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Generalized Quantum Lagrangian

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Abstract

The paper concerns the formulation of a Lagrangian function compliant with classical, quantum and relativistic outcomes. The literature Lagrangians are reported with modified local Lorentz transformations, or with potentials inferred directly from the relativistic metric or with geometrical meaning. In this paper the Lagrangian is formulated via the concept of quantum uncertainty only, which allows a non-deterministic approach. This theoretical frame is proven useful to merge without additional hypotheses quantum and relativistic outcomes in a straightforward way.

Keywords

Quantum Theory, Relativity, Vector Fields

1. Introduction

The present paper proposes a possible theoretical frame to define and implement the Lagrange equation. Various theoretical models are reported in the literature to formulate Lagrangians with modified local Lorentz transformations [1], or with potentials formulated directly from the relativistic metric [2] or with geometrical meaning [3]. Considering for simplicity but without loss of generality a one-dimensional frame, the standard Lagrange equation is defined by

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}; \tag{1.1}$$

as it is known

$$\frac{\partial L}{\partial \dot{x}} = p_x, \quad \frac{\partial L}{\partial x} = F_x. \tag{1.2}$$

These equations are deterministic, as the notations ∂ signify infinitesimal ranges of the respective generalized dynamical variables: in fact, besides the mere mathematical features of the definitions (1.1) and (1.2), via L the local coordinates x and t are defined as a function of which p_x and F_x are calculated.

This implies in turn the physical meaning of all these dynamical variables.

Just this conclusion, clearly unphysical because it violates the Heisenberg principle, emphasizes the crucial reason of the difficulty of unifying relativity and quantum physics [4]. In fact, the relativity is basically classical physics crucially enriched by four fundamental concepts [5]: 4-dimensional geometry of non-Euclidean space time, equivalence principle, finite and invariant light speed in vacuum, covariance principle. However, the mathematical tool to implement and quantify these ideas, the space time metrics, is deterministic: the tensor calculus describes the space time as a classical entity whose local geometry, flat or curved by the presence of mass, is exactly knowable as a function of coordinates, momenta and energy. The conceptual difficulty of merging the quantum non-locality and non-reality on the one side with the relativistic determinism on the other side is evident. As long as the specific physical problems do not involve quantum constraints, e.g. the perihelion precession of planets or the light beam bending, everything works well: it's worth the same idea that the macroscopic stability of a building is successfully calculable without caring about the Schrödinger equation and its conceptual basis. In fact, however, the E.P.R. paradox [6] that should demonstrate the incompleteness of the quantum mechanics, has instead evidenced just the incompleteness of the general relativity: the concept of entanglement able to explain interactions at superluminal distances demonstrated that quantum ideas are required to save the validity of one of the foundations of relativity, *i.e.* the finite limit value of light speed.

Moreover, it is shown in [7] [8] that the metric of the special relativity allows calculating the Lorentz transformation properties of the three classical components of the angular momentum; it is crucial the fact that this further conflict of the relativistic mathematical frame with the requirements of quantum physics is not due to the lucid reasoning exposed in the quoted textbooks, but to the unjustified application of the relativistic metrics outside the appropriate frame of classical physics.

Now consider in this respect finite coordinate and momentum component ranges $\delta x = x_2 - x_1$ and $\delta p_x = p_{x2} - p_{x1}$ requiring that *any* coordinate $x_1 \le x \le x_2$ and *any* corresponding momentum $p_{x1} \le p_x \le p_{x2}$ in the respective ranges should be equivalently and indistinguishably considered in calculating the allowed states of a physical system; in turn it means accrediting physical meaning to the whole ranges of values, and not to the random local dynamical variables they enclose. The agnosticism of the quantum uncertainty removes the link between an arbitrary x in its allowed range and the corresponding p_x and F_x in its allowed range.

All of this means excluding $\delta x \to 0$ for $x_2 \to x_1$, because this limit would admit the physical meaning of local conjugate dynamical variables. These considerations hold of course for (1.1) and (1.2). Appears thus appropriate the idea of a "quantum Lagrangian" inherently incorporating itself the requirements of the physical context it implements, in particular the features of a non-local and non-real quantum frame. In other words, it appears sensible to modify the starting point (1.1) in order that the probabilistic character of the quantum world becomes since the beginning compliant with the appropriate formulation of its descriptive tool.

Previous papers [4] have evidenced that introducing the concept of quantum uncertainty helps not only to solve quantum problems but also to find relativistic results in a straightforward and simple way.

All of this suggests that (1.1) and (1.2), despite their huge classical importance, are actually improvable to overcome the conceptual gap between classical physics, relativity and quantum world.

