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A Note on Sharp Affine Poincaré-Sobolev Inequalities and Exact in Minimization of Zhang's Energy on Bounded Variation and Exactness

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Abstract

As for the affine energy, Edir Junior and Ferreira Leite establish the existence of minimizers for particular restricted subcritical and critical variational issues on $BV(\Omega)$. Similar functionals exhibit deeper weak* topological traits including lower semicontinuity and affine compactness, and their geometry is non-coercive. Our work also proves the result that extremal functions exist for certain affine Poincaré-Sobolev inequalities.

Keywords

Affine Energy, Affine Sobolev Inequality, Compactness of Affine Immersion, Constrained Minimization

1. Introduction

The variational issues have been extensively researched within the domain of boundedly variable functions $BV(\Omega)$. This has mostly been in connection with the availability of solutions where the 1-Laplace operator is present, such as in the well-known Cheeger's problem [1]. See, among other places, [2]-[7] for contributions along this line of thought. A portion of them place a greater emphasis, more especially, on the challenge of reducing the functional

$$\Phi\left(u_j\right)=\sum\left|Du_j\right|(\Omega)+\int_\Omega\sum a\left|u_j\right|\mathrm{d}x+\int_{\partial\Omega}\sum b\left|\tilde{u}_j\right|\mathrm{d}\mathcal{H}^{1+\epsilon}$$

where is a bounded open in $\mathbb{R}^{2+\epsilon}$ with Lipschitz border, $\epsilon\geq 0, a\in L^\infty\left(\Omega\right)$, and $b\in L^\infty\left(\partial\Omega\right)$, and either on the full $BV(\Omega)$ space or restricted to some portion of it. Total variation measure of the sequence u_j, its trace on $\partial\Omega$, and its Hausdorff measure in dimension $(1+\epsilon)$ are denoted by $|Du_j|(\Omega), \tilde{u}_j$ and $\mathcal{H}^{1+\epsilon}$, respectively.

Two subsets of $BV(\Omega)$ typically considered are:

$$X = \left\{u_j \in BV(\Omega) : \int_\Omega \sum |u_j|^{1+\epsilon} \mathrm{d}x = 1\right\},$$

$$Y = \left\{u_j \in BV(\Omega) : u_j \in X, \int_\Omega \sum |u_j|^{2\epsilon} u_j \mathrm{d}x = 0\right\}$$

The associated minimization issue involves proving the existence of minimizers for the least amount of energy.

The relevant issue of minimizing consists of determining whether or not there are minimizers for exponents with the lowest possible quantities of energy. $\epsilon \geq 0$:

$$c = \inf_{u_j \in X} \sum \Phi(u_j) \text{ and } d = \inf_{u_j \in Y} \sum \Phi(u_j).$$

Other non-critical examples are discussed in [5], whereas some crucial cases have been the subject of research in [2] [3] [8] [9] [10]. [5] has been cited for its work.

The reducing of Φ in the sets X and Y (with $\epsilon = 0$) is the fact that several classical functional inequalities, including the [11], have nonzero solutions (extremal functions) is another driving force, and $L^{1+\epsilon}$ [11] for $\epsilon \geq 0$. More specifically, their respective sharp versions on $BV(\Omega)$ state that

1) Poincaré inequality ($\mathcal{P}$):

If $\lambda_1 > 0$ such that $\lambda_1 \left\|\sum u_j\right\|_{L^1(\Omega)} \leq \sum |Du_j|(\Omega) + \sum \|\tilde{u}_j\|_{L^1(\partial\Omega)}$;

2) Poincaré-Wirtinger inequality ($\mathcal{PW}$):

There exists an optimal constant $\mu_1 > 0$ such that

$$\mu_1 \left\|\sum \left(u_j - (u_j)_\Omega\right)\right\|_{L^1(\Omega)} \leq \sum |Du_j|(\Omega);$$

3) Poincaré-Sobolev inequality ($\mathcal{PS}$):

Then $\lambda_{1+\epsilon} > 0$ such that $\lambda_{1+\epsilon} \left\|\sum u_j\right\|_{L^{1+\epsilon}(\Omega)} \leq \sum |Du_j|(\Omega) + \sum \|\tilde{u}_j\|_{L^1(\partial\Omega)}$;

4) Poincaré-Wirtinger-Sobolev inequality ($\mathcal{PPS}$):

Then $\mu_{1+\epsilon} > 0$ such that

$$\mu_{1+\epsilon} \left\|\sum \left(u_j - (u_j)_\Omega\right)\right\|_{L^{1+\epsilon}(\Omega)} \leq \sum |Du_j|(\Omega),$$

where $(u_j)_\Omega$ denotes the average of u_j over Ω, see [12] [13] [14] [15].

Several weaker inequalities than those listed above have been proposed to explain the presence of extremal functions for their sharp affine counterparts, and this serves as inspiration for our own theory of minimization for functional in which the term $|Du_j|(\Omega)$ gives place to the Zhang's affine energy.

In [8], Zhang introduced the affine L^1 energy (or functional) for functions $u_j \in W^{1,1}(\mathbb{R}^{2+\epsilon})$ given by

$$\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(u_j) = \alpha_{2+\epsilon} \left(\int_{\mathbb{S}_{1+\epsilon}} \left(\int_{\mathbb{R}^{2+\epsilon}} \sum |\nabla_\xi u_j(x)| \mathrm{d}x\right)^{-(2+\epsilon)} \mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}},$$

where $\alpha_{2+\epsilon} = (2\omega_{1+\epsilon})^{-1} \left((2+\epsilon)\omega_{2+\epsilon}\right)^{1+1/2+\epsilon}$. Here, $\nabla_\xi \sum u_j(x) = \sum \nabla u_j(x) \cdot \xi$

and ω_k in $\mathbb{R}^k$. The property $\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(u_j \circ T\right) = \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(u_j\right)$ for every $T \in SL(2+\epsilon)$, where $SL(2+\epsilon)$ denotes the special linear group of $(2+\epsilon)\times(2+\epsilon)$ matrices with determinant equal to 1.

The result of [16] ensures that the sharp Sobolev-Zhang inequality

$$(2+\epsilon)\,\omega_{2+\epsilon}^{1/2+\epsilon}\left\|u_j\right\|_{L^{(1+3\epsilon)}\left(\mathbb{R}^{2+\epsilon}\right)} \le \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(u_j\right) \tag{1}$$

holds for all $u_j \in W^{1,1}\left(\mathbb{R}^{2+\epsilon}\right)$, under invertible $(2+\epsilon)\times(2+\epsilon)$ matrices. Actually, characteristic functions are not in $W^{1,1}\left(\mathbb{R}^{2+\epsilon}\right)$, but rather belong to $BV\left(\mathbb{R}^{2+\epsilon}\right)$.

The Sobolev-Zhang inequality (II) is weaker than the classical sharp L^1 Sobolev inequality

$$(2+\epsilon)\,\omega_{2+\epsilon}^{1/2+\epsilon}\left\|\sum u_j\right\|_{L^{(1+3\epsilon)}\left(\mathbb{R}^{2+\epsilon}\right)} \le \sum\left\|\nabla u_j\right\|_{L^1\left(\mathbb{R}^{2+\epsilon}\right)}, \tag{2}$$

since

$$\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(u_j\right) \le \sum\left\|\nabla u_j\right\|_{L^1\left(\mathbb{R}^{2+\epsilon}\right)} \tag{3}$$

(see page 194 of [14]) as well as (3) being unyielding on non-spherical ellipsoid features. In addition, Zhang said that the Petty projection inequality (e.g. [17] [18]) is the underlying geometric inequality for (1), whereas the traditional isoperimetric inequality is the underlying geometric inequality for (2). In further work, Wang [19] proved that the Sobolev-Zhang inequality generalizes to functions $u_j \in BV\left(\mathbb{R}^{2+\epsilon}\right)$, where the affine BV energy is naturally represented by, just as it does in the Sobolev case (e.g. [20]).

$$\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(u_j\right) = \alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}}\left(\int_{\mathbb{R}^{2+\epsilon}}\sum\left|\sigma_{u_j}(x)\cdot\xi\right|\mathrm{d}\left(\left|Du_j\right|\right)(x)\right)^{-(2+\epsilon)}\mathrm{d}\xi\right)^{-1/2+\epsilon},$$

where $\sigma_{u_j}:\Omega\to\mathbb{R}^{2+\epsilon}$ the Radon-Nikodym derivative of Du_j, the total variation $\left|Du_j\right|$ on Ω, which satisfies $\left|\sigma_{u_j}\right|=1$ almost everywhere in Ω (w.r.t. $\left|Du_j\right|$). Moreover, equality in (1) is exactly accomplished by multiples of ellipsoid characteristic functions, and even after being translated to (3), it retains its inferiority to the classical predecessor

$$\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(u_j\right) \le \sum\left|Du_j\right|\left(\mathbb{R}^{2+\epsilon}\right). \tag{4}$$

After Zhang's first breakthrough, a wealth of further literature was produced detailing several refinements and new affine functional inequalities. The majority of the contributions are available at [21]-[38]. Given a function $u_j \in BV(\Omega)$, denote by $\bar{u}_j$ its zero extension outside of Ω. The Lipschitz regularity of $\partial\Omega$ guarantees that $\bar{u}_j \in BV\left(\mathbb{R}^{2+\epsilon}\right)$,

$$\sum\left|D\bar{u}_j\right|\left(\mathbb{R}^{2+\epsilon}\right) = \sum\left|Du_j\right|(\Omega) + \sum\left\|\tilde{u}_j\right\|_{L^1(\partial\Omega)} \tag{5}$$

and $\mathrm{d}\left(D\bar{u}_j\right) = \tilde{u}_j\nu_j\,\mathrm{d}\mathcal{H}^{1+\epsilon}\mathcal{H}^{1+\epsilon}$-almost everywhere on $\partial\Omega$, where ν_j denotes the unit outward normal to $\partial\Omega$ (see e.g. page 38 of [27]). Implies that

$$\begin{aligned}\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right) = \alpha_{2+\epsilon}\Bigg(\int_{\mathbb{S}_{1+\epsilon}}\sum\Bigg(&\int_{\Omega}\left|\sigma_{u_j}(x)\cdot\xi\right|\mathrm{d}\left(\left|Du_j\right|\right)(x)\\ &+\int_{\partial\Omega}\left|\tilde{u}_j(x)\right|\left|\nu_j(x)\cdot\xi\right|\mathrm{d}\mathcal{H}^{1+\epsilon}(x)\Bigg)^{-(2+\epsilon)}\mathrm{d}\xi\Bigg)^{-1/2+\epsilon}\end{aligned}$$

