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The Existence of Ground State Solutions for Schrödinger-Kirchhoff Equations Involving the Potential without a Positive Lower Bound

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Abstract

In this paper, we study the following Schrödinger-Kirchhoff equation $-\left(a+b\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+V(x)u=f(u), u\in H^1(\mathbb{R}^2)$, where $V(x)\geq 0$ and vanishes on an open set of $\mathbb{R}^2$ and *f* has critical exponential growth. By using a version of Trudinger-Moser inequality and variational methods, we obtain the existence of ground state solutions for this problem.

Keywords

Schrödinger-Kirchhoff Equations, Critical Exponential Growth, Ground State Solution, Degenerate Potential

1. Introduction

In this paper, we study the existence of ground state solutions for the following Schrödinger-Kirchhoff equation:

$$-\left(a+b\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+V(x)u=f(u), u\in H^1(\mathbb{R}^2), \tag{1.1}$$

where $a,b>0$ and the potential $V(x)\geq 0$ satisfying:

(V_1) $V(x)=0$ at $B_\delta(0)$ and $V(x)\geq C_0$ in $\mathbb{R}^2\setminus B_{2\delta}(0)$ for some $C_0,\delta>0$.

(V_2) There holds true;

$$\sup_{x\in\mathbb{R}^2}V(x)=\lim_{|x|\to\infty}V(x)=\gamma>0, \tag{1.2}$$

and the nonlinear term $f(t)$ is a continuous function on $\mathbb{R}$. Moreover, we impose the following conditions on the nonlinearity $f(t)$;

(f_0) $f(t)=0$ for all $t\leq 0$;

(f_1) critical exponential growth; there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to+\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0 & \text{for } a > a_0, \\ +\infty & \text{for } a < a_0; \end{cases} \tag{1.3}$$

(f_2) there exists $\mu > 4$ such that

$$0 < \mu F(t) = \mu \int_0^t f(s)\,ds \le t f(t) \quad \forall t \in \mathbb{R}\setminus\{0\}. \tag{1.4}$$

(f_3) there exist $t_0 > 0$ and $M_0 > 0$ such that $F(t) \le M_0 |f(t)|$ for any $|t| \ge t_0$;

(f_4) $f(t) = o(t)$ and $f(0) = 0$;

(f_5) $f(t) \in C^1(\mathbb{R})$ and $\frac{f(t)}{t^3}$ is increasing.

Without losing generality, we suppose that $a = b = 1$. So we may rewrite problem (1.1) in the following form:

$$-\left(1 + \int_{\mathbb{R}^3} |\nabla u|^2 \,dx\right)\Delta u + V(x)u = f(u), u \in H^1(\mathbb{R}^3). \tag{1.5}$$

Remark 1.1 *The condition* (f_2) *implies that* $F(t) = o(t^2)$ *as* $t \to 0$. *Indeed, the condition* (f_2) *implies that*

$$\left(\frac{F(t)}{t^\mu}\right)' > 0, \tag{1.6}$$

from which we can promptly obtain $F(t) = o(t^2)$ *as* $t \to 0$. *From the condition* (f_1), (f_2) *and* (f_4), *we can get the following growth condition for* $f(t)$; *for any* $\varepsilon > 0$ *and* $\beta_0 > \alpha_0$, *there exists* C_ε *such that*

$$|f(t)| \le \varepsilon |t| + C_\varepsilon t^\mu \left(e^{\beta_0 t^2} - 1\right), \quad \forall t \in \mathbb{R}. \tag{1.7}$$

From the condition (f_5), *we can also easily check that the function* $f(t)t - 4F(t)$ *is increasing.*

The corresponding Dirichlet problem for (1.1) on a smooth domain $\Omega \subset \mathbb{R}^2$,

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \,dx\right)\Delta u = f(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.8}$$

is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 \,dx\right)\Delta u = f(x,u), \tag{1.9}$$

which was first proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Problem (1.9) has attracted considerable attention after Lions [2] introduced an abstract framework to the problem.

The above problem is nonlocal as the appearance of the term $\int_\Omega |\nabla u|^2 \,dx$ implies that (1.5) is not a pointwise identity. This phenomenon provokes some mathematical difficulties, which make the study of such class of problems particularly interesting. For more details on the physical and mathematical background of this problem to [2] [3] [4] [5].

Since we will work with critical exponential growth, we need to review the Trudinger-Moser inequality. For one thing, let Ω denote a smooth bounded domain in $\mathbb{R}^N\left(N\geq 2\right)$. N. Trudinger [6] proved that there exists $\alpha>0$ such that $W_0^{1,N}\left(\Omega\right)$ is embedded in the Orlicz space $L_{\varphi_\alpha}\left(\Omega\right)$ determined by the Young function $\varphi_\alpha=\mathrm{e}^{\alpha|t|^{\frac{N}{N-1}}}$. It was sharpened by J. Moser [7] who found the best exponent $\alpha_n=n\omega_{n-1}^{\frac{1}{n-1}}$, where ω_{n-1} is the surface measure of the unit sphere in $\mathbb{R}^N$. For another, the Trudinger-Moser inequality was extended for unbounded domains by D. M. Cao [8] in $\mathbb{R}^2$ and for any dimension $N\leq 2$ by J. M. do Ó [9]. Moreover, J. M. do Ó *et al.* [10] established a sharp Concentration-compactness principle associated with the singular Trudinger-Moser inequality in $\mathbb{R}^N$.