The Lagrangian is the basic foundation of the standard model [9]; clearly, the idea of formulating quantum problems simply implementing a form of (1.2) and (1.1) modified since the beginning according to the quantum uncertainty appears reasonable and rational. The present paper aims to highlight how to demonstrate the physical validity of these introductory premises and implement a non-deterministic Lagrangian to solve quantum and relativistic problems.

2. Preliminary Considerations

Redraft now (1.1) according to the uncertainty equation

$$\delta p_x \delta x = n\hbar = \delta\varepsilon\delta t \tag{2.1}$$

writing instead

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta \dot{x}} = \frac{\delta L}{\delta x}, \quad \frac{\delta L}{\delta \dot{x}} = p_x, \quad \frac{\delta L}{\delta x} = F_x \tag{2.2}$$

and then waiving the limits $\delta \to \partial$. Several papers have shown that (2.1) are direct corollary of the operative definition of space time [10]

$$\frac{\hbar G}{c^2} = \frac{length^3}{time} \tag{2.3}$$

and allow merging relativity and quantum physics; here the crucial point is to replace the deterministic metrics with the non-local and non-real character of (2.1) [4]. Seemingly (2.2) is a replica of (1.2) simply with different notation. Instead, replacing ∂ with δ introduces a huge conceptual gap between (2.2) and (1.1): the former symbol implies local values of any function $f(x)$ at x and $x + \mathrm{d}x$, the latter waives in fact the coordinates being both x and $x + \delta x$ arbitrary and unknown along with $\delta x \neq 0$ itself.

Are here remined only a few remarks on (2.1) to justify some algebraic steps introduced below; further details about this agnostic theoretical background and its implications for quantum systems are reported in [11] [12]. Here it is worth remarking that: 1) the ranges of dynamical variables concerned in (2.1) are arbitrary and unknown, 2) the concept of derivative is replaced by that of mere ratio of ranges, 3) any approach based on (2.1) necessarily assumes a space time conceptual frame and 4) actually (2.1) is not linked to a specific reference system where are defined all ranges. The last item is so crucial to deserve being shortly sketched here for completeness. Regard the range sizes as multiple of the respective Planck units, e.g. let be $\delta x = n_x^* \sqrt{\hbar G/c^3}$ and $\delta p = n_p^* \sqrt{\hbar c^3/G}$; then $\delta p_x \delta x = n_x^* n_p^* \hbar$ [4]: the starred symbols are arbitrary real numbers, n of (2.1) is arbitrary integer with the meaning of number of allowed quantum states. An analogous result holds for $\delta\varepsilon\delta t$, as it is immediate to verify, so (2.1) reads with obvious meaning of symbols

$$n_x^* n_p^* = n = n_\varepsilon^* n_t^* . \tag{2.4}$$

This means that formulating any physical problem after having replaced systematically the local dynamical variables via the corresponding uncertainty ranges, the resulting equations bypass by definition the necessity of specifying the reference system. The relevance of this statement in relativity is self-evident, likewise the quantization introduced by n. In effect, let $\delta p_x \delta x = n\hbar$ and $\delta p'_x \delta x' = n'\hbar$ be defined in different inertial reference system in relative constant motion. As n and n' symbolize sequences of integers, and not a specific number, it appears that the primed and unprimed products of conjugate variables are actually indistinguishable; in both reference systems, the products in (2.1) yield sequences of numbers, $n = 1, 2, \cdots$ and $n' = 1, 2, \cdots$, arbitrary and unknowable by definition. So just the quantization justifies why different reference systems become in fact indistinguishable. Even the concept of derivative takes in the present model the mere meaning of ratio of uncertainty ranges; previous papers have addressed this subtle point [4], which is further explained in the following as it is essential to emphasize the validity of this statement. Here it is worth considering that the classical concept of local velocity $v = \mathrm{d}x/\mathrm{d}t$ implies that both differentials $\mathrm{d}x$ and $\mathrm{d}t$ must tend concurrently to zero to define v somewhere in the space time. According to (2.1), instead, v is definable via two separate and independent uncertainty ranges δx and δt about which nothing is known: in fact the classical requirement of their concurrent tending to any value, e.g. both to zero, would imply introducing deterministic information and thus would violate their total agnosticism compelled by the quantum uncertainty. For this reason $\delta x/\delta t$ is mere ratio of two independent ranges, which does not exclude that both range sizes can be arbitrarily small although anyway finite. So the usual differential $\mathrm{d}f = (\partial f/\partial x)\mathrm{d}x$ reads $\delta f = (\delta f/\delta x)\delta x$ in the present model, which holds at the first order approximation for δx in principle finite and fulfilling (2.1).