The preceding Formulas (4) and (5), and a reverse Minkowski inequality compare the affine BV energy of zero extended functions to local expressions:

(C1) $\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right) \leq \sum \left|Du_j\right|(\Omega) + \sum \left\|\tilde{u}_j\right\|_{L^1(\partial\Omega)}$ for all $u_j \in BV(\Omega)$;

(C2) $\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right) = \sum \mathcal{E}_{\Omega}\left(u_j\right)$ for all $u_j \in BV_0(\Omega)$;

(C3) $\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right) \geq \sum \mathcal{E}_{\Omega}\left(u_j\right) + \sum \mathcal{E}_{\partial\Omega}\left(\tilde{u}_j\right)$ for all $u_j \in BV(\Omega)$ with $\tilde{u}_j \neq 0$ on $\partial\Omega$ (a.e.) or limitless potential so long as is non-flat in the sense that $\sum \nu_j(x)\cdot\xi \neq 0$ on $\partial\Omega$ (a.e.) for every $\xi \in \mathbb{S}^{1+\epsilon}$, where $BV_0(\Omega)$ denotes the subspace of $BV(\Omega)$ of functions with zero trace on $\partial\Omega$,

$$\sum \mathcal{E}_{\Omega}\left(u_j\right) = \alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}}\left(\int_{\Omega}\sum\left|\sigma_{u_j}(x)\cdot\xi\right|\mathrm{d}\left(\left|Du_j\right|\right)(x)\right)^{-(2+\epsilon)}\mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}}$$

and

$$\sum \mathcal{E}_{\partial\Omega}\left(\tilde{u}_j\right) = \alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}}\left(\int_{\partial\Omega}\sum\left|\tilde{u}_j(x)\right|\left|\nu_j(x)\cdot\xi\right|\mathrm{d}\mathcal{H}^{1+\epsilon}(x)\right)^{-(2+\epsilon)}\mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}}$$

(C3) is more complicated than (C1) and (C2) (Corollary 3.1). Ball domains satisfy the geometric requirement. They are affine invariants.

$\sum \mathcal{E}_{\Omega}\left(u_j \circ T\right) = \sum \mathcal{E}_{T(\Omega)}\left(u_j\right)$ and $\sum \mathcal{E}_{\partial\Omega}\left(\tilde{u}_j \circ T\right) = \sum \mathcal{E}_{\partial T(\Omega)}\left(\tilde{u}_j\right)$ for every $T \in SL(2+\epsilon)$.

From (C1), the term $\mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right)$ weakens the right-hand side of ($\mathcal{P}$) and ($\mathcal{PS}$), encouraging us to study the new functional. $\Phi_{\mathcal{A}} : BV(\Omega) \to \mathbb{R}$,

$$\sum \Phi_{\mathcal{A}}\left(u_j\right) = \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right) + \int_{\Omega}\sum a\left|u_j\right|\mathrm{d}x + \int_{\partial\Omega}\sum b\left|\tilde{u}_j\right|\mathrm{d}\mathcal{H}^{1+\epsilon}.$$

The well-definedness of $\Phi_{\mathcal{A}}$ for limited weights *a* and *b* may be shown by invoking the trace embedding and (4).

Consider the least energy levels of $\Phi_{\mathcal{A}}$ on *X* and *Y*:

$$c_{\mathcal{A}} = \inf_{u\in X}\sum \Phi_{\mathcal{A}}\left(u_j\right) \text{ and } d_{\mathcal{A}} = \inf_{u\in Y}\sum \Phi_{\mathcal{A}}\left(u_j\right).$$

Theorem 1.1. The levels $c_{\mathcal{A}}$ and $d_{\mathcal{A}}$ are attained for any $\epsilon \geq 0$.

The next one covers critical cases.

Theorem 1.2. The levels $c_{\mathcal{A}}$ and $d_{\mathcal{A}}$ are attained for any $\epsilon \geq 0$, provided that $0 < c_{\mathcal{A}} < (2+\epsilon)\omega_{2+\epsilon}^{1/2+\epsilon}$ and $0 < d_{\mathcal{A}} < (2+\epsilon)\omega_{2+\epsilon}^{1/2+\epsilon}$, respectively.

As a result of its weak* closure in $BV_0(\Omega)$,, the logic used to prove Theorems 1.1 and 1.2 yields equivalent assertions on the space $BV(\Omega)$ (Proposition 3.2). To be more specific, when applied to functions with zero trace in $BV(\Omega)$, the functional $\Phi_{\mathcal{A}}$ calculated using (C2) yields

$$\sum \Phi_{\mathcal{A}}\left(u_j\right) = \sum \mathcal{E}_{\Omega}\left(u_j\right) + \int_{\Omega}\sum a\left|u_j\right|\mathrm{d}x.$$

Denote by $c_{\mathcal{A},0}$ and $d_{\mathcal{A},0}$ the respective least energy levels of $\Phi_{\mathcal{A}}$ on the sets $X_0 = X \cap BV_0(\Omega)$ and $Y_0 = Y \cap BV_0(\Omega)$.

Theorem 1.3. The levels $c_{\mathcal{A},0}$ and $d_{\mathcal{A},0}$ are attained for any $\epsilon \geq 0$.

Theorem 1.4. The levels $c_{\mathcal{A},0}$ and $d_{\mathcal{A},0}$ are attained for any $\epsilon \geq 0$, provided that $0 < c_{\mathcal{A},0} < (2+\epsilon)\omega_{2+\epsilon}^{1/2+\epsilon}$ and $0 < d_{\mathcal{A},0} < (2+\epsilon)\omega_{2+\epsilon}^{1/2+\epsilon}$, respectively.

The Sobolev-Zhang inequality on $BV\left(\mathbb{R}^{2+\epsilon}\right)$ yields the sharp affine variants of ($\mathcal{P}$) and ($\mathcal{PW}$) and also of ($\mathcal{PS}$) and ($\mathcal{PWS}$) for $\epsilon \geq 0$:

1) Inequality of Poincaré affineness ($\mathcal{AP}$):

A best constant may be found. $\lambda_1^{\mathcal{A}} > 0$ such that $\lambda_1^{\mathcal{A}} \left\|\sum u_j\right\|_{L^1(\Omega)} \leq \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right)$;

2) Inequality via the affine Poincaré-Wirtinger transform ($\mathcal{APW}$):

There exists an optimal constant $\mu_1^{\mathcal{A}} > 0$ such that

$\mu_1^{\mathcal{A}} \left\|\sum\left(u_j - \left(u_j\right)_\Omega\right)\right\|_{L^1(\Omega)} \leq \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right)$;

3) The Poincaré-Sobolev inequality ($\mathcal{APS}$):

A best constant may be found $\lambda_{1+\epsilon}^{\mathcal{A}} > 0$ such that

$\lambda_{1+\epsilon}^{\mathcal{A}} \left\|\sum u_j\right\|_{L^{1+\epsilon}(\Omega)} \leq \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right)$;

4) Affine Poincaré-Wirtinger-Sobolev inequality ($\mathcal{APWS}$):

A best constant may be found $\mu_{1+\epsilon}^{\mathcal{A}} > 0$ such that

$\mu_{1+\epsilon}^{\mathcal{A}} \left\|\sum\left(u_j - \left(u_j\right)_\Omega\right)\right\|_{L^{1+\epsilon}(\Omega)} \leq \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right)$.

It also deserves to be noticed that $\mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right)$ and $\left|Du_j\right|(\Omega)$ are incomparable via a one-way inequality in $BV(\Omega)$. In effect, since $\mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{\chi}_\Omega\right) = \mathcal{E}_{\partial\Omega}(1) > 0$ and $\left|D\chi_\Omega\right|(\Omega) = 0$, there is no constant $C > 0$ such that $\mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right) \leq C\left|Du_j\right|(\Omega)$ holds for all $u_j \in BV(\Omega)$. On the other hand, a reverse inequality also fails in view of the example of [38] in $BV_0(\Omega)$. Accordingly, ($\mathcal{APW}$) and ($\mathcal{APWS}$) seem to be natural affine counterparts of ($\mathcal{PW}$) and ($\mathcal{PWS}$), respectively.

Nonetheless, the term $\mathcal{E}_\Omega\left(u_j\right)$ appears on the right-hand side when we restrict ourselves to functions in $BV_0(\Omega)$. In this space, we denote the respective inequalities by ($\mathcal{AP}_0$), ($\mathcal{APW}_0$), ($\mathcal{APS}_0$) and ($\mathcal{APWS}_0$).

A direct application of Theorems 1.1 and 1.3 for $\epsilon = 0$ is as follows:

Theorem 1.5. The inequalities ($\mathcal{AP}$) and ($\mathcal{APW}$) and also ($\mathcal{APS}$) and ($\mathcal{APWS}$) with $\epsilon \geq 0$ admit extremal functions in $BV(\Omega)$. The same conclusion holds true in $BV_0(\Omega)$ for ($\mathcal{AP}_0$), ($\mathcal{APW}_0$), ($\mathcal{APS}_0$) and ($\mathcal{APWS}_0$).

