u)$ is bijective, then performing a change of the unknown one puts the problem into the doubly nonlinear framework in the form $u + \operatorname{div}\left(f(u) - \nabla\phi(u)\right) = s$. Existence and uniqueness follows (see [3]).

2) If $b(u)$ independent of t and u is solution of (S) it is also solution of (P) with source term $s - b(u)$. Then, we can deduce from Definition 2.1 and Proposition 2.3 their equivalent form for the stationary problem.

Definition 3.7. A measurable function u taking values in [0, 1] is an entropy solution of (S) if u is a weak solution of (S) and $\phi(u) \in H^1(\Omega)$ and for all $\xi \in \mathcal{C}^\infty\left(\mathbb{R}^\ell\right)^+$, $\forall k \in [0, u_c] \cup [u_s, 1]$,

$$\begin{aligned} &\int_\Omega sign(u-k)\left(f(u) - f(k) - \nabla\phi(u)\right)\cdot\nabla\xi \mathrm{d}x + \int_{\partial\Omega} \left|f(k)\cdot\eta(y)\right|\xi \mathrm{d}\mathcal{H}^{\ell-1}(x) \\ &-\int_\Omega sign(u-k)\left(b(u) + \psi(u) - s\right)\xi \mathrm{d}x \geq 0. \end{aligned} \tag{3.17}$$

Proposition 3.7. Let $\xi \in \mathcal{C}^\infty\left(\mathbb{R}^\ell\right)$; then for all $k \in [u_s, 1]$, for all $D \in \mathbb{R}^\ell$, for all entropy solution u of (S), we have:

$$\begin{aligned} &\int_\Omega sign(u-k)\left(f(u) - f(k) - \nabla\phi(u) + D\right)\cdot\nabla\xi \mathrm{d}x + \int_{\partial\Omega} \left|\left(f(k) - D\right)\eta\right|\xi \mathrm{d}\mathcal{H}^{\ell-1}(x) \\ &-\int_\Omega sign(u-k)\left(b(u) + \psi(u) - s\right)\xi \mathrm{d}x \geq \overline{\lim_{\sigma\to 0}} \frac{1}{\sigma} \int_{\Omega\cap A_\phi^\sigma} \nabla\phi(u)\left(\nabla\phi(u) - D\right)\xi. \end{aligned} \tag{3.18}$$

In the next subsection, we give a Definition of so called "trace regular entropy solution"

3.2.2. Trace Regular Entropy Solution

Definition 3.8. An entropy solution is called trace regular solution of (P) if the normal component of the total flux $\left(f(u) - \nabla\phi(u)\right)\cdot\eta$, has L^1 strong trace $\gamma\left(f(u) - \nabla\phi(u)\right)$ at boundary of Lipschitz domain *i.e.*: for $\xi \in L^\infty(\partial\Omega)$

$$\lim_{\sigma\to 0} \frac{1}{\sigma} \int_0^1 \int_{\partial\Omega} \xi \left|\left(f(u) - \nabla\phi(u)\right)\cdot\eta(x) - \gamma\left(f(u) - \nabla\phi(u)\right)\cdot\eta(x,\sigma)\right| \mathrm{d}x \mathrm{d}\tau = 0. \tag{3.19}$$

The difficulty is that the regularized zero-flux boundary condition does not permit control over the tangential derivatives (with respect to $\partial\Omega$) of the solution. Thus, boundary flux traces of solution seem hard to obtain and we need the concepts of domains with Lipschitz deformable boundaries and traces (see [15], [16] for more details).

Remark 3.9. Notice that if the normal component of the flux

$\left(f\left(u\right)-\nabla\phi\left(u\right)\right)\cdot\eta$ is continuous function then it satisfy (3.19).

From now on, we will suppose that $\Omega=\left(a,b\right)$ is a bounded interval of $\mathbb{R}$. We have this property

Proposition 3.10. For all $s\in\left[0,1\right]$, the problem (S) admits a solution u such that $\left(f\left(u\right)-\phi\left(u\right)_y\right)$ is continuous up the boundary, *i.e.*, $\left(f\left(u\right)-\phi\left(u\right)_y\right)\in\mathcal{C}\left(\left[a,b\right]\right)$.

Moreover, $f\left(u\right)-\phi\left(u\right)_y$ is zero at $y=a$ and $y=b$.