Consider now that (2.1) implies

$$\frac{\delta\varepsilon}{\delta p_x} = v_x = \frac{\delta x}{\delta t} \quad \textit{i.e.} \quad \frac{\delta}{\delta t}\frac{\delta\varepsilon}{\delta p_x} = \frac{\delta}{\delta t}\frac{\delta x}{\delta t} = a_x , \tag{2.5}$$

being a_x acceleration by dimensional reasons; also

$$\delta\dot{x} \equiv \frac{\delta(\delta x)}{\delta t} = \left(\frac{\delta x|_{x_2,t_2}}{\delta t} - \frac{\delta x|_{x_1,t_1}}{\delta t} \right) = v_x(x_2,t_2) - v_x(x_1,t_1) = \delta v_x , \tag{2.6}$$

where $\delta(\delta x)$ is the change of range size δx during a given time lapse δt at which are defined both $\delta x|_{x_2,t_2}$ and $\delta x|_{x_1,t_1}$. Note that $\delta\dot{x}$ is the change rate of δx, whereas v_x is the velocity component of the particle concerned by ε and p_x; the fact that in (2.6) $\delta v_x = \delta \dot{x}$, is due to the total delocalization of *m* in δx whatever the size of this latter might be. So the change $\delta(\delta x)$ of δx requires changing v_x of *m* by δv_x for the particle to fit entirely the new confinement range; as the rate at which moves the confined particle is $\le c$, it follows that $\delta \dot{x} \le c$ as well. This has to do with the finite velocity with which propagates any perturbation/interaction throughout the space time described by (2.1). It follows from the second (2.5)

$$\frac{\delta}{\delta t}\frac{\delta\varepsilon}{\delta p_x} = a_x = \frac{F_x}{m} \tag{2.7}$$

being *m* a constant mass and a_x acceleration. Moreover, it is also true that (2.1) implies

$$\frac{\delta p_x}{\delta t} = \frac{n\hbar/\delta t}{\delta x} = F_x; \tag{2.8}$$

i.e. according to (2.5) just this property of all variable delocalization range is equivalent in general to the rising of a force field F_x within δx, whose particular physical meaning does not need being specified. In other words, the fact that the force is mere consequence of the stretching/shrinking of the space time delocalization range is nothing else but the generalization of the Einstein concept of space time curvature responsible of and appearing as the gravity: here however, owing to the form of (2.1), the concept of force is relatable according to (2.8) to any physical reason that implies modification rate, not necessarily just curvature, of the space time uncertainty ranges. The Einstein intuition is found here as a corollary of the quantum uncertainty along with the equivalence principle [4]. So, (2.7) yields

$$\frac{\delta}{\delta t}\left(\frac{\delta\varepsilon}{\delta p_x/m}\right) = \frac{\delta}{\delta t}\left(\frac{\delta\varepsilon}{\delta v_x}\right) = F_x \tag{2.9}$$

that in turn reads with the help of (2.6)

$$\frac{\delta}{\delta t}\frac{\delta\varepsilon}{\delta v_x} = \frac{\delta}{\delta t}\frac{\delta\varepsilon}{\delta\dot{x}} = F_x, \quad \delta p_x = m\delta v_x, \quad p_x = mv_x. \tag{2.10}$$

On the one hand (2.8) reads

$$\frac{\delta p_x}{\delta t} = F_x = -\frac{\delta U}{\delta x}, \quad \delta U = \frac{n\hbar}{\delta x}\delta\dot{x} = n\hbar\frac{\delta\dot{x}}{\delta x} = n\hbar\frac{\delta v_x}{\delta x}; \tag{2.11}$$

on the other hand, it is reasonable to expect that F_x is related to the change of kinetic energy of *m*, *i.e.* $\delta\varepsilon_{kin} = v_x \delta p_x = \delta\left(mv_x^2/2\right)$ according to (2.5) and (2.10). So $\delta\varepsilon_{kin}/\delta v_x = \delta T/\delta v_x = mv_x$. Hence

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta v_x} = \frac{\delta}{\delta t}\frac{\delta L}{\delta\dot{x}} = \frac{\delta L}{\delta x}, \quad L = T(v_x) - U(\delta x). \tag{2.12}$$

This result merges (2.10) and (2.11) and is therefore the sought Lagrange equation compliant with the non-deterministic definition (2.12) of *L*; moreover

$$\frac{\delta L}{\delta v_x} = \frac{\delta L}{\delta \dot{x}} = p_x, \tag{2.13}$$

which is just the expected result in agreement with (2.10).

In summary the Lagrange Equations (2.12) are related to (2.1); is significant that p_x is no longer a local value, as stated in the introduction, since now $p_{x1} \leq p_x \leq p_{x2}$. The fact that the uncertainty ranges can be unknowably infinitely large or infinitely small does not prevent approaching the same result of the classical approach via deterministic derivatives; yet these latter are mere extrapolation of a more general approach fulfilling the Heisenberg principle.

Three quantum and relativistic final remarks deserve attention, already concerned in [4] and sketched here.