Recent work has focused on finding extremal functions for local affine $L^{1+\epsilon}$ -Sobolev type inequalities, and to our knowledge, this topic has only been discussed in the context of functions with zero trace in the publications [38] and [39]. In particular, the first one provides extremals for the affine L2-Sobolev inequality on $W_0^{1,2}(\Omega)$,, while the second one provides extremals for the affine L(1+)-Poincaré inequality on $W_0^{1,1+\epsilon}(\Omega)$ for any $\epsilon > 0$ and on $BV_0(\Omega)$ for $\epsilon = 0$. For example, in [38], the authors provide a different demonstration of Theorem 1.5 for on $BV_0(\Omega)$ using an elegant method based on their Lemma 1 and Theorem 9.

In the critical case $\epsilon = 0$, one knows from (1) that characteristic functions of ellipsoids in Ω are extremals of ($\mathcal{APS}$), however, exist no extremal for ($\mathcal{APS}_0$). The usual argument of nonexistence consists in showing, by means of a standard rescaling, that the optimal constant corresponding to $BV_0(\Omega)$ is also $(2+\epsilon)\omega_{2+\epsilon}^{1/2+\epsilon}$. The key points are the strict continuity of $u_j \mapsto \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(u_j\right)$ on $BV\left(\mathbb{R}^{2+\epsilon}\right)$ (Theorem 4.4 of [19]) and the density of $BV_c^\infty\left(\mathbb{R}^{2+\epsilon}\right)$ in ($\mathbb{R}^{2+\epsilon}$) (Corollary 3.2 of [40]), where $BV_c^\infty\left(\mathbb{R}^{2+\epsilon}\right)$ denotes the space of bounded functions in $BV\left(\mathbb{R}^{2+\epsilon}\right)$ with compact support.

We close the introduction with an application of Theorems 1.1 and 1.3 for $\epsilon > 0$.

We point out that ($\mathcal{APW}$), ($\mathcal{APWS}$), ($\mathcal{APW}_0$) and ($\mathcal{APWS}_0$) are prototypes of more general affine inequalities depending on $(1+\epsilon)$ and $(1+2\epsilon)$. Precisely, for each $\epsilon \ge 0$, let $m_{1+2\epsilon} : L^{1+2\epsilon}(\Omega) \to \mathbb{R}$ be the unique function that satisfies

$$\int_\Omega \sum \left|u_j - m_{1+2\epsilon}(u_j)\right|^{2\epsilon}\left(u_j - m_{1+2\epsilon}(u_j)\right)\mathrm{d}x = 0$$

for all $u_j \in L^{1+2\epsilon}(\Omega)$. It is important to note that $m_{1+2\epsilon}$ is continuous, 1-homogeneous and bounded on bounded subsets of $L^{1+2\epsilon}(\Omega)$. Of course,
$m_1(u_j) = (u_j)_\Omega$ for $(1+2\epsilon) = 1$. The construction of $m_{1+2\epsilon}$ is canonical and makes use of basic results as the mean value theorem and dominated and monotone convergence theorems.

The properties satisfied by $m_{1+2\epsilon}$ together with (1) produce two new affine inequalities for $\epsilon \ge 0$ that extend ($\mathcal{APW}$), ($\mathcal{APWS}$), ($\mathcal{APW}_0$) and ($\mathcal{APWS}_0$).

1) Generalized affine [11] inequality ($\mathcal{GAPWS}$) on $BV(\Omega)$:

There exists an optimal constant $\mu^{\mathcal{A}}_{1+\epsilon,1+2\epsilon} > 0$ such that
$\mu^{\mathcal{A}}_{1+\epsilon,1+2\epsilon}\left\|\sum\left(u_j - m_{1+2\epsilon}(u_j)\right)\right\|_{L^{1+2\epsilon}(\Omega)} \le \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}(\bar{u}_j)$.

2) Generalized affine [11] inequality ($\mathcal{GAPWS}_0$) on $BV_0(\Omega)$:

There exists an optimal constant $\mu^{\mathcal{A},0}_{1+\epsilon,1+2\epsilon} > 0$ such that
$\mu^{\mathcal{A},0}_{1+\epsilon,1+2\epsilon}\left\|\sum\left(u_j - m_{1+2\epsilon}(u_j)\right)\right\|_{L^{1+\epsilon}(\Omega)}$ criteria that follow.

2. Background on the Space $BV(\Omega)$

We talk about some basic definitions and old results about functions with limited changes. Books [13] [20] [40] are good places to look for more information on the subject.

Let be a part of $\mathbb{R}^{2+\epsilon}$ that is open with $\epsilon \ge 0$. A function $u_j \in L^1(\Omega)$ is said to have bounded variation in if its distributional derivative is a Radon measure with a vector value. Du $(D_1 u_j, \cdots, D_{2+\epsilon} u_j)$ in Ω, that is, $D_i u_j$ is a Radon measure fulfilling

$$\int_\Omega \sum \varphi D_i u_j = -\int_\Omega \sum \frac{\partial \varphi}{\partial x_i} u_j \mathrm{d}x$$

for every $u_j \in C_0^\infty(\Omega)$. $BV(\Omega)$ stands for the vector space of all functions with bounded variation in u_j total dispersion is characterized by

$$\begin{aligned}\left|Du_j\right|(\Omega) &= \sup\left\{\sum_{i=1}^{2+\epsilon}\int_\Omega \sum \psi_i D_i u_j \mathrm{d}x : \psi = (\psi_1, \cdots, \psi_{2+\epsilon}) \in C_0^\infty(\Omega, \mathbb{R}^{2+\epsilon}), |\psi| \le 1\right\}\\ &= \sup\left\{-\int_\Omega \sum u_j \operatorname{div}\psi \mathrm{d}x : \psi \in C_0^\infty(\Omega, \mathbb{R}^{2+\epsilon}), |\psi| \le 1\right\},\end{aligned}$$

where $|\psi| = \left(\psi_1^2 + \cdots + \psi_{2+\epsilon}^2\right)^{1/2}$. The variation $\left|Du_j\right|$ is a positive Radon measure on Ω. Denote by σ_{u_j} the Radon Nikodym derivative of Du_j with respect to $\left|Du_j\right|$. Then, $\sigma_{u_j} : \Omega \to \mathbb{R}^{2+\epsilon}$ is a measurable field satisfying $\left|\sigma_{u_j}\right| = 1$ almost everywhere in Ω (w.r.t. $\left|Du_j\right|$) and $\mathrm{d}(Du_j) = \sigma_{u_j}\mathrm{d}(\left|Du_j\right|)$.

For $u_j \in BV(\Omega)$, the Lebesgue-Radon-Nikodym decomposition of the measure Du_j is given by

$$Du_j = \sum \nabla u_j \mathcal{L} + \sum \sigma^s_{u_j} \left|D^s u_j\right|,$$

where ∇u_j and $D^s u_j$ denote respectively the (density) absolutely continuous part and the singular part of Du_j with respect to the $(2+\epsilon)$-dimensional Lebesgue measure $\mathcal{L}$ and $\sigma^s_{u_j}$ is the Radon-Nikodym derivative of $D^s u_j$ with respect to its total variation measure $\left|D^s u_j\right|$. In particular,

$$\sum \left|Du_j\right| = \sum \left|\nabla u_j\right| \mathcal{L} + \sum \left|D^s u_j\right|.$$

The space $BV(\Omega)$ is Banach with respect to the norm

$$\sum \left\|u_j\right\|_{BV(\Omega)} = \sum \left\|u_j\right\|_{L^1(\Omega)} + \sum \left|Du_j\right|(\Omega),$$

however, it is neither separable nor reflexive.

The strict (intermediate) topology is induced by the metric

$$d\left(u_j, v_j\right) = \sum \left\|Du_j\right|(\Omega) - \left|Dv_j\right|(\Omega)\right| + \sum \left\|u_j - v_j\right\|_{L^1(\Omega)}.$$

The weak* topology, the weakest of the three ones, is quite appropriate for dealing with minimization problems. A sequence $\left(u_j\right)_k$ converges weakly* to u_j in $BV(\Omega)$, if $\sum \left(u_j\right)_k \to \sum u_j$ strongly in $L^1(\Omega)$ and $\sum D\left(u_j\right)_k \to \sum Du_j$ weakly in the measure sense, that is,

$$\int_\Omega \sum \varphi D\left(u_j\right)_k \to \int_\Omega \sum \varphi Du_j$$

for every $\varphi \in C_0^\infty(\Omega)$.

Let's pretend is a Lipschitz-bounded bounded open. Listed below are some of the most well-known characteristics that will be used later on:

1) Every $BV(\Omega)$ admits a weakly* convergent subsequence;

2) Every weakly* in $BV(\Omega)$ is bounded;

3) $BV(\Omega)$ is embedded continuously into $L^{1+\epsilon}(\Omega)$ for $\epsilon \ge 0$ and compactly for $\epsilon \ge 0$;

4) Each function $u_j \in BV(\Omega)$ admits a boundary trace $\tilde{u}_j$ in $L^1(\partial\Omega)$ and the trace operator $u_j \mapsto \tilde{u}_j$ is continuous on $BV(\Omega)$ with respect to the strict topology;

5) For any function $u_j \in BV(\Omega)$, its zero extension $\bar{u}_j$ outside of Ω belongs to $BV\left(\mathbb{R}^{2+\epsilon}\right)$;

6) $\sum \left\|u_j\right\|'_{BV(\Omega)} = \sum \left|Du_j\right|(\Omega) + \sum \left\|\tilde{u}_j\right\|_{L^1(\partial\Omega)}$ defines a norm on $BV(\Omega)$ equivalent to the usual norm $\left\|u_j\right\|_{BV(\Omega)}$;

7) $W^{1,1}\left(\mathbb{R}^{2+\epsilon}\right)$ is dense in $BV\left(\mathbb{R}^{2+\epsilon}\right)$ with respect to the strict topology.