Since ψ is bijective, the proof of Proposition 3.11 is identical to the proof of Proposition 4.8 of [3].

The main result of this section is the following theorem:

Theorem 3.11. Suppose that $\Omega=\left(a,b\right)$ is a bounded interval of $\mathbb{R}$, then (P) admits a unique $b\left(u\right)$ such that u is entropy solution of (P).

3.2.3. Abstract Evolution Problem

We present now the problem (P) under the abstract form of an evolution equation governed by an accretive operator, in order to apply classical results of the nonlinear semigroup theory (see, e.g., [17]).

Let us define the (possibly multivalued) operator $A^b_{f,\phi,\psi}$ by it resolvent

$$\left(u,z\right)\in A^b_{f,\phi,\psi}=\left\{u\setminus u\ \text{ entropy solution of }\left(\mathrm{S}\right),\ \text{with } z=s-b\left(u\right)-\psi\left(u\right)\right\}.$$

Consider the abstract equation:

$$b\left(u\right)_t+A^b_{f,\phi,\psi}\left(u\right)\ni 0,\quad b\left(u\right)\left(t=0\right)=b_0 \tag{E}$$

For an operator $A:L^1\left(\Omega\right)\to L^1\left(\Omega\right)$, denote by $R\left(A\right)$ its range, by $D\left(A\right)$ its domain and by $\overline{R\left(A\right)}$, $\overline{D\left(A\right)}$ their closures in $L^1\left(\Omega\right)$ respectively

Let us stress that for $u\in D\left(A^b_{f,\phi,\psi}\right)$, $f\left(u\right)-\phi\left(u\right)_y\in\mathcal{C}_0\left(\left[a,b\right]\right)$ due to Proposition 3.11.

Recall (cf. [17]) that an operator A is accretive if $\left[\beta-\hat{\beta},\alpha-\hat{\alpha}\right]_{L^1(\Omega)}\geq 0$ for all $\left(\beta,\alpha\right),\left(\hat{\beta},\hat{\alpha}\right)\in A$, where for $\beta,\alpha\in L^1\left(\Omega\right)$ the bracket $\left[.,.\right]_{L^1(\Omega)}$ is defined by $\left[\beta,\alpha\right]_{L^1(\Omega)}=\int_\Omega sign\left(\beta\right)\alpha+\int_{[\beta=0]}\left|\alpha\right|$.

If A is accretive and $R\left(I+\lambda A\right)=L^1\left(\Omega\right)$ for some $\lambda>0$, then A is m-accretive.

Proposition 3.13. Let $\left(u,z\right)\in A^b_{f,\phi,\psi}$, $\left(\hat{u},\hat{z}\right)\in A^b_{f,\phi,\psi}$. Then for $\xi\in\mathcal{C}^\infty\left(\overline{\Omega}\right)^+$

$$\begin{aligned}&\int_\Omega\left|b\left(u\right)-b\left(\hat{u}\right)\right|\xi\mathrm{d}y+\int_\Omega\left|\psi\left(u\right)-\psi\left(\hat{u}\right)\right|\xi\mathrm{d}y\\&+\int_\Omega sign\left(u-\hat{u}\right)\left(f\left(u\right)-f\left(\hat{u}\right)-\phi\left(u\right)_y+\phi\left(\hat{u}\right)_y\right)\cdot\xi_y\mathrm{d}y\\&\leq\int_\Omega sign\left(u-\hat{u}\right)\left(s-\hat{s}\right)\xi\mathrm{d}y+\int_{[u=\hat{u}]}\left|s-\hat{s}\right|\xi\mathrm{d}y=\left[u-\hat{u},s-\hat{s}\right]_{L^1(\Omega)}.\end{aligned} \tag{3.20}$$

Proof. (Sketched) The proof of Proposition 3.13 is actually contained in the proof of Theorem 3.17 below, due to Remark 2.2. Actually a simpler argument applies, because both $f\left(\hat{u}\right)-\phi\left(\hat{u}\right)_y$ and $f\left(u\right)-\phi\left(u\right)_y$ have strong trace in the context of the stationary problem (S). ■

Somewhat abusively, we will write $L^1(\Omega;[0,1])$ for the set of all measurable functions from [*a*, *b*] to [0, 1].

Proposition 3.14. The following properties hold true.

1) $A^b_{f,\phi,\psi}$ is accretive in $L^1(\Omega)$.