-Note first why the uncertainty implies itself as a corollary the quantum indistinguishability of identical particles: (2.1) does not concern explicitly the particle but its phase space. The properties of the latter determine that of the former, not vice versa. So once having assessed motivation and usefulness of neglecting the local dynamical variables, systematically replaced by their corresponding ranges in describing any fundamental physical law like the Lagrangian, the attempt to distinguish two identical quantum particles about which nothing is known becomes intuitively unphysical.

-Also note why (2.1) imply the existence of an upper limit of velocity in the space time. Write the first (2.5) as $\delta\varepsilon = v_x \delta p_x = \boldsymbol{v} \cdot \delta \boldsymbol{p}$; so any finite $\delta\varepsilon$ should yield $\delta\varepsilon / v_x \to 0$ for $v_x \to \infty$ and thus $\delta p_x \to 0$ too. But this is a contradiction, because it would imply a local value p_x corresponding to the matching value of $p_{x2} \to p_{x1}$; clearly the single value p_x cannot correspond to the range of values $\varepsilon_2/v_x - \varepsilon_1/v_x$ about which no specific hypothesis is admissible without appropriate information about ε_1 and ε_2. Is instead admissible $p_x = \varepsilon_2/c - \varepsilon_1/c$, with ε_1 and ε_2 arbitrary along with p_x arbitrary as well. So, the finite c prevents the determinism of a single p_x.

-In the previous equations δp_x has been expressed as $m\delta v_x$ in (2.9) and, whatever δv_x might be, via $n\hbar/\delta x$ in (2.8) owing to (2.1). Note now that $\hbar/\delta x = (\hbar/\delta t)/(\delta x/\delta t) = \varepsilon/v_x$, having taken $n = 1$ for simplicity: is it possible to merge these results? Write identically

$$\delta(p_x c) = mc\delta v_x = mc^2 \delta \frac{v_x}{c} = \varepsilon \frac{\delta v_x}{c} \tag{2.14}$$

being ε an energy by dimensional reasons; so one finds

$$\delta p_x = \varepsilon \delta \left(\frac{v_x}{c^2} \right) \tag{2.15}$$

that in general is compliant with the form

$$p_x = \frac{\varepsilon v_x}{c^2}. \tag{2.16}$$

This is the well-known relativistic form of momentum obtained without special hypotheses. Of course this result is sensible because of the existence of a finite velocity c required according to the previous point.

-The scalar $\boldsymbol{v}\cdot\delta\boldsymbol{p}$, although concerned here as $v_x\delta p_x$ for brevity, corresponds actually to any number of physical dimensions possible in the space time here symbolized generically by the subscript x: thus all considerations about (2.1) are by definition compatible with theories requiring extra dimensions [13]. The fact that δx could be for example a radial distance or the component of any range size $\delta\boldsymbol{x}$ or any extra dimension range $\delta\boldsymbol{s}$, agrees with the actual idea of truly generalized dynamical variables characterizing the Lagrange Equation (2.2).

It is worth emphasizing eventually that the present approach is not relativistic, likewise the previous ones e.g. [4], but typically quantum: it does not implement the metric, but the Equation (2.1) as a conceptual basis.

This statement makes sense reminding the purposes that historically motivated the birth of quantum and relativistic theories: simplifying this statement as much as possible, the former aimed to explain why the electron does not fall into the nucleus, the latter to find a covariant physics. Whether or not (2.1) surrogate the premises of relativity, it is a subtle point already checked in previous papers and still in progress; even this model attempts to verify the chance of obtaining contextually relativistic results in a mere quantum frame. This expectation, legitimated by the previous four points 1) to 4) emphasizing the intrinsic features of (2.1), is in fact confirmed by the next results.

3. Generalization of the Equation (2.12)

To simplify the notations, the components v_x and p_x of $\boldsymbol{v}$ and $\boldsymbol{p}$ will be shortened simply as v and p in the following.

The Lagrange Equation (2.12) has been obtained via (2.1) only, yet with the help of some classical simplifications, *i.e.* to regard m independent of v and to assume explicitly $T=T(v)$ and $U=U(\delta x)$. In fact these assumptions are also introduced in the standard derivations of the classical Lagrangian for a conservative field: *i.e.* in the current literature $\delta x\to 0$ is local generalized coordinate, $\delta x/\delta t\to \mathrm{d}x/\mathrm{d}t$ is local generalized velocity. In effect, it is confirmed below that the previous definitions of generalized coordinate ranges and velocities still hold in the present model without requiring specific hypotheses; with such agnostic premises any consequent model results intuitively as general as possible. In principle, nothing hinders to think $U=U(\delta x,v)$; for example consider that actually in relativity the rest mass is a constant by definition whereas the dynamic mass depends on v, as

$$m_{dyn}=\frac{m}{\beta},\quad \beta=\sqrt{1-\frac{v^2}{c^2}},\quad m=const. \tag{3.1}$$

Even this result has been obtained in the frame of a physical model based on (2.1) [4].