3. Lower Weak* Semi Continuity of $\mathcal{E}_{\mathbb{R}^n}$

For an open subset $\Omega \subset \mathbb{R}^{2+\epsilon}$ and $u_j \in BV(\Omega)$, consider the affine BV energy

$$\mathcal{E}_\Omega\left(u_j\right) = \alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}} \sum \left(\int_\Omega \left|\sigma_{u_j}(x)\cdot\xi\right| \mathrm{d}\left(\left|Du_j\right|\right)(x)\right)^{-(2+\epsilon)} \mathrm{d}\xi\right)^{-1/2+\epsilon}.$$

We start by giving an answer to the question:

When is the affine energy $\mathcal{E}_\Omega(u_j)$ zero?

For each $\xi \in \mathbb{S}^{1+\epsilon}$, denote by Ψ_ξ the functional on $BV(\Omega)$,

$$\Psi_\xi(u_j) = \int_\Omega \sum \left|\sigma_{u_j}(x)\cdot\xi\right| \mathrm{d}\left(\left|Du_j\right|\right)(x).$$

Theorem 3.1. (See [14]) Let $u_j \in BV(\Omega)$. Then, $\sum \mathcal{E}_\Omega(u_j) = 0$ if, and only if, $\sum \Psi_{\tilde{\xi}}(u_j) = 0$ for some $\tilde{\xi} \in \mathbb{S}^{(2+\epsilon)-1}$.

Proof. The sufficiency is the easy part. In fact, assume that $\sum \Psi_\xi(u_j) > 0$ for all $\xi \in \mathbb{S}^{1+\epsilon}$. Thanks to the continuity of $\xi \in \mathbb{S}^{1+\epsilon} \mapsto \Psi_\xi(u_j)$, there exists a constant $c > 0$ so that $\sum \Psi_\xi(u_j) \ge c$ for all $\xi \in \mathbb{S}^{1+\epsilon}$. But this lower bound immediately yields $\sum \mathcal{E}_\Omega(u_j) \ge c\alpha_{2+\epsilon}\left((2+\epsilon)\omega_{2+\epsilon}\right)^{-1/2+\epsilon} > 0$.

Conversely, we prove that $\sum \mathcal{E}_\Omega(u_j) = 0$ whenever $\sum \Psi_{\tilde{\xi}}(u_j) = 0$ for some $\tilde{\xi} \in \mathbb{S}^{1+\epsilon}$. Let $m \in \mathbb{N}$ be the maximum number of linearly independent vectors $\xi \in \mathbb{S}^{1+\epsilon}$ such that $\sum \Psi_\xi(u_j) = 0$. If $m = (2+\epsilon)$, then clearly $\sum Du_j = 0$ in Ω and thus, by (4), we have $\sum \mathcal{E}_\Omega(u_j) = 0$. Else, choose an orthonormal basis $\{\xi_1, \cdots, \xi_{2+\epsilon}\}$ of $\mathbb{R}^{2+\epsilon}$ so that $\sum \Psi_{\xi_i}(u_j) = 0$ for $i = (2+\epsilon) - m + 1, \cdots, 2+\epsilon$, which correspond to the last m vectors of basis with $0 < m < (2+\epsilon)$.

For $x \in \Omega$ and $\xi \in \mathbb{S}^{1+\epsilon}$, write

$$\sigma_{u_j}(x) = \sigma_1(x)\xi_1 + \cdots + \sigma_{2+\epsilon}(x)\xi_{2+\epsilon} \text{ and } \xi = a_1\xi_1 + \cdots + a_{2+\epsilon}\xi_{2+\epsilon}.$$

The condition $\sum \Psi_{\xi_i}(u_j) = 0$ implies that $\sigma_i(x) = 0$ for $i = (2+\epsilon) - m + 1, \cdots, (2+\epsilon)$. So, the Cauchy-Schwarz inequality gives

$$\left|\sigma_{u_j}(x)\cdot\xi\right| = \left|\sigma_1(x)a_1 + \cdots + \sigma_{(2+\epsilon)-m}(x)a_{(2+\epsilon)-m}\right| \le \left(a_1^2 + \cdots + a_{(2+\epsilon)-m}^2\right)^{1/2}.$$

Set $a(\xi) = \left(a_1, \cdots, a_{(2+\epsilon)-m}\right)$ and $a'(\xi) = \left(a_{(2+\epsilon)-m+1}, \cdots, a_{2+\epsilon}\right)$. Since $0 < m < (2+\epsilon)$, we get

$$\begin{aligned}
&\int_{\mathbb{S}^{1+\epsilon}} \sum \left(\int_\Omega \left|\sigma_{u_j}(x)\cdot\xi\right| \mathrm{d}\left(\left|Du_j\right|\right)(x)\right)^{-(2+\epsilon)} \mathrm{d}\xi \\
&\ge \sum \left|Du_j\right|(\Omega)^{-(2+\epsilon)} \int_{\mathbb{S}^{1+\epsilon}} \left|a(\xi)\right|^{-(2+\epsilon)} \mathrm{d}\xi \\
&\ge \sum \left|Du_j\right|(\Omega)^{-(2+\epsilon)} \int_{|a(\xi)| \le \sqrt{3}/2} \left|a(\xi)\right|^{-(2+\epsilon)} \mathrm{d}\xi \\
&\ge \sum \frac{m\omega_m}{2^{m-1}} \left|Du_j\right|(\Omega)^{-(2+\epsilon)} \int_{|a(\xi)| \le \sqrt{3}/2} \left|a(\xi)\right|^{-(2+\epsilon)} \mathrm{d}a(\xi) \\
&= \sum \left((2+\epsilon) - m\right)\omega_{(2+\epsilon)-m} \frac{m\omega_m}{2^{m-1}} \left|Du_j\right|(\Omega)^{-(2+\epsilon)} \int_0^{\sqrt{3}/2} \rho^{-m-1} \mathrm{d}\rho \\
&= \infty,
\end{aligned}$$

and hence $\sum \mathcal{E}_\Omega(u_j) = 0$.

An interesting application of Theorem 3.1 of independent interest, is (see [14])

Corollary 3.1. Let $\Omega \subset \mathbb{R}^{2+\epsilon}$ be a bounded open with Lipschitz boundary. Then,

$$\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}(\bar{u}_j) \ge \sum \mathcal{E}_\Omega(u_j) + \sum \mathcal{E}_{\partial\Omega}(\tilde{u}_j)$$

for all $u_j \in BV(\Omega)$ with $\tilde{u}_j \neq 0$ on $\partial\Omega$ (a.e.) or without any restriction in case $\partial\Omega$ is non-flat, where the definitions of $\sum \mathcal{E}_{\partial\Omega}(\tilde{u}_j)$ and non-flat boundary were given in the comparison (C3) of the introduction.

Proof. Firstly, the identity

$$\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}(\bar{u}_j) = \alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}} \sum \left(\int_\Omega \left|\sigma_{u_j}(x)\cdot\xi\right| \mathrm{d}\left(\left|Du_j\right|\right)(x) + \int_{\partial\Omega} \sum \left|\tilde{u}_j(x)\right|\left|\nu_j(x)\cdot\xi\right| \mathrm{d}\mathcal{H}^{1+\epsilon}(x)\right)^{-(2+\epsilon)} \mathrm{d}\xi\right)^{-1/2+\epsilon}$$

gives $\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}(\tilde{u}_j) \geq \sum \mathcal{E}_\Omega(u_j \nu_j)$ and $\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}(\bar{u}_j) \geq \sum \mathcal{E}_{\partial\Omega}(\tilde{u}_j)$. Therefore, if $\sum \mathcal{E}_\Omega(u_j) = 0$ or $\sum \mathcal{E}_{\partial\Omega}(\tilde{u}_j) = 0$, the conclusion follows.

Assume that $\mathcal{E}_\Omega(u_j)$ and $\mathcal{E}_{\partial\Omega}(\tilde{u}_j)$ are nonzero. Set $\sum g_j(\xi) = \sum \Psi_\xi(u_j)$ and $\sum \tilde{g}_j(\xi) = \sum \tilde{\Psi}_\xi(u_j)$, where

$$\sum \tilde{\Psi}_\xi(u_j) = \int_{\partial\Omega} \sum \left|\tilde{u}_j(x)\right|\left|\nu_j(x)\cdot\xi\right| \mathrm{d}\mathcal{H}^{1+\epsilon}(x).$$

By Theorem 3.1 we have $g_j(\xi) > 0$ for all $\xi \in \mathbb{S}^{1+\epsilon}$. If $\mathcal{E}_{\partial\Omega}(\tilde{u}_j) \neq 0$ and the assuming the statement imply that $\tilde{g}_j(\xi) > 0$ for all $\xi \in \mathbb{S}^{1+\epsilon}$. Then, we get

$$\begin{aligned}
\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(\bar{u}_j) &= \alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}} \sum \left(g_j(\xi) + \tilde{g}_j(\xi)\right)^{-(2+\epsilon)} \mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}} \\
&\geq \alpha_{2+\epsilon}\left(\int_{1+\epsilon} \sum g_j(\xi)^{-2+\epsilon} \mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}} + \alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}} \sum \tilde{g}_j(\xi)^{-(2+\epsilon)} \mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}} \\
&= \sum \mathcal{E}_\Omega(u_j) + \sum \mathcal{E}_{\partial\Omega}(\tilde{u}_j)
\end{aligned}$$

The next step is to prove that of $\mathcal{E}_{\mathbb{R}^{2+\epsilon}}$ on $L^1_{loc}(\mathbb{R}^{2+\epsilon})$ is weak* continuous below uniform bounds on the total variation. Outside of Theorem 3.1, the proof relies on pivotal conclusions by Goffman and Serrin (Theorems 2 and 3 of [39]). For different enhancements and expansions of [41], we also refer to [42] and [43], as well as references therein.