2) For all λ sufficiently small, $R\left(I+\lambda A^b_{f,\phi,\psi}\right)$ contains $L^1(\Omega;[0,1])$.

3) $\overline{D\left(A^b_{f,\phi,\psi}\right)} = L^1(\Omega;[0,1])$.

Proof.

1) Let $(u,z)\in A^b_{f,\phi,\psi}$, $(\hat{u},\hat{z})\in A^b_{f,\phi,\psi}$. Applying Proposition 3.13 with $\xi=1$ in (3.20) and the standard properties of the bracket (see [17]), we get

$$\begin{aligned}&\|b(u)-b(\hat{u})\|_{L^1(\Omega)}+\|\psi(u)-\psi(\hat{u})\|_{L^1(\Omega)}\le[u-\hat{u},s-\hat{s}]_{L^1(\Omega)}\\&\le\left[u-\hat{u},b(u)-b(\hat{u})+\psi(u)-\psi(\hat{u})+z-\hat{z}\right]_{L^1(\Omega)}\\&\le\|b(u)-b(\hat{u})\|_{L^1(\Omega)}+\|\psi(u)-\psi(\hat{u})\|_{L^1(\Omega)}+[u-\hat{u},z-\hat{z}]_{L^1(\Omega)}.\end{aligned}$$

We deduce that $[u-\hat{u},z-\hat{z}]_{L^1(\Omega)}\ge 0$, so that $A^b_{f,\phi,\psi}$ is accretive.

2) For $\lambda>0$, consider the problem

$$\begin{cases}b(u_\lambda)+\lambda\left(f(u_\lambda)-(\phi(u_\lambda))_y\right)_y+\psi(u_\lambda)=s & \text{in } \Omega,\\ \lambda\left(f(u_\lambda)-\phi(u_\lambda)_y\right)\cdot\eta(y)=0 & \text{on } \partial\Omega\end{cases}\qquad (\mathrm{S}_\lambda)$$

Notice that the notion of solution for (S_λ) is like the Definition 3.7. Let $s\in L^1(\Omega;[0,1])$ and $\lambda>0$ then, there exists u_λ entropy solution of (S_λ) (see Proposition 3.11) such that $\left(u_\lambda,\dfrac{s-b(u_\lambda)-\psi(u_\lambda)}{\lambda}\right)\in A^b_{f,\phi,\psi}$.

Hence $s\in R\left(I+\lambda A^b_{f,\phi,\psi}\right)$ and therefore $R\left(I+\lambda A^b_{f,\phi,\psi}\right)\supset L^1(\Omega;[0,1])$ which was to be shown.

3) Let $PC([a,b];[0,1])$ be the set of piecewise constant functions from [*a*, *b*] to [0, 1]. Then $PC([a,b];[0,1])$ is dense in $L^1([a,b];[0,1])$. Take $s\in PC([a,b];[0,1])$, $s=\sum_i s_i\mathbf{1}_{(a_i,b_i)}$ where the (a_i,b_i) are disjoint intervals. There exists $u_n\in L^\infty(a,b)$ entropy solution of $(S_{1/n})$, *i.e.*, we have $\left(u_n,n\left(s-b(u_n)-\psi(u_n)\right)\right)\in A^b_{f,\phi,\psi}$. For $k\in[0,u_c]\cup[u_s,1]$, for all $\xi\in C_0^\infty(\mathbb{R})$. We get

$$\begin{aligned}&\frac{1}{n}\int_a^b sign(u_n-k)\left(f(u_n)-f(k)-\partial_y\phi(u_n)\right)\cdot\partial_y\xi\mathrm{d}y\\&+\int_a^b sign(u_n-k)\left(s-b(u_n)-\psi(u_n)\right)\xi\mathrm{d}y+\frac{1}{n}\int_{[a,b]}\left|f(k)\cdot\eta(y)\right|\xi\mathrm{d}\sigma\ge 0\end{aligned}\qquad(3.21)$$

For every *i*, one can construct ξ_i^n such that $\xi_i^n\to\mathbf{1}_{(a_i,b_i)}$, as $n\to\infty$, supp $\xi_i^n\subset(a_i,b_i)$, $\left\|\partial_y\xi_i^n\right\|_{L^\infty}\le 2\sqrt[3]{n}$ and $\xi_i^n\equiv 1$ in $\left(a_i+\delta_n^i,b_i-\delta_n^i\right)$ with $\delta_n^i=\dfrac{b_i-a_i}{2\sqrt[3]{n}}$. Take $k=c_i$ and ξ_i^n in (3.21).