This section aims thus to generalize significantly via (2.1) the meaning of the result (2.12) itself with the help of the definition (2.5) of generalized velocity.

Define then

$$L = f\left(v^k\right),\quad v = \frac{\delta x}{\delta t}, \tag{3.2}$$

being k an arbitrary exponent. It is necessary to calculate in agreement with (2.6) the ratios

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta v}\right),\quad \frac{\delta L}{\delta x} \tag{3.3}$$

and demonstrate that

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta v}\right) = \frac{\delta L}{\delta x},\quad \delta v = \delta \dot{x}. \tag{3.4}$$

Calculate foremost

$$\frac{\delta L}{\delta v} = \frac{\delta L}{\delta f}\frac{\delta f}{\delta v^k} k v^{k-1} = Y: \tag{3.5}$$

the first (3.3) yields directly

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta v}\right) = \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta f}\frac{\delta f}{\delta v^k} k v^{k-1}\right) = \frac{\delta Y}{\delta t}, \tag{3.6}$$

whereas the second (3.3) yields implementing v of (3.2) and (2.16)

$$\frac{\delta L}{\delta x} = \frac{\delta L}{\delta v}\frac{\delta v}{v}\frac{v}{\delta x} = Y\frac{\delta v}{v}\frac{1}{\delta t} = \frac{\delta Y}{\delta t},\quad \delta Y = \delta v\frac{Y}{v}. \tag{3.7}$$

In turn, since in general for any function f holds $\delta f \equiv \delta\left(f + const\right)$, (3.7) reads with the help of (3.5)

$$\frac{\delta L}{\delta x} = Y\delta\left(\log\left(v\right)\right)\frac{1}{\delta t} = Y\frac{\delta\left(\log\left(v\right) + const\right)}{\delta t} = Y\frac{\delta\left(\log\left(mv\right)\right)}{\delta t},\quad const = \log\left(m\right). \tag{3.8}$$

Merge now (3.6) and (3.8). By subtracting side by side

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta v}\right) = \frac{\delta Y}{\delta t},\quad \frac{\delta L}{\delta x} = Y\frac{\delta\left(\log\left(mv\right)\right)}{\delta t} \tag{3.9}$$

the result is

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta v}\right) - \frac{\delta L}{\delta x} = \frac{\delta Y}{\delta t} - \frac{Y}{\delta t}\delta\left(\log\left(mv\right) - \log\left(m_0 v_0\right)\right) \tag{3.10}$$

because of course $\delta\left(m_0 v_0\right) = 0$. Hence

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta v}\right) - \frac{\delta L}{\delta x} = \frac{\delta Y}{\delta t} - \frac{Y}{\delta t}\delta\log\left(\frac{p}{p_0}\right) = 0 \tag{3.11}$$

because the right hand side reads, according to the second (3.7),

$$\frac{Y}{\delta t}\left(\frac{\delta Y}{Y} - \delta\log\left(\frac{p}{p_0}\right)\right) = \frac{Y}{\delta t}\delta\left(\log\left(\frac{v}{v_0}\right) - \log\left(\frac{p}{p_0}\right)\right) = 0,\quad \frac{Y}{Y_0} = \frac{v}{v_0} = \frac{p}{p_0}. \tag{3.12}$$

The result is thus, whatever f and the exponent k of (3.5) might be,

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta v}\right) - \frac{\delta L}{\delta x} = \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{x}}\right) - \frac{\delta L}{\delta x} = 0. \tag{3.13}$$

Since (3.2) involves v^k only, with v defined by the uncertainty ranges δx and δt only, one infers through (2.4) that L holds regardless of the reference system where is defined v.

It is possible to think at this point that the physical worth of the definition (2.2) of L effectively rests on the generality of (3.2). To demonstrate in fact this conclusion, let us start from a known result [7]: let the Lagrangian of a free neutral particle be

$$L = -\zeta\beta, \quad \beta = \left(1 - \frac{v^2}{c^2}\right)^{1/2}, \tag{3.14}$$

being ζ an appropriate constant energy in agreement with (2.15), and check how this function fits (3.4) in the conceptual frame so far introduced. In effect the previous definitions (3.2) and (3.5) imply

$$L = \zeta f\left(v^k\right), \quad f = -\beta, \quad k = 2, \quad v = v(\delta x, \delta t) = \frac{\delta x}{\delta t}, \tag{3.15}$$

so that the given value of k yields now in particular

$$Y = \frac{\delta L}{\delta f}\frac{\delta f}{\delta\left(v^k\right)}kv^{k-1} = -\zeta\frac{\delta\beta}{\delta\left(v^2\right)}2v = \frac{\zeta}{c^2}\frac{v}{\beta}; \tag{3.16}$$

then trivial calculations yield

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta v}\right) = \frac{\delta Y}{\delta t} = \frac{\zeta}{c^2}\frac{\delta(v/\beta)}{\delta t}, \tag{3.17}$$

being

$$\frac{\delta(v/\beta)}{\delta t} = \frac{1}{\beta^3}\frac{\delta v}{\delta t}. \tag{3.18}$$