Let $f_j : \mathbb{R}^{2+\epsilon} \to \mathbb{R}$ be a nonnegative convex function with linear growth, that is, $f_j(w) \leq M(|w|+1)$ for all $w \in \mathbb{R}^{2+\epsilon}$, where $M > 0$ is a constant. Define the recession function $(f_j)_\infty : \mathbb{R}^{2+\epsilon} \to \mathbb{R}$ associated to f_j by

$$(f_j)_\infty(w) = \limsup_{t\to\infty} \sum \frac{f_j(tw)}{t}.$$

For $u_j \in BV(\mathbb{R}^{2+\epsilon})$, write $Du_j = \sum \nabla u_j \mathcal{L} + \sum \sigma^s_{u_j}\left|D^s u_j\right|$ and let $\Psi : BV(\mathbb{R}^{2+\epsilon}) \to \mathbb{R}$ defined by

$$\Psi(u_j) = \int_{\mathbb{R}^{2+\epsilon}} \sum f_j(\nabla u_j(x))\mathrm{d}x + \int_{\mathbb{R}^{2+\epsilon}} \sum (f_j)_\infty\left(\sigma^s_{u_j}(x)\right)\mathrm{d}\left(\left|D^s u_j\right|\right)(x).$$

Proposition 3.1

The functional Ψ is strongly lower semicontinuous on $L^1_{loc}(\mathbb{R}^{2+\epsilon})$.

Theorem 3.2. (See [14]) If $(u_j)_k \to (u_j)_0$ strongly in $L^1_{loc}(\mathbb{R}^{2+\epsilon})$ and $\left|D(u_j)_k\right|(\mathbb{R}^{2+\epsilon})$ is bounded, then

$$\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left((u_j)_0\right) \leq \liminf_{k\to\infty} \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left((u_j)_k\right).$$

Proof. Let $(u_j)_k$ be a sequence converging strongly to $(u_j)_0$ in $L^1_{loc}(\mathbb{R}^{2+\epsilon})$ such that $|D(u_j)_k|(\mathbb{R}^{2+\epsilon})$ is bounded. If $\sum \Psi_{\tilde{\xi}}((u_j)_0)=0$ for some $\tilde{\xi}\in\mathbb{S}^{1+\epsilon}$, by Theorem 3.1, we have $\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}((u_j)_0)=0$ and the conclusion follows trivially.

It then suffices to assume that $\sum \Psi_{\xi}((u_j)_0)>0$ for all $\xi\in\mathbb{S}^{1+\epsilon}$. Set $f_j^{\xi}(w)=|w\cdot\xi|$ for any $\xi\in\mathbb{S}^{1+\epsilon}$. Since f_j^{ξ} is convex, nonnegative, 1-homogeneous and $\sum (f_j)_\infty^{\xi}=\sum f_j^{\xi}$, we have

$$\begin{aligned}
\Psi_{\xi}(u_j) &= \int_{\mathbb{R}^{2+\epsilon}} \sum \left|\sigma_{u_j}(x)\cdot\xi\right| \mathrm{d}\left(|Du_j|\right)(x)\\
&= \int_{\mathbb{R}^{2+\epsilon}} \sum f_j^{\xi}\left(\frac{\nabla u_j(x)}{|\nabla u_j(x)|}\right)|\nabla u_j(x)|\mathrm{d}x + \int_{\mathbb{R}^{2+\epsilon}} \sum f_j^{\xi}\left(\sigma_{u_j}^{s}(x)\right)\mathrm{d}\left(|D^s u_j|\right)(x)\\
&= \int_{\mathbb{R}^{2+\epsilon}} \sum f_j^{\xi}(\nabla u_j(x))\mathrm{d}x + \int_{\mathbb{R}^{2+\epsilon}} \sum (f_j)_\infty^{\xi}\left(\sigma_{u_j}^{s}(x)\right)\mathrm{d}\left(|D^s u_j|\right)(x).
\end{aligned}$$

Hence, by Proposition 3.1 Ψ_{ξ} is strongly lower semicontinuous on $L^1_{loc}(\mathbb{R}^{2+\epsilon})$, and so

$$\begin{aligned}
&\int_{\mathbb{R}^{2+\epsilon}} \sum \left|\sigma_{(u_j)_0}(x)\cdot\xi\right| \mathrm{d}\left(|D(u_j)_0|\right)(x)\\
&\le \liminf_{k\to\infty} \int_{\mathbb{R}^{2+\epsilon}} \sum \left|\sigma_{(u_j)_k}(x)\cdot\xi\right| \mathrm{d}\left(|D(u_j)_k|\right)(x).
\end{aligned} \tag{6}$$

We now ensure the existence of a constant $c_0>0$ and an integer $k_0\in\mathbb{N}$, both independent of $\xi\in\mathbb{S}^{1+\epsilon}$, such that, for any $k\ge k_0$,

$$\int_{\mathbb{R}^{2+\epsilon}} \sum \left|\sigma_{(u_j)_k}(x)\cdot\xi\right| \mathrm{d}\left(|D(u_j)_k|\right)(x) \ge c_0. \tag{7}$$

Otherwise, module a renaming of indexes, we get a sequence $\xi_k\in\mathbb{S}^{1+\epsilon}$ such that $\xi_k\to\tilde{\xi}$ and

$$\int_{\mathbb{R}^{2+\epsilon}} \sum \left|\sigma_{(u_j)_k}(x)\cdot\xi_k\right| \mathrm{d}\left(|D(u_j)_k|\right)(x) \le \frac{1}{k}.$$

Using the assumption that $|D(u_j)_k|(\mathbb{R}^{2+\epsilon})$ is bounded, we find a constant $\epsilon\ge 0$ such that

$$\int_{\mathbb{R}^{2+\epsilon}} \sum \left|\sigma_{(u_j)_k}(x)\cdot\tilde{\xi}\right| \mathrm{d}\left(|D(u_j)_k|\right)(x) \le C_1\left\|\xi_k-\tilde{\xi}\right\| + \frac{1}{k} \to 0.$$

Then, by (6), we get $\sum \Psi_{\tilde{\xi}}((u_j)_0)=0$.

Finally, combining (6), (7) and Fatou's lemma, we derive

$$\begin{aligned}
&\int_{\mathbb{S}^{1+\epsilon}} \sum \left(\int_{\mathbb{R}^{2+\epsilon}} \left|\sigma_{(u_j)_0}(x)\cdot\xi\right| \mathrm{d}\left(|D(u_j)_0|\right)(x)\right)^{-2+\epsilon} \mathrm{d}\xi\\
&\ge \int_{\mathbb{S}^{1+\epsilon}} \limsup_{k\to\infty} \sum \left(\int_{\mathbb{R}^{2+\epsilon}} \left|\sigma_{(u_j)_k}(x)\cdot\xi\right| \mathrm{d}\left(|D(u_j)_k|\right)(x)\right)^{-(2+\epsilon)} \mathrm{d}\xi\\
&\ge \limsup_{k\to\infty} \sum \int_{\mathbb{S}^{1+\epsilon}} \left(\int_{\mathbb{R}^{2+\epsilon}} \left|\sigma_{(u_j)_k}(x)\cdot\xi\right| \mathrm{d}\left(|D(u_j)_k|\right)(x)\right)^{-(2+\epsilon)} \mathrm{d}\xi,
\end{aligned}$$

and thus

$$\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\left(u_j\right)_0\right) = \left(\int_{\mathbb{S}^{1+\epsilon}} \sum \left(\int_{\mathbb{R}^{2+\epsilon}} \left|\sigma_{\left(u_j\right)_0}(x)\cdot\xi\right| \mathrm{d}\left(\left|D\left(u_j\right)_0\right|\right)(x)\right)^{-(2+\epsilon)} \mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}}$$

$$\le \liminf_{k\to\infty} \sum \left(\int_{\mathbb{S}^{1+\epsilon}} \left(\int_{\mathbb{R}^{2+\epsilon}} \left|\sigma_{\left(u_j\right)_k}(x)\cdot\xi\right| \mathrm{d}\left(\left|D\left(u_j\right)_k\right|\right)(x)\right)^{-(2+\epsilon)} \mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}}$$

$$= \liminf_{k\to\infty} \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\left(u_j\right)_k\right).$$

As an immediate consequence of Theorem 3.2 we have:

Corollary 3.2. If $\sum\left(u_j\right)_k \to \sum\left(u_j\right)_0$ weakly* in $BV(\Omega)$, then

$$\sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\left(\bar{u}_j\right)_0\right) \le \liminf_{k\to\infty} \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\left(\bar{u}_j\right)_k\right).$$

This result is the key point towards the lower weak* semicontinuity of the functional $\Phi_{\mathcal{A}} : BV(\Omega) \to \mathbb{R}$. We recall that

$$\Phi_{\mathcal{A}}\left(u_j\right) = \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right) + \int_\Omega \sum a\left|u_j\right| \mathrm{d}x + \int_{\partial\Omega} \sum b\left|\tilde{u}_j\right| \mathrm{d}\mathcal{H}^{1+\epsilon},$$

where $a \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega) \ge 0$. Since the integral functional on Ω is clearly weakly* continuous on $BV(\Omega)$, it only remains to discuss the semicontinuity of the boundary integral term.