$$\int_{a_i+\delta_n^i}^{b_i-\delta_n^i} \left| b(u_n)+\psi(u_n)-c_i \right| \mathrm{d}y$$
$$\leq \frac{1}{n}\int_{a_i+\delta_n^i}^{b_i-\delta_n^i} sign(u_n-c_i)\left(f(u_n)-f(c_i)-\partial_y\phi(u_n)\right)\partial_y\xi_i^n \mathrm{d}y$$
$$\leq \frac{2}{n}|b-a|\|f\|_{L^\infty}\cdot\left\|\partial_y\xi_i^n\right\|_{L^1}+|\left\|\frac{1}{\sqrt{n}}\partial_y\phi(u_n)\right\|_{L^2}\cdot\left\|\partial_y\xi_i^n\right\|_{L^2}.$$

Then, for all $\delta>\delta_n^i$, $u_n\to g$ a.e. on $\bigcup_i(a_i+\delta,b_i-\delta)$. We conclude by the Lebesgue theorem that $u_n\to g$ in $L^1([a,b])$. In conclusion, $D\left(A^b_{f,\phi,\psi}\right)$ is dense in $PC([a,b];[0,1])$ and therefore, it is also dense in $L^1(\Omega;[0,1])$. ■

3.2.4. Integral Solution and Uniqueness

Now, we can exploit the notion of integral solution (see, e.g., [7] [17]).

Definition 3.15. Suppose that $u_0\in L^1(\Omega)$ function $b(v)\in\mathcal{C}\left([0,T];L^1\left([a,b];[\underline{\alpha},\overline{\alpha}]\right)\right)$ is an integral solution of (E) if

$b(v)(0,.)=b_0(.)$ and for all $(u,z)\in A^b_{f,\phi,\psi}$

$$\left\|b(v)(t)-b(\hat{v})(t)\right\|_{L^1}+\left\|\psi(v)(t)-\psi(u)\right\|_{L^1(\Omega)}\leq\left\|b(u_0)-b(\hat{u}_0)\right\|_{L^1}.$$

In particular, the integral solution is unique.

Theorem 3.17. Let $\Omega=[a,b]$. Let v be an entropy solution of (P) and u be an entropy solution of (S). Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|b(v)(t)-b(u)\right\|_{L^1(\Omega)}+\left\|\psi(v)-\psi(u)\right\|_{L^1(\Omega)}\\&\leq\int_\Omega sign(v-u)\left(b(u)-s\right)\mathrm{d}x \text{ in } \mathcal{D}'(0,T).\end{aligned}\tag{3.22}$$

In particular, $b(v)$ is an integral solution of (E).

Proof of Theorem 3.17 We adopt the doubling of variables of Kruzkhov [1] in the sense of [3] [18]. We compare regular solution and entropy solution. Keep in mind that by the result of [19] an entropy solution v of (S) is automatically time-continuous with values in $L^1(\Omega;[0,1])$. We consider $v=v(t,x)$ an entropy solution of (P) and $u=u(y)$ an entropy solution of (S). Consider nonnegative function $\xi=\xi(t,x,y)$ having the property that $\xi(.,.,y)\in\mathcal{C}^\infty\left([0,T)\times\overline{\Omega}\right)$ for each $y\in\overline{\Omega}$, $\xi(t,x,.)\in\mathcal{C}_0^\infty\left(\overline{\Omega}\right)$ for each $(t,x)\in[0,T)\times\overline{\Omega}$. As in [3], we denote $\Omega_x=\{x\in\Omega;v(t,x)\in[0,u_c]\}$, $\Omega_y=\{y\in\Omega;u(y)\in[0,u_c]\}$ and Ω_x^c, Ω_y^c their complementaries in Ω. To simplify the notations, take $\mathcal{F}[v]=f(v)-\phi(v)_x$, $\mathcal{F}[u]=f(u)-\phi(u)_y$ and $\Phi(w(r),\hat{w}(s))=\phi(w)_r\left(\phi(w)_r-\phi(\hat{w})_s\right)$.