Let the constant ζ be for example m_0c^2, according to (2.16), or any possible generalization of it depending on the specific physical problem. If so, since Y is momentum owing to (3.12), then (3.16) reads

$$Y = momentum = \frac{m_0 v}{\beta} = \frac{m_0c^2}{\beta}\frac{v}{c^2} = \frac{\varepsilon v}{c^2}, \quad \varepsilon = \frac{m_0c^2}{\beta}:$$

these equations are well known outcomes of special relativity. Examine now two corollaries of (3.16).

-On the one hand (3.16) reads $\delta L = Y\delta\left(v^k\right)/kv^{k-1}$, so that (3.7) yields

$$\frac{\delta L}{\delta x} = \frac{Y}{\delta t}\delta\log(v);$$

thus, likewise to (3.8), (3.11) and (3.12),

$$\frac{\delta L}{\delta x} = \frac{Y}{\delta t}\log\left(\frac{mv}{m_0v_0}\right) = \frac{\delta Y}{\delta t} = \frac{\delta}{\delta t}\left(\frac{\delta L}{\delta v}\right) \tag{3.19}$$

that of course is again the Lagrange equation. In turn it also follows

$$\delta L = Y\frac{\delta x}{\delta t}\delta\log(v) = Yv\delta\log(v) = Yv_0\frac{v}{v_0}\log\left(\frac{v}{v_0}\right);$$

this result is interesting because it has an entropic form. The energy ratio $\delta L/Yv_0$ equals $-S = w\log w$ if the ratio v/v_0 has probabilistic meaning for the number of allowed quantum states pertinent to the given value of the ratio, for example putting $v_0 = c$. This last result holds regardless of specific *f* and *k*, as it depends on the resulting generalized momentum Y only; so the arbitrary constant v_0 allows regarding $Yv_0 = k_B T$ and write $\delta L = -TS$ summing over all states of the system. Then
$\delta L = L - L_0 = E - U - L_0$ yields

$$E = U - TS + L_0:$$

i.e. the generalized Lagrangian is compatible with the Helmholtz free energy, with $L_0 = 0$ in particular, or with the Gibbs free energy putting $L_0 = PV$.

-On the other hand $\delta L = \delta x \delta Y/\delta t$ of (3.19) fits (2.1), noting that L is anyway an energy whereas Y is anyway a momentum; therefore $\delta L \delta t = \delta Y \delta x$ is nothing else but (2.1). Thus just (3.19) also yields, owing to (3.18),

$$\delta L = \frac{m_0}{\beta}\frac{\delta x}{\beta^2 \delta t}\delta v = m_{dyn}\frac{\delta x/\beta}{\beta \delta t} = m_{dyn}\delta x_0 \delta t_0, \quad \delta x_0 = \frac{\delta x}{\beta}, \quad \delta t_0 = \beta \delta t, \quad m_{dyn} = \frac{m_0}{\beta}: \quad (3.20)$$

i.e., if at the right hand side of the first equation the product $\delta x_0 \delta t_0$ space and time ranges is independent of *v*, then δx is the Lorentz contraction of the proper space range size δx_0 where one confined particle is at rest, whereas δt is the time dilation of the proper time range size δt_0 of the particle at rest. Eventually, as it also appears that

$$\delta x \delta t = \delta x_0 \delta t_0 = \delta x' \delta t', \quad (3.21)$$

one infers that $\delta x \delta t$ must be invariant in different reference systems in reciprocal motion; indeed, even the primed quantities are also referable to the same proper length and time.

A further remark about the force $\delta L/\delta x = ma_x$; it is sensible to expect that this force is someway deducible from the definition of *Y*; indeed, since by definition $F_x = \delta p_x/\delta t$, (3.14) and (3.16) yield

$$F_x = \frac{\delta L}{\delta x} = \frac{\delta Y}{\delta t} = \frac{\zeta}{c^2}\frac{\delta}{\delta t}\left(\frac{v}{\beta}\right) = m_0 a_x, \quad a_x = \frac{\delta}{\delta t}\frac{v}{\sqrt{1 - v^2/c^2}}.$$

In effect also this result is well known; it is concerned in [7] in particular for $a_x = const$ and does not need further comments here; it is worth emphasizing that after having integrated the right hand side as a function of the left hand side $v_a = a_x t$, the further integral over time of the velocity v_a tends to *c* at increasing δt.