Proposition 3.2. (See [44]) If $\sum\left(u_j\right)_k \to \sum\left(u_j\right)_0$ weakly* in $BV(\Omega)$, then

$$\int_{\partial\Omega} \sum b\left|\left(\tilde{u}_j\right)_0\right| \mathrm{d}\mathcal{H}^{1+\epsilon} \le \liminf_{k\to\infty} \int_{\partial\Omega} \sum b\left|\left(\tilde{u}_j\right)_k\right| \mathrm{d}\mathcal{H}^{1+\epsilon}.$$

Proof. Let $\left(u_j\right)_k$ be a sequence converging weakly* to $\left(u_j\right)_0$ in $BV(\Omega)$. For each $\varepsilon > 0$, we consider the norm $\|\cdot\|_\varepsilon$ on $BV(\Omega)$

$$\left\|u_j\right\|_\varepsilon = \varepsilon \sum \left|Du_j\right|(\Omega) + \int_{\partial\Omega} \sum (b+\varepsilon)\left|\tilde{u}_j\right| \mathrm{d}\mathcal{H}^{1+\epsilon}.$$

Since ≥ 0, $\|\cdot\|_\varepsilon$ is equivalent to $\|\cdot\|_{BV(\Omega)}$ and $\|\cdot\|'_{BV(\Omega)}$, and so

$$\left\|\left(u_j\right)_0\right\|_\varepsilon \le \liminf_{k\to\infty} \sum \left\|\left(u_j\right)_k\right\|_\varepsilon. \tag{8}$$

Take a constant $\epsilon \ge 0$ so that $\left\|\left(u_j\right)_k\right\|'_{BV(\Omega)} \le C$ and a subsequence $\left(u_j\right)_{k_j}$ such that

$$\lim_{j\to\infty} \int_{\partial\Omega} \sum b\left|\left(\tilde{u}_j\right)_{k_j}\right| \mathrm{d}\mathcal{H}^{1+\epsilon} = \liminf_{k\to\infty} \int_{\partial\Omega} \sum b\left|\left(\tilde{u}_j\right)_k\right| \mathrm{d}\mathcal{H}^{1+\epsilon}.$$

By (8), for j large, we get

$$\varepsilon \sum \left|D\left(u_j\right)_0\right|(\Omega) + \int_{\partial\Omega} \sum (b+\varepsilon)\left|\left(\tilde{u}_j\right)_0\right| \mathrm{d}\mathcal{H}^{1+\epsilon} - \varepsilon$$
$$\le C\varepsilon + \int_{\partial\Omega} \sum b\left|\left(\tilde{u}_j\right)_{k_j}\right| \mathrm{d}\mathcal{H}^{1+\epsilon}.$$

Letting $j \to \infty$ and after $\varepsilon \to 0$, the statement follows as wished.

Finally, Corollary 3.2 and Proposition 3.2 lead to

Corollary 3.3. The functional $\Phi_{\mathcal{A}}$ is lower weakly* semicontinuous on $BV(\Omega)$.

4. Subcritical Minimizations with Constraints on *BV*(Ω)

Theorems 1.1 and 1.3 are proved. Corollary 3.3 and the Rellich Kondrachov compactness theorem are key:

Theorem 4.1. The affine ball $B_{\mathcal{A}}(\Omega)$ is compact in $L^{1+\epsilon}(\Omega)$ for any $\epsilon \geq 0$.

Proof requires two preliminary outcomes. First, weak* convergence of displacements of limited sequences in $BV(\mathbb{R}^{2+\epsilon})$ and strong convergence in $L^{1+\epsilon}(\mathbb{R}^{2+\epsilon})$. In other spaces, embedding co-compactness has been extensively investigated [45] [46] [47]. This proves the completeness.

Proposition 4.1. (See [14]) Let $(u_j)_k$ be a bounded sequence in $BV(\mathbb{R}^{2+\epsilon})$. Then, $\sum(u_j)_k(\cdot-(y_j)_k) \to 0$ locally weakly* in $BV(\mathbb{R}^{2+\epsilon})$ for any sequence $(y_j)_k$ in $\mathbb{R}^{2+\epsilon}$ if, and only if, $(u_j)_k \to 0$ strongly in $L^{1+\epsilon}(\mathbb{R}^{2+\epsilon})$ for any $\epsilon > 0$.

Proof. Assume first that $\sum(u_j)_k \to 0$ strongly in $L^{1+\epsilon}(\mathbb{R}^{2+\epsilon})$ for some $\epsilon > 0$. If $\sum(v_j)_k = \sum(u_j)_k(\cdot-(y_j)_k)$ doesn't converge locally weakly* to zero in $BV(\mathbb{R}^{2+\epsilon})$ for some sequence $(y_j)_k$ in $\mathbb{R}^{2+\epsilon}$, then there is a bounded open subset Ω of $\mathbb{R}^{2+\epsilon}$ and $\epsilon > 0$ such that, module a subsequence,

$\sum\|(v_j)_k\|_{L^1(\Omega)} \geq \varepsilon$ or $\sum|\mathrm{d}(v_j)_k(\varphi)| \geq \varepsilon$ for some $\varphi \in C_0^\infty(\Omega)$, where

$\mathrm{d}v_j(\varphi) = \int_\Omega \sum \varphi \mathrm{d}v_j$. Since $(v_j)_k$ is bounded in $BV(\mathbb{R}^{2+\epsilon})$, one may assume that $\sum(v_j)_k \to \sum v_j$ weakly* in $BV(\Omega)$. Thus, letting $k \to \infty$ in the two cases, one gets $\sum\|v_j\|_{L^1(\Omega)} \geq \varepsilon$ or $|\mathrm{d}v_j(\varphi)| \geq \varepsilon$. On the other hand, one knows that $\sum(v_j)_k \to 0$ strongly in $L^{1+\epsilon}(\mathbb{R}^{2+\epsilon})$ and $\sum(v_j)_k \to \sum v_j$ strongly in $L^1(\Omega)$, so $\sum v_j = 0$ in Ω. But this contradicts the last two inequalities.

Conversely, assume that $\sum(u_j)_k(\cdot-(y_j)_k) \to 0$ locally weakly* in $BV(\mathbb{R}^{2+\epsilon})$ for any sequence $(y_j)_k$ in $\mathbb{R}^{2+\epsilon}$. Choose a fixed $\epsilon > 0$ and consider the $(2+\epsilon)$-cube $Q = (0,1)^{2+\epsilon}$. Using the continuity of the Sobolev immersion $BV(Q) \hookrightarrow L^{1+\epsilon}(Q)$, we deduce that

$$\begin{aligned}
\int_{Q+y} \sum|(u_j)_k|^{1+\epsilon} \mathrm{d}x &= \int_Q \sum|(u_j)_k(x-y_j)|^{1+\epsilon} \mathrm{d}x \\
&\leq C\sum\|(u_j)_k(\cdot-y_j)\|_{BV(Q)}\left(\int_Q |(u_j)_k(x-y_j)|^{1+\epsilon} \mathrm{d}x\right)^{\frac{\epsilon}{1+\epsilon}} \\
&= C\sum\|(u_j)_k\|_{BV(Q+y_j)}\left(\int_Q |(u_j)_k(x-y_j)|^{1+\epsilon} \mathrm{d}x\right)^{\frac{\epsilon}{1+\epsilon}}
\end{aligned}$$

for every $y_j \in \mathbb{R}^{2+\epsilon}$, where C is a constant independent of y_j.

By adding the inequality over $y_j \in \mathbb{Z}^{2+\epsilon}$, we obtain

$$\int_{\mathbb{R}^{2+\epsilon}} \sum|(u_j)_k|^{1+\epsilon} \mathrm{d}x \leq C\sum\|(u_j)_k\|_{BV(\mathbb{R}^{2+\epsilon})} \sup_{y_j \in \mathbb{Z}^{2+\epsilon}} \left(\int_Q |(u_j)_k(x-y_j)|^{1+\epsilon} \mathrm{d}x\right)^{\frac{\epsilon}{1+\epsilon}}. \quad (9)$$

Right-hand side of (9) is finite. $BV(\mathbb{R}^{2+\epsilon})$, bounds u k, hence it also bounds $L^1(\mathbb{R}^{2+\epsilon})$ and in $L^{1+\epsilon}(\mathbb{R}^{2+\epsilon})$ by Sobolev inequality. Then, $\epsilon > 0$ and a simple interpolation.

Choose $(y_j)_k \in \mathbb{Z}^{2+\epsilon}$ so that

$$\left(\int_Q \sum|(u_j)_k(x-(y_j)_k)|^{1+\epsilon} \mathrm{d}x\right)^{\frac{\epsilon}{1+\epsilon}} \geq \frac{1}{2} \sup_{y_j \in \mathbb{Z}^{2+\epsilon}} \sum\left(\int_Q |(u_j)_k(x-y_j)|^{1+\epsilon} \mathrm{d}x\right)^{\frac{\epsilon}{1+\epsilon}}.$$

Hence, (9) gives

$$\int_{\mathbb{R}^{2+\epsilon}} \sum \left|\left(u_j\right)_k\right|^{1+\epsilon} \mathrm{d}x \le 2C_1 \sum \left(\int_Q \left|\left(u_j\right)_k \left(x-\left(y_j\right)_k\right)\right|^{1+\epsilon} \mathrm{d}x\right)^{\frac{\epsilon}{1+\epsilon}} \tag{10}$$

for some constant C_1 independent of *k*.

However, the rigorous condition $\epsilon > 0$ lets us apply the Rellich-Kondrachov compactness theorem to the embedding $BV(Q) \hookrightarrow L^{1+\epsilon}(Q)$ In order to make a rough calculation of the right side of (10). In fact, module a subsequence, we have $\sum\left(v_j\right)_k = \sum\left(u_j\right)_k\left(\cdot-\left(y_j\right)_k\right) \to \sum v_j$ strongly in $L^{1+\epsilon}(Q)$. But, by assumption, $\sum\left(v_j\right)_k \to 0$ locally weakly* in $BV\left(\mathbb{R}^{2+\epsilon}\right)$, and so $\left(v_j\right)_k \to 0$ strongly in $L^1(Q)$. Therefore, $\sum v_j = 0$ in Q and, since $\epsilon = 0$, we deduce from (10) that $\left(u_j\right)_k \to 0$ strongly in $L^{1+\epsilon}\left(\mathbb{R}^{2+\epsilon}\right)$.