In (2.6), take $\xi=\xi(t,x,y)$, $k=u(y)$, $D=\phi(u)_y$ and integrate over Ω_y^c. We get

$$\begin{aligned}&\int_{\Omega_y^c}\int_0^T\int_\Omega\left\{\left|b(v)-b(u)\right|\xi_t+sign(v-u)\left(\mathcal{F}[v]-\mathcal{F}[u]\right)\cdot\xi_x\right\}\mathrm{d}x\mathrm{d}t\mathrm{d}y\\&+\int_{\Omega_y^c}\int_0^T\int_{x\in\partial\Omega}\left|\mathcal{F}[u]\cdot\eta(x)\right|\xi\mathrm{d}\sigma\mathrm{d}t\mathrm{d}y+\int_{\Omega_y^c}\int_\Omega\left|b_0-b(u)\right|\xi(0,x,y)\\&-\int_{\Omega_y^c}\int_0^T\int_\Omega sign(v-u)\psi(v)\xi\mathrm{d}x\mathrm{d}t\mathrm{d}y\geq\overline{\lim_{\sigma\to0}}\frac{1}{\sigma}\int_{\Omega_y^c}\int_0^T\int_{x\in\Omega\cap A_\phi^\sigma}\Phi(v,u)\xi\end{aligned}\tag{3.23}$$

In the same way, in (2.2) take $\xi=\xi(t,x,y)$, $k=u(y)$, integrate over Ω_y, and use the fact that $\phi(u)_y=0$ in Ω_y. We get

$$\begin{aligned}&\int_{\Omega_y}\int_0^T\int_\Omega\left\{\left|b(v)-b(u)\right|\xi_t+sign(v-u)\left(\mathcal{F}[v]-\mathcal{F}[u]\right)\cdot\xi_x\right\}\\&+\int_{\Omega_y}\int_0^T\int_{x\in\partial\Omega}\left|\mathcal{F}[u]\cdot\eta(x)\right|\xi+\int_{\Omega_y}\int_\Omega\left|b_0-b(u)\right|\xi(0,x,y)\\&-\int_{\Omega_y}\int_0^T\int_\Omega sign(v-u)\psi(v)\xi\mathrm{d}x\mathrm{d}t\mathrm{d}y\ge 0.\end{aligned}\tag{3.24}$$

Since $\Omega=\Omega_x\cup\Omega_x^c$, by adding (3.23) to (3.24) we obtain

$$\begin{aligned}&\int_\Omega\int_0^T\int_\Omega\left\{\left|b(v)-b(u)\right|\xi_t+sign(v-u)\left(\mathcal{F}[v]-\mathcal{F}[u]\right)\cdot\xi_x\right\}\mathrm{d}x\mathrm{d}t\mathrm{d}y\\&+\int_\Omega\int_0^T\int_{x\in\partial\Omega}\left|\mathcal{F}[u]\cdot\eta(x)\right|\xi\mathrm{d}\sigma\mathrm{d}t\mathrm{d}y+\int_\Omega\int_\Omega\left|b_0-b(u)\right|\xi(0,x,y)\mathrm{d}x\mathrm{d}y\\&-\int_\Omega\int_0^T\int_\Omega sign(v-u)\psi(v)\xi\ge\overline{\lim_{\sigma\to0}}\frac{1}{\sigma}\int_{\Omega_y^c}\int_0^T\int_{x\in\Omega\cap A_\phi^\sigma}\Phi(v,u)\xi\end{aligned}\tag{3.25}$$

In (3.18), take $\xi=\xi(t,x,y)$, $k=v(t,x)$, $D=\phi(v)_x$ and integrate over $(0,T)\times\Omega_x^c$

$$\begin{aligned}&\int_0^T\int_{\Omega_x^c}\int_\Omega sign(v-u)\left(\mathcal{F}[v]-\mathcal{F}[u]\right)\cdot\xi_y\mathrm{d}y\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_{\Omega_x^c}\int_{y\in\partial\Omega}\left|\mathcal{F}[v]\cdot\eta(y)\right|\xi\mathrm{d}\sigma\mathrm{d}x\mathrm{d}t+\int_\Omega\int_0^T\int_\Omega sign(v-u)\psi(u)\xi\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_{\Omega_x^c}\int_\Omega sign(v-u)\left(b(u)-s(y)\right)\xi\ge\overline{\lim_{\sigma\to0}}\frac{1}{\sigma}\int_{\Omega_x^c}\int_0^T\int_{y\in\Omega\cap A_\phi^\sigma}\Phi(u,v)\xi\end{aligned}\tag{3.26}$$