This last relativistic result is once more crucial to confirm the validity of the approach so far followed.

4. Discussion

In principle, the uncertainty ranges can be understood thinking to the confidence intervals unavoidably inherent any measurement process. Reasonably these intervals, and not the random outcomes of the single measurements, are reliable starting points to infer physical information; this simple analogy and the fact that the standard wave formalism is also obtainable as a byproduct of (2.1) [4], are enough to realize why the chance to infer contextually both quantum and relativistic results is greatly simplified implementing the concept of uncertainty in its most agnostic form. Equation (2.1) have been tested in various quantum problems [11] [12].

An example is so short to deserve being mentioned explicitly here. Infer the unique component $M_u = \boldsymbol{M}\cdot\boldsymbol{u}$ of quantum angular momentum merely knowing its classical definition $(\boldsymbol{r}\times\boldsymbol{p})\cdot\boldsymbol{u}$: write $(\delta\boldsymbol{r}\times\delta\boldsymbol{p})\cdot\boldsymbol{u}$, which reads $M_u = (\boldsymbol{u}\times\delta\boldsymbol{r})\cdot\delta\boldsymbol{p} = \delta\boldsymbol{w}\cdot\delta\boldsymbol{p} = \pm\delta p_w\delta w$ with $\delta p_w = \delta\boldsymbol{p}\cdot\delta\boldsymbol{w}/|\delta\boldsymbol{w}|$ and $\delta\boldsymbol{w} = \boldsymbol{u}\times\delta\boldsymbol{r}$. So (2.1) imply $M_u = \pm l\hbar$ with $l = 0$ or $l = integer$ depending on whether $\delta\boldsymbol{w}\perp\delta\boldsymbol{p}$ or not.

The values of l agree with the fact that here the quantum numbers of wave mechanics take the physical meaning of numbers of allowed quantum states of the angular momentum component, if any. This is the only possible result; changing $\boldsymbol{u}$ to find a further component $M_{u'}$ would trivially meaning repeating the same, unique information.

The chance of obtaining contextually a wide variety of quantum and relativistic results, already asserted in [4], is confirmed here through the consistency of (2.2) with that concept of Lagrangian and helps understanding why (2.1) inferred from (2.3) have actual physical meaning, rather than the deterministic local values of the dynamical variables falling in these intervals.

It is instructive to verify now the results obtainable introducing in the conceptual frame so far outlined a different Lagrangian function L' defined now by $\beta^2 = 1 - v^2/c^2$ instead of the well known one of (3.14). Repeating the steps (3.15) to (3.18), the previous definitions (3.2), (3.5) and (3.7) imply now

$$L' = -\zeta' f', \quad f' = \beta^2, \quad k = 2, \tag{4.1}$$

being again ζ' an appropriate constant energy; so (3.16) becomes

$$Y' = \frac{\delta L'}{\delta f'}\frac{\delta f'}{\delta v^k} k v^{k-1} = -\zeta'\frac{\delta\beta^2}{\delta v^2}2v = \zeta'\frac{1}{c^2}2v. \tag{4.2}$$

Then on the one hand

$$\frac{\delta}{\delta t}\left(\frac{\delta L'}{\delta v}\right) = \frac{\delta Y'}{\delta t} = \frac{2\zeta'}{c^2}\frac{\delta v}{\delta t}, \tag{4.3}$$

whereas on the other hand, by calculating $\delta L'/\delta x$ similarly to (3.7) and (3.8), (4.2) yields now

$$\frac{\delta L'}{\delta x} = \zeta'\frac{2v}{c^2}\frac{\delta v}{\delta x} = \frac{2\zeta'}{c^2}\delta v\frac{1}{\delta t} \tag{4.4}$$

whence, comparing (4.3) and (4.4), one infers recalling (2.6)

$$\frac{\delta}{\delta t}\left(\frac{\delta L'}{\delta v}\right) = \frac{\delta}{\delta t}\left(\frac{\delta L'}{\delta \dot{x}}\right) = \frac{\delta L'}{\delta x}. \tag{4.5}$$

There is new information in (4.4) with respect to (3.14) of the free neutral particle, as it results rewriting the first equality as

$$\frac{\delta L'}{\delta x} = \zeta' \frac{2v}{c^2}\frac{\delta v}{\delta x} = \frac{\zeta'}{c^2}\frac{\delta\left(v^2\right)}{\delta x} = \frac{\zeta'}{c^2}\frac{\delta\left(\boldsymbol{v}\cdot\boldsymbol{v}\right)}{\delta x};$$

the last term can be rewritten as follows

$$\frac{\delta L'}{\delta x} = \frac{\delta\left(\zeta'\boldsymbol{A}\cdot\boldsymbol{v}\right)}{\delta x}, \quad \zeta'\boldsymbol{A} = \frac{\zeta'}{c^2}\left(\boldsymbol{v}+\boldsymbol{v}_0\right), \quad \boldsymbol{v}\cdot\boldsymbol{v}_0 = 0 .$$