As noted, exist no upper bound for $\left|Du_j\right|\left(\mathbb{R}^{2+\epsilon}\right)$ in terms of $\mathcal{E}_{\mathbb{R}^{2+\epsilon}} u_j$ on $BV\left(\mathbb{R}^{2+\epsilon}\right)$.

However, Huang and Li (Theorem 1.2 of [48]) established that such an estimate holds true for functions $u_j \in W^{1,1}\left(\mathbb{R}^{2+\epsilon}\right)$ unless an acceptable affine transformation T depends on u_j. Wang's [49] tools allow the finding to apply to $BV\left(\mathbb{R}^{2+\epsilon}\right)$.

Proposition 4.2 (Huang-Li Theorem). For any $u_j \in BV\left(\mathbb{R}^{2+\epsilon}\right)$, one has

$$d_0 \min_{T \in SL(2+\epsilon)} \left|\sum D\left(u_j \circ T\right)\right|\left(\mathbb{R}^{2+\epsilon}\right) \le \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}} u_j,$$

where $d_0 = 4^{-1}\pi\Gamma\left(\frac{3+\epsilon}{2}\right)\Gamma(3+\epsilon)^{\frac{1}{2+\epsilon}}\Gamma\left(\frac{4+\epsilon}{2}\right)^{-\left(\frac{1}{2+\epsilon}\right)-1}$.

Proof of Theorem 4.1. (See [44]) Let $\left(u_j\right)_k$ be a sequence in $B_{\mathcal{A}}(\Omega)$. By Proposition 4.2 there is a matrix $T_k \in SL(2+\epsilon)$ such that $d_0\left|\sum D\left(\left(\bar{u}_j\right)_k \circ T_k\right)\right|\left(\mathbb{R}^{2+\epsilon}\right) \le \sum \mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right)_k$. Note also that $\sum\left\|\left(\bar{u}_j\right)_k \circ T_k\right\|_{L^1\left(\mathbb{R}^{2+\epsilon}\right)} = \sum\left\|\left(u_j\right)_k\right\|_{L^1(\Omega)}$, so $\sum\left(v_j\right)_k = \sum\left(\bar{u}_j\right)_k \circ T_k$ is bounded in $BV\left(\mathbb{R}^{2+\epsilon}\right)$. We now analyze two possibilities.

Assume first that $\left|T_k\right| \to \infty$. Let $\left(y_j\right)_k$ be an arbitrary sequence in $\mathbb{R}^{2+\epsilon}$. The boundedness of $\sum\left(v_j\right)_k\left(\cdot-\left(y_j\right)_k\right)$ in $BV\left(\mathbb{R}^{2+\epsilon}\right)$ implies, module a subsequence, that $\sum\left(v_j\right)_k\left(\cdot-\left(y_j\right)_k\right) \to \sum \bar{v}_j$ locally weakly* in $BV\left(\mathbb{R}^{2+\epsilon}\right)$. Since $1+\epsilon < \frac{2+\epsilon}{1+\epsilon}$, the Rellich Kondrachov compactness theorem also gives $\sum\left(v_j\right)_k\left(\cdot-\left(y_j\right)_k\right) \to \sum \bar{v}_j$ strongly in $L^{1+\epsilon}_{loc}\left(\mathbb{R}^{2+\epsilon}\right)$ and $\sum\left(v_j\right)_k\left(x-\left(y_j\right)_k\right) \to \sum \bar{v}_j(x)$ almost everywhere in $\mathbb{R}^{2+\epsilon}$, up to a subsequence. Consider the set

$$X = \liminf \sum T_k^{-1}\left(\Omega + T_k\left(\left(y_j\right)_k\right)\right) = \sum \bigcup_{m \ge 1} \bigcap_{k \ge m} T_k^{-1}\left(\Omega + T_k\left(\left(y_j\right)_k\right)\right).$$

Since $\left|T_k\right| \to \infty$ and Ω is bounded, X has zero Lebesgue measure (e.g. page 7 of [46]). For $x \notin X$, we have $x \notin \bigcap_{k \geq m} T_k^{-1}\left(\Omega + T_k\left(\left(y_j\right)_k\right)\right)$ for any $m \geq 1$, which yields $T_k\left(x - \left(y_j\right)_k\right) \notin \Omega$ for every *k*, up to a subsequence. Thus, $\sum \bar{v}_j(x) = \lim_{k\to\infty} \sum \left(v_j\right)_k \left(x - \left(y_j\right)_k\right) = \lim_{k\to\infty} \sum \left(\bar{u}_j\right)_k \left(T_k\left(x - \left(y_j\right)_k\right)\right) = 0$ and hence $\sum \left(v_j\right)_k \left(\cdot - \left(y_j\right)_k\right) \to 0$ locally weakly in $BV\left(\mathbb{R}^{2+\epsilon}\right)$ for any sequence $\left(y_j\right)_k$ in $\mathbb{R}^{2+\epsilon}$. By Proposition 4.1, $\sum \left(\bar{u}_j\right)_k \to 0$ strongly in $L^{1+\epsilon}\left(\mathbb{R}^{2+\epsilon}\right)$ and so $\left(u_j\right)_k \to 0$ strongly in $L^{1+\epsilon}(\Omega)$.

If $\left|T_k\right| \nrightarrow \infty$, then one may assume that T_k converges to some $T \in SL(2+\epsilon)$. Choose $R > 0$ large enough so that $T^{-1}(\Omega) \subset B_R$ and $T_k^{-1}(\Omega) \subset B_R$ for every *k*. Module a subsequence, we know that $\sum \left(v_j\right)_k \to \sum \left(v_j\right)_0$ weakly* in $BV\left(B_R\right)$ and $\left(v_j\right)_k \to \left(v_j\right)_0$ strongly in $L^{1+\epsilon}\left(B_R\right)$.

Set $\sum \left(u_j\right)_0 = \sum \left(v_j\right)_0 \circ T^{-1}$ in Ω. Notice that $u_0 \in BV(\Omega)$ once $T^{-1}(\Omega) \subset B_R$. Let $\left(\bar{u}_j\right)_0 \in BV\left(\mathbb{R}^{2+\epsilon}\right)$ be the extension of $\left(u_j\right)_0$ by zero outside of Ω. Since $T \circ T_k^{-1}$ converges to the identity *I*, by the generalized dominated convergence theorem, it follows that $\sum \left\|\left(\bar{u}_j\right)_0 \circ T \circ T_k^{-1} - \left(u_j\right)_0\right\|_{L^{1+\epsilon}(\Omega)} \to 0$. Consequently, since $T_k^{-1}(\Omega) \subset B_R$, we have

$$
\begin{aligned}
&\left\|\sum\left(\left(u_j\right)_k - \left(u_j\right)_0\right)\right\|_{L^{1+\epsilon}(\Omega)} \\
&\leq \sum \left\|\left(v_j\right)_k \circ T_k^{-1} - \left(v_j\right)_0 \circ T_k^{-1}\right\|_{L^{1+\epsilon}(\Omega)} + \sum \left\|\left(\bar{u}_j\right)_0 \circ T \circ T_k^{-1} - \left(u_j\right)_0\right\|_{L^{1+\epsilon}(\Omega)} \\
&\leq \sum \left\|\left(v_j\right)_k - \left(v_j\right)_0\right\|_{L^{1+\epsilon}(B_R)} + \sum \left\|\left(\bar{u}_j\right)_0 \circ T \circ T_k^{-1} - \left(u_j\right)_0\right\|_{L^{1+\epsilon}(\Omega)} \to 0.
\end{aligned}
$$

A fact that follows from the proof and deserves to be highlighted is

Corollary 4.1. Let $\left(u_j\right)_k$ be a sequence in $B_{\mathcal{A}}(\Omega)$ such that $\sum \left(u_j\right)_k \to \sum \left(u_j\right)_0$ strongly in $L^{1+\epsilon}(\Omega)$ for some $\epsilon \geq 0$. If $\sum \left(u_j\right)_0 \neq 0$, then $\left(u_j\right)_k$ is bounded in $BV(\Omega)$.

Proof of Theorem 1.1. (See [13]) Let $\left(u_j\right)_k$, $\Phi_{\mathcal{A}}$ in *X*. By Hölder's inequality, $\left(u_j\right)_k$ is bounded in $L^1(\Omega)$, $b \geq 0$ on $\partial\Omega$, the affine energy $\mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\bar{u}_j\right)_k$ is also bounded. Therefore, by Theorem 4.1 there exists $\left(u_j\right)_0 \in BV(\Omega)$ such that $\sum \left(u_j\right)_k \to \sum \left(u_j\right)_0$ strongly in $L^{1+\epsilon}(\Omega)$. Therefore, $\left(u_j\right)_0 \in X$ and, by Corollary 4.1, $\left(u_j\right)_k$ is bounded in $BV(\Omega)$.

Passing to a subsequence, if necessary, one may assume that $\sum \left(u_j\right)_k \to \sum \left(u_j\right)_0$ weakly* in $BV(\Omega)$. Then, by Corollary 3.3 we derive

$$
\sum \Phi_{\mathcal{A}}\left(\left(u_j\right)_0\right) \leq \liminf_{k\to\infty} \sum \Phi_{\mathcal{A}}\left(\left(u_j\right)_k\right) = c_{\mathcal{A}},
$$

and thus $\left(u_j\right)_0$ minimizes $\Phi_{\mathcal{A}}$ in *X*.

The same argument also works for a minimizing sequence $\left(u_j\right)_k$ of $\Phi_{\mathcal{A}}$ in *Y*. So, $\sum \left(u_j\right)_k \to \sum \left(u_j\right)_0$ weakly* in $BV(\Omega)$ and $\sum \left(u_j\right)_k \to \sum \left(u_j\right)_0$ strongly in $L^{1+\epsilon}(\Omega)$, module a subsequence, and thus $\left(u_j\right)_0 \in X$ and

$$
\sum \Phi_{\mathcal{A}}\left(\left(u_j\right)_0\right) \leq \liminf_{k\to\infty} \Phi_{\mathcal{A}}\left(\left(u_j\right)_k\right) = d_{\mathcal{A}}.
$$

It remains to check that $u_0 \in Y$, which it follows readily from Theorem 4.1 applied to $L^{1+2\epsilon}(\Omega)$ for $\epsilon \geq 0$.