Since $u(y)$ is entropy solution, then take in (3.17) $\xi=\xi(t,x,y)$, integrate over and use the fact that in $\phi(v)_x=0$ in $(0,T)\times\Omega_x$.

$$\begin{aligned}&\int_0^T\int_{\Omega_x}\int_\Omega sign(v-u)\left(\mathcal{F}[v]-\mathcal{F}[u]\right)\cdot\xi_y\mathrm{d}y\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_{\Omega_x}\int_{y\in\partial\Omega}\left|\mathcal{F}[v]\cdot\eta(y)\right|\xi\mathrm{d}\sigma\mathrm{d}x\mathrm{d}t+\int_\Omega\int_0^T\int_\Omega sign(v-u)\psi(u)\xi\mathrm{d}y\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_{\Omega_x}\int_\Omega sign(v-u)\left(b(u)-s(y)\right)\xi\mathrm{d}x\mathrm{d}t\mathrm{d}y\ge0.\end{aligned}\tag{3.27}$$

By adding (3.26) to (3.27), we obtain

$$\begin{aligned}&\int_0^T\int_\Omega\int_\Omega sign(v-u)\left(\mathcal{F}[v]-\mathcal{F}[u]\right)\cdot\xi_y\mathrm{d}y\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega\int_{y\in\partial\Omega}\left|\mathcal{F}[v]\cdot\eta(y)\right|\xi\mathrm{d}\sigma\mathrm{d}x\mathrm{d}t+\int_\Omega\int_0^T\int_\Omega sign(v-u)\psi(u)\xi\mathrm{d}y\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega\int_\Omega sign(v-u)\left(b(u)-s\right)\xi\ge\overline{\lim_{\sigma\to0}}\frac{1}{\sigma}\int_{\Omega_x^c}\int_0^T\int_{y\in\Omega\cap A_\phi^\sigma}\Phi(u,v)\xi.\end{aligned}\tag{3.28}$$

Now, sum (3.25) and (3.28) to obtain

$$\begin{aligned}
&\int_0^T\int_\Omega\int_\Omega |b(v)-b(u)|\xi_t \mathrm{d}y\mathrm{d}x\mathrm{d}t+\int_\Omega\int_\Omega |b_0-b(u)|\xi(0,x,y)\mathrm{d}x\mathrm{d}y \\
&+\int_0^T\int_\Omega\int_\Omega sign(v-u)(\mathcal{F}[v]-\mathcal{F}[u])\cdot(\xi_x+\xi_y)\mathrm{d}y\mathrm{d}x\mathrm{d}t \\
&+\int_0^T\int_{x\in\partial\Omega}\int_\Omega |\mathcal{F}[u]\cdot\eta(x)|\xi\mathrm{d}\sigma\mathrm{d}t\mathrm{d}y+\int_0^T\int_\Omega\int_{y\in\partial\Omega}|\mathcal{F}[v]\cdot\eta(y)|\xi\mathrm{d}y\mathrm{d}\sigma\mathrm{d}t \qquad (3.29)\\
&+\int_0^T\int_\Omega\int_\Omega sign(v-u)(b(u)-s(y))\xi-\int_\Omega\int_0^T\int_\Omega|\psi(v)-\psi(u)|\xi\mathrm{d}x\mathrm{d}t \\
&\geq \overline{\lim_{\sigma\to 0}}\frac{1}{\sigma}\int_0^T\int\int_{\Omega_x^c\times\Omega_y^c\cap A_\phi^\sigma}\left|\phi(v)_x-\phi(u)_y\right|^2\xi\mathrm{d}y\mathrm{d}x\mathrm{d}t\geq 0.
\end{aligned}$$

Next, following the idea of [3] we consider the test function $\xi(t,x,y)=\theta(t)\rho_n(x,y)$, where $\theta\in\mathcal{C}_0^\infty(0,T)$, $\theta\geq 0$, $\rho_n(x,y)=\delta_n(\Delta)$ and $\Delta=\left(1-\frac{1}{n(b-a)}\right)x-y+\frac{a+b}{2n(b-a)}$. Then, $\rho_n\in\mathcal{D}(\bar{\Omega}\times\bar{\Omega})$ and $\rho_{n_{|\Omega\times\partial\Omega}}(x,y)=0$.