Then owing to (4.5) and to the Helmholtz decomposition theorem of vector calculus, which states that a 3D field rapidly decaying is equivalent to the sum of an irrotational and a solenoidal field, the right hand side can be also rewritten for sake of generality as

$$\frac{\delta L'}{\delta x} = \frac{\delta\left(\zeta'\boldsymbol{A}\cdot\boldsymbol{v}-\varphi\right)}{\delta x}, \quad \frac{\delta\varphi}{\delta x} = \frac{\delta}{\delta t}\left(\frac{\delta\varphi}{\delta v}\right), \tag{4.6}$$

being φ an appropriate scalar function fulfilling the second condition: inserting the first (4.6) in (4.5) is still fulfilled (3.26) thanks to the second condition in agreement with (3.29). It indicates that the new choice (4.1) of the Lagrangian function is compatible with the new L', *i.e.* $L' = \zeta'\boldsymbol{A}\cdot\boldsymbol{v}-\varphi$. As the Lagrange equations are linear in L it is possible to write

$$L_{tot} = \zeta L + \zeta' L' = \zeta\left(1-\frac{v^2}{c^2}\right)^{1/2} + \zeta'\left(1-\frac{v^2}{c^2}\right) = \zeta\left(1-\frac{v^2}{c^2}\right)^{1/2} + \zeta'\left(\boldsymbol{A}\cdot\boldsymbol{v}-\varphi\right). \tag{4.7}$$

This result is well known to be the Lagrangian of a charged particle in an electromagnetic field [7].

Eventually note that the Lagrange equation can be inferred uniquely via quantum considerations. Equation (2.1) imply inherently

$$\frac{\delta p_x}{\delta t} = \frac{n\hbar/\delta x}{\delta t} = \frac{n\hbar\omega}{\delta x} = \frac{\delta L}{\delta x}, \quad \omega = \frac{1}{\delta t};$$

this result confirms (3.19) and yields in turn

$$\delta L = n\hbar\omega = \frac{\delta x\delta p_x}{\delta t} = v_x\delta p_x \quad \textit{i.e.} \quad v_x = \frac{\delta L}{\delta p_x}.$$

Hence, owing to (2.6) and (2.5),

$$\frac{\delta}{\delta t}v_x = \frac{\delta}{\delta t}\frac{\delta L}{\delta p_x} = \frac{\delta}{\delta t}\frac{\delta L}{\delta\left(mv_x\right)} = \frac{\delta}{\delta t}\frac{\delta L}{m\delta\dot{x}} = a_x = \frac{F_x}{m} = \frac{\delta L}{m\delta x},$$

being m a constant mass, whence via (2.6)

$$\frac{\delta}{\delta t}\frac{\delta L}{\delta\dot{x}} = \frac{F_x}{m} = \frac{\delta L}{\delta x}, \quad \frac{\delta L}{\delta\dot{x}} = mv_x, \quad \frac{\delta L}{\delta x} = F_x . \tag{4.8}$$

This generalized δL can be reasonably handled with (3.2) and implementing even $k \neq 2$, despite so far only the chance $k = 2$ has been concerned explicitly to demonstrate the connection of the present approach with the known Lagrangian of special relativity.

The usefulness of the present model also appears considering that the mere Newtonian potential, having the form $const/\delta x$, admits actually a much more general formulation [4].

So new perspectives appear plausible even in the extended case of non-Newtonian potentials $U = U\left(\delta x^k, \delta \dot{x}^{k'}\right)$; for example, in the most general case where the particle is not free, e.g. because it interacts with several different fields, the functional dependence of L_k on *v* would imply v^k or a sum $L_\Sigma = \sum_k \zeta_k L_k$ corresponding to the superposition of several quantum *k*-states. Of course momentum and force of each L_k, pertinent to the form of the respective f_k, are still given by (3.5) and (3.7). Work is in progress on this point.

5. Conclusions

The model defines a Lagrange function directly relatable to classical, thermodynamic, relativistic and quantum results.

The chance of generalizing further the few cases highlighted in present paper is possible to appear, and in principle simple, by regarding appropriately (4.7): the sum of two terms possibly suggests a more general series expansion of *L* as a function of higher powers of Lorentz factors β.

If so, then some open questions arise concerning in particular the Lagrangian of the standard model: what kind of phenomena could describe these higher order terms of L_Σ with different *f* and *k*? Are the terms of this conceivable series expansion L_Σ related to the Feynman diagrams of the respective situations? Do these terms require new hypotheses to introduce the mass, or the mass is introducible via dimensional considerations only as in (4.8)?

Activity is in progress to clarify the actual physical implications of the generalization (2.2).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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