Proof of Theorem 1.3. (See [14]) Applying Proposition 3.2 with $b = 1$, we conclude that the space $BV_0(\Omega)$ is weakly* closed in $BV(\Omega)$. Then, the limitation of $\Phi_{\mathcal{A}}$ to $BV_0(\Omega)$ may be shown as before.

5. Critical Minimizations under Constraints on *BV*(Ω)

Consider the truncation for $h>0$:

$$T_h(s)=\min(\max(s,-h),h) \text{ and } R_h(s)=s-T_h(s).$$

Proposition 2.3 of [2] ensures that

$\sum|Du_j|(\mathbb{R}^{2+\epsilon})=\sum|DT_hu_j|(\mathbb{R}^{2+\epsilon})+\sum|DR_hu_j|(\mathbb{R}^{2+\epsilon})$ for every $u_j\in BV(\mathbb{R}^{2+\epsilon})$.

Proposition 5.1. (See [14]) For any $u_j\in BV(\mathbb{R}^{2+\epsilon})$,

$$\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(u_j)\geq\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(T_hu_j)+\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(R_hu_j).$$

Proof. We first prove the inequality for functions $u_j\in W^{1,1}(\mathbb{R}^{2+\epsilon})$. From the definition of $T_h(s)$, we have $T_hu_j,R_hu_j\in W^{1,1}(\mathbb{R}^{2+\epsilon})$ and

$\sum\Psi_\xi(u_j)=\sum\Psi_\xi(T_hu_j)+\sum\Psi_\xi(R_hu_j)$ for all $\xi\in\mathbb{S}^{2+\epsilon-1}$, where

$$\Psi_\xi(u_j)=\int_{\mathbb{R}^{2+\epsilon}}\sum|\nabla_\xi u_j(x)|\,\mathrm{d}x.$$

Note that this decomposition implies $\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(u_j)\geq\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(T_hu_j)$ and $\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(u_j)\geq\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(R_hu_j)$. Thus, the statement follows if $\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(T_hu_j)=0$ or $\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(R_hu_j)=0$.

Assuming that $\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(T_hu_j)$ and $\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(R_hu_j)$ are nonzero, by Theorem 3.1, we have $\sum\Psi_\xi(T_hu_j),\sum\Psi_\xi(R_hu_j)>0$ for all $\xi\in\mathbb{S}^{1+\epsilon}$. So, by the Minkowski's inequality for negative exponents, we get

$$\begin{aligned}\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(u_j)&=\alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}}\sum\left(\Psi_\xi(T_hu_j)+\Psi_\xi(R_hu_j)\right)^{-(2+\epsilon)}\mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}}\\&\geq\alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}}\sum\left(\Psi_\xi(T_hu_j)\right)^{-(2+\epsilon)}\mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}}\\&\quad+\alpha_{2+\epsilon}\left(\int_{\mathbb{S}_{1+\epsilon}}\sum\left(\Psi_\xi(R_hu_j)\right)^{-(2+\epsilon)}\mathrm{d}\xi\right)^{-\frac{1}{2+\epsilon}}\\&=\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(T_hu_j)+\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(R_hu_j).\end{aligned}$$

Finally, the inequality extends to $BV(\mathbb{R}^{2+\epsilon})$ by using both the density of $W^{1,1}(\mathbb{R}^{2+\epsilon})$ in $BV(\mathbb{R}^{2+\epsilon})$ and the continuity of $u_j\in BV(\mathbb{R}^{2+\epsilon})\mapsto\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(u_j)$ with respect to the strict topology.

Proof of Theorems 1.2 and 1.4. (See [14]) Thanks to the weak* closure of $BV_0(\Omega)$ in $BV(\Omega)$, it is enough to just prove Theorem 1.2.

Let $(u_j)_k$ be a minimizing sequence of $\Phi_{\mathcal{A}}$ in *X*. Proceeding as in the proof of Theorem 1.1 by Theorem 4.1, have $\sum(u_j)_k\to\sum(u_j)_0$ strongly in $L^1(\Omega)$, module a subsequence. One may also assume that $\sum(u_j)_k\to\sum(u_j)_0$ almost everywhere in Ω and $\sum T_h(u_j)_k\to\sum T_h(u_j)_0$ weakly in $L^{\frac{2+\epsilon}{1+\epsilon}}(\Omega)$.

Using the Sobolev-Zhang inequality on $BV(\mathbb{R}^{2+\epsilon})$,

$$(2+\epsilon)\,\omega_{2+\epsilon}^{1/2+\epsilon}\left(\int_{\mathbb{R}^{2+\epsilon}}\sum|u_j|^{\frac{2+\epsilon}{1+\epsilon}}\mathrm{d}x\right)^{\frac{1+\epsilon}{2+\epsilon}}\leq\sum\mathcal{E}_{\mathbb{R}^{2+\epsilon}}(u_j),$$

and that b is nonnegative, we derive

$$\begin{aligned} c_{\mathcal{A}} &= \lim_{k\to\infty}\sum\left(\mathcal{E}_{\mathbb{R}^{2+\epsilon}}\left(\left(\bar{u}_j\right)_k\right)+\int_\Omega a\left|\left(u_j\right)_k\right|\mathrm{d}x+\int_{\partial\Omega} b\left|\left(\tilde{u}_j\right)_k\right|\mathrm{d}\mathcal{H}^{1+\epsilon}\right) \\ &\geq (2+\epsilon)\,\omega_{2+\epsilon}^{1/2+\epsilon}+\int_\Omega\sum a\left|\left(u_j\right)_0\right|\mathrm{d}x, \end{aligned}$$

so the condition $c_{\mathcal{A}} < (2+\epsilon)\,\omega_{2+\epsilon}^{1/2+\epsilon}$ implies that $\sum\left(u_j\right)_0 \neq 0$. Hence, by Corollaries 3.3 and 4.1 we have $\sum\left(u_j\right)_k \to \sum\left(u_j\right)_0$ weakly* in $BV(\Omega)$ and $\sum\Phi_{\mathcal{A}}\left(\left(u_j\right)_0\right) \leq c_{\mathcal{A}}$. It only remains to show that $\left(u_j\right)_0 \in X$.

By Proposition 5.1, we easily deduce that

$$\begin{aligned} c_{\mathcal{A}} &= \lim_{k\to\infty}\sum\Phi_{\mathcal{A}}\left(\left(u_j\right)_k\right) \geq \lim_{k\to\infty}\sum\left(\Phi_{\mathcal{A}}\left(T_h\left(u_j\right)_k\right)+\Phi_{\mathcal{A}}\left(R_h\left(u_j\right)_k\right)\right) \\ &\geq c_{\mathcal{A}}\lim_{k\to\infty}\sum\left(\left\|T_h\left(u_j\right)_k\right\|_{L^{\frac{2+\epsilon}{1+\epsilon}}(\Omega)}+\left\|R_h\left(u_j\right)_k\right\|_{L^{\frac{2+\epsilon}{1+\epsilon}}(\Omega)}\right). \end{aligned}$$

Applying Lemma 3.1 of [2], we have

$$c_{\mathcal{A}} \geq c_{\mathcal{A}}\sum\left[\left\|T_h\left(u_j\right)_0\right\|_{\frac{2+\epsilon}{1+\epsilon}}+\left(1+\left\|R_h\left(u_j\right)_0\right\|_{\frac{2+\epsilon}{1+\epsilon}}^{1+\epsilon}-\left\|\left(u_j\right)_0\right\|_{\frac{2+\epsilon}{1+\epsilon}}^{\frac{2+\epsilon}{\frac{1+\epsilon}{2+\epsilon}}}\right)^{\frac{1+\epsilon}{2+\epsilon}}\right].$$

Using the condition $c_{\mathcal{A}} > 0$ and letting $h \to \infty$, one obtains

$$1 \geq \sum\left(\left\|\left(u_j\right)_0\right\|_{\frac{2+\epsilon}{1+\epsilon}}^{\frac{2+\epsilon}{\frac{1+\epsilon}{2+\epsilon}}}\right)^{\frac{1+\epsilon}{2+\epsilon}}+\left(1-\left\|\left(u_j\right)_0\right\|_{\frac{2+\epsilon}{1+\epsilon}}^{1+\epsilon}\right)^{\frac{1+\epsilon}{2+\epsilon}},$$

and thus $\left(u_j\right)_0 \in X$ because $\sum\left(u_j\right)_0 \neq 0$.

The sequence $\left(u_j\right)_k$ of $\Phi_{\mathcal{A}}$ is taken in Y, then $\sum\left(u_j\right)_k \to \sum\left(u_j\right)_0$ almost everywhere in Ω, $\left(u_j\right)_0 \in X$ and $\sum\Phi_{\mathcal{A}}\left(\left(u_j\right)_0\right) \leq d_{\mathcal{A}}$. On the other hand, the first two properties along with Brezis-Lieb Lemma imply that $\sum\left(u_j\right)_k \to \sum\left(u_j\right)_0$ strongly in $L^{\frac{2+\epsilon}{1+\epsilon}}(\Omega)$. Finally, since $\epsilon \geq 0$, it follows that $\left(u_j\right)_0 \in Y$.

6. Conclusion

We establish the existence of minimizers for a class of restricted variational problems on $BV(\Omega)$ using the affine energy first presented by Zhang in [16], both for subcritical and critical limitations. Functionals that are related to this one have non-coercive geometry, and further in the weak* topology you'll find features like lower semicontinuity and affine compactness. Our work also proves the existence of extremal functions for certain classes of affine Poincaré-Sobolev inequalities.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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