Due to this choice $\int_0^T\int_{x\in\Omega}\int_{y\in\partial\Omega}|\mathcal{F}[v]\cdot\eta(y)|\rho_n\theta\mathrm{d}y\mathrm{d}\sigma\mathrm{d}t=0$.

By Proposition 3.11, $\left(f(u)-\phi(u)_y\right)\in\mathcal{C}_0([a,b])$. Therefore we have $|\mathcal{F}[u]\cdot\eta(x)|\to 0$ when $x\to y$, *i.e.*, as $n\to\infty$. We conclude that

$$\lim_{n\to\infty}\int_0^T\int_{x\in\partial\Omega}\int_{y\in\Omega}|\mathcal{F}[u]\cdot\eta(x)|\rho_n\theta\mathrm{d}y\mathrm{d}\sigma\mathrm{d}t=0.$$

It remains to study the limit, as $n\to\infty$

$$I_n=\int_0^T\int_\Omega\int_\Omega \theta sign(v-u)(\mathcal{F}[v]-\mathcal{F}[u])\cdot\left((\rho_n)_x+(\rho_n)_y\right)\mathrm{d}y\mathrm{d}x\mathrm{d}t$$

We use the change of variable $(x,y)\mapsto(x,z)$ with $z=n(x-y)-\frac{1}{b-a}x+\frac{a+b}{b-a}$,

$$\begin{aligned}
I_n&=\frac{2}{b-a}\int_{-1}^1\int_0^T\int_\Omega sign(v-u)(\mathcal{F}[v]-\mathcal{F}[u])\cdot\delta_n'(z)\theta \\
&=\frac{2}{b-a}\int_{-1}^1\int_0^T\int_a^b sign(v(t,x)-u_n(x,z))(p(t,x)-q_n(x,z))\delta_n'(z)\theta(t),
\end{aligned}$$

where $u_n(x,z):=u(y)$, $p(t,x):=\mathcal{F}[v]$ and $q_n:=\mathcal{F}[u]$.

For z given, $u_n(.,z)$ converges to $u(.)$ in L^1 and $q_n(.,z)$ converges to $q(.):=f(u)-\phi(u)_x$ in L^1. From Lemma 4.14 of [3], we deduce that for all $z\in[-1,1]$

$$K_n(z):=\int_Q sign(v_n(t,x,z))h_n(t,x,z)\mathrm{d}x\mathrm{d}t\to_{n\to\infty}\int_Q sign(v)h\mathrm{d}x\mathrm{d}t=:K,$$

where $v_n:=v-u_n$, $h_n:=p-q_n$ and $h:=p-q$.

Then $K_n(.)$ converges to $K = \text{const}$ independently of *z*. Moreover, from the definition of K_n one finds easily the uniform L^∞ bound $|K_n| \le 2\left(\|p\|_{L^1(Q)} + T\|q\|_{L^1(\Omega)}\right)$, for *n* large enough. Hence by the Lebesgue theorem,

$$\lim_{n\to\infty}\int_{-1}^{1} K_n(z)\delta'(z) = K\int_{-1}^{1}\delta'(z) = 0.$$

We have shown that the limit of I_n equals zero. The passage to the limit in other terms in (3.29) is straightforward. Finally (3.29) gives for $n\to\infty$

$$\int_0^T\int_\Omega \left|b(v(t,x)) - b(u(y))\right|\theta'(t)\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_\Omega |\psi(v)-\psi(u)|\theta$$
$$+\int_0^T\int_\Omega sign(v-u)(b(u)-s)\theta \ge 0.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|b(v)(t)-b(u)\|_{L^1(\Omega)}$$
$$\le \int_\Omega \left[sign(v-u)(b(u)-s) - |\psi(v)-\psi(u)|\right]\mathrm{d}x \text{ in } \mathcal{D}'(0,T).$$

Thus, $b(v)$ is an integral solution of (E).

Now, the claim of Theorem 3.12 is a direct consequence of the fact that if *u* is the entropy solution then $b(u)$ is an integral solution, and of Corollary 3.16. ∎

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Kruzhkov, S.N. (1970) First Order Quasilinear Equations with Several Independent Variables. *Matematicheskii Sbornik*, **81**, 228-255.

[2] Carrillo, J. (1999) Entropy Solutions for Nonlinear Degenerate Problems. *Archive for Rational Mechanics and Analysis*, **147**, 269-361. https://doi.org/10.1007/s002050050152

[3] Andreianov, B. and Karimou Gazibo, M. (2013) Entropy Formulation of Degenerate Parabolic Equation with Zero-Flux Boundary Condition. *Zeitschrift für Angewandte Mathematik und Physik*, **64**, 1471-1491. https://doi.org/10.1007/s00033-012-0297-6

[4] Bürger, R., Frid, H., and Karlsen, K.H. (2007) On the Well-Posedness of Entropy Solution to Conservation Laws with a Zero Fux Boundary Condition. *Journal of Mathematical Analysis and Applications*, **326**, 108-120. https://doi.org/10.1016/j.jmaa.2006.02.072

[5] Ouaro, S. and Touré, H. (2007) Uniqueness of Entropy Solution to Nonlinear Elliptic-Parabolic Problems. *Electronic Journal of Differential Equations*, No. 82, 1-15.

[6] Ouaro, S. (2007) Entropy Solutions of Nonlinear Elliptic-Parabolic-Hyperbolic Degenerate Problems in One Dimension. *International Journal of Evolution Equations*, **3**, 1-17.

[7] Bénilan, Ph. and Touré, H. (1995) Sur l'équation générale $u_t = a\left(., u, \phi(., u)_x\right)_x + v$ dans L^1. I. Etude du problème stationnaire. In: *Evolution Equations* (*Baton Rouge*, *LA*, 1992), Dekker, New York, 35-62.

[8] Andreianov, B. and Wittbold, P. (2012) Convergence of Approximate Solutions to an Elliptic-Parabolic Equation without Structure Condition. *Nonlinear Differential Equations and Applications*, **19**, 695-717. https://doi.org/10.1007/s00030-011-0148-8

[9] Ammar, K. and Wittbold, P. (2003) Existence of Renormalized Solutions of Degenerate Elliptic Parabolic Problems. *Proceedings of the Royal Society of Edinburgh Section A*, **133**, 477-496. https://doi.org/10.1017/S0308210500002493

[10] Andreianov, B., Bendahmane, M., Karlsen, K.H. and Ouaro, S. (2009) Well-Posedness Results for Triply Nonlinear Degenerate Parabolic Equations. *Journal of Differential Equations*, **247**, 277-302. https://doi.org/10.1016/j.jde.2009.03.001

[11] Andreianov, B. and Gazibo, M.K. (2016) Explicit Formulation for the Dirichlet Problem for Parabolic-Hyperbolic Conservation Laws. *Networks and Heterogeneous Media*, **11**, 203-222. https://doi.org/10.3934/nhm.2016.11.203

[12] Alt, H.W. and Luckhaus, S. (1981) Quasilinear Elliptic-Parabolic Differential Equations. *Mathematische Zeitschrift*, **183**, 311-341. https://doi.org/10.1007/BF01176474

[13] Ammar, K., Carrillo, J. and Willbold, P. (2006) Scalar Conservation Laws with General Boundary Condition and Continuous Flux Function. *Journal of Differential Equations*, **228**, 111-139. https://doi.org/10.1016/j.jde.2006.05.002

[14] Ammar, K. and Redwane, H. (2008) Degenerate Stationary Problems with Homogeneous Boundary Conditions. *Electronic Journal of Differential Equations*, **30**, 1-18.

[15] Vasseur, A. (2001) Strong Traces for Solutions of Multidimensional Scalar Conservation Laws. *Archive for Rational Mechanics and Analysis*, **160**, 181-193. https://doi.org/10.1007/s002050100157

[16] Gazibo Karimou, M. (2023) Degenerate Parabolic Equation with Zero Flux Boundary Condition and Its Approximations. Preprint.

[17] Bénilan, Ph., Crandall, M.G. and Pazy, A. (1994) Nonlinear Evolution Equations in Banach Spaces. Preprint Book.

[18] Andreianov, B. and Bouhsiss, F. (2004) Uniqueness for an Elliptic-Parabolic Problem with Neumann Boundary Condition. *Journal of Evolution Equations*, **4**, 273-295. https://doi.org/10.1007/978-3-0348-7924-8_37

[19] Cances, C. and Gallouet, Th. (2011) On the Time Continuity of Entropy Solutions. *Journal of Evolution Equations*, **11**, 43-45. https://doi.org/10.1007/s00028-010-0080-0