Many significant research results about (1.1) have been obtained. For example, in [11], X. Wu studied the nontrivial solutions and high energy solutions of problem (1.1) if $V\left(x\right)$ has a positive constant lower bound and the nonlinearities term with 4-superlinear growth at infinity. In [12], the authors studied the following Schrödinger-Kirchhoff type equation

$$M\left(\int_{\mathbb{R}^2}\left|\nabla u\right|^2+V\left(x\right)u^2\mathrm{d}x\right)\left(-\Delta u+V\left(x\right)u\right)=A\left(x\right)f\left(u\right)\text{ in }\mathbb{R}^2 \tag{1.10}$$

where M is a Kirchoff-type function and $V\left(x\right)\geq V_0$ is a continuous function, A is locally bounded and the function f has critical exponential growth. Applying variational methods beside a new Trudinger-Moser type inequality, they get the of ground state solution. Moreover, in the the local case $M\equiv 1$, they also get some relevant results. We emphasize that in these papers, the potential $V\left(x\right)$ have a positive constant lower bound. Some studies of the Kirchhoff equation with critical exponential growth may refer [13] [14] [15] [16].

In [17], the author establishes a class of Trudinger-Moser inequality and proves the existence of the ground state solution to a class of Schrödinger equation with critical exponential growth. In addition, a class of quasilinear n-Laplace Schrödinger equations with degenerate potentials and of exponential growth is also studied. But to the best of our knowledge, the Schrödinger-Kirchhof equation that satisfies condition (V_1), (V_2) doesn't seem to have been studied. Different from the first two results, the appearance of the term $\int_{\mathbb{R}^2}\left|\nabla u\right|^2\mathrm{d}x$, Some proof methods in the original text are invalid, so we have to find other methods, for the details see Lemmas 3.2 and 3.9.

Motivated by [17], we can prove the existence of the ground state solution to problem (1.5) as in [17]. In order to get the result we want, we use a version of Trudinger-Moser inequality.

Lemma 1.2. (*Trudinger-Morse inequality* [17]) *Assume that the potential* $V\left(x\right)\geq 0$ *satisfies that* $V\left(x\right)=0$ *at the ball* $B_\delta\left(0\right)$ *centered at the origin with the radius* δ *and* $V\left(x\right)\geq C_0$ *in* $\mathbb{R}^2\setminus B_{2\delta}\left(0\right)$ *for some* $\delta>0$. *Then*

$$\sup_{u\in H^1\left(\mathbb{R}^2\right),\int_{\mathbb{R}^2}\left|\nabla u\right|^2+V\left(x\right)u^2\mathrm{d}x\leq 1}\int_{\mathbb{R}^2}\left(\mathrm{e}^{4\pi u^2}-1\right)\mathrm{d}x<\infty. \tag{1.11}$$

Lemma 1.2 will be used to obtain the existence of ground state solution of the following Schrödinger-Kirchhof equation;

$$\begin{cases} -\left(1+\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+V(x)u=f(u) \text{ in } \mathbb{R}^2, \\ u\in H^1(\mathbb{R}^2), \end{cases} \tag{1.12}$$

Lemma 1.3. (*Fatou's Lemma*) *Let* (X,B,μ) *be a measure space, and* $\{f_n : X\to[0,\infty]\}$ *be a sequence of non-negative measurable functions. Then the function* $\liminf_{n\to\infty} f_n$ *is measurable and*

$$\int_X \liminf f_n \mathrm{d}\mu \le \liminf \int_X f_n \mathrm{d}\mu. \tag{1.13}$$

Now, we are ready to state the main results of this paper.

Theorem 1.4. *Suppose that* (V_1), (V_2) *and* f_0 - f_5 *hold. If we further assume that*

$$\lim_{t\to+\infty}\frac{F(t)t^2}{\mathrm{e}^{\alpha_0 t^2}}=\infty, \tag{1.14}$$

then (*1.12*) *admits a positive ground state solution.*

2. Preliminaries

In this section, we give some useful notions and lemmas, which are used to prove our results.

Now, we introduce some notations. For any $1\le r<\infty$, $L^r(\mathbb{R}^2)$ is the usual Lebesgue space with the norm

$$\|u\|_r=\left(\int_{\mathbb{R}^2}|u|^r\right)^{\frac{1}{r}}.$$

$H^1(\mathbb{R}^2)$ is the usual Sobolev space with the norm

$$\|u\|^2_{H^1(\mathbb{R}^2)}=\int_{\mathbb{R}^2}\left(|\nabla u|^2+|u|^2\right)\mathrm{d}x.$$

Lemma 2.1. ([17]) *Assume that* $u\in H^1(\mathbb{R}^2)$ *such that*

$$\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x<+\infty,$$

where $V(x)$ *satisfies the assumption* (V_1). *Then there exists some constant* $c>0$ *depending on* δ *and* C_0 *such that*

$$\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}x<c\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x.$$

which was proved in [17].

Remark 2.2. *If we define* $H_V(\mathbb{R}^2)$ *as the completion of* $C_0^\infty(\mathbb{R}^2)$ *under the norm*

$$\left(\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x\right)^{\frac{1}{2}},$$

then Lemma 2.1 implies an result;

$$H_V(\mathbb{R}^2)=H^1(\mathbb{R}^2).$$

The problem (1.5) associated functional is

$$I_V(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)dx+\frac{1}{4}\left(\int_{\mathbb{R}^2}|\nabla u|^2 dx\right)^2-\int_{\mathbb{R}^2}F(u)dx.$$

where $F(t)=\int_0^t f(s)ds$, and its Nehari manifold is

$$\mathcal{N}_V(u)=\left\{u\in H^1(\mathbb{R}^2)\,|\,u\neq 0,\langle I_V'(u),u\rangle=0\right\},$$

where

$$\langle I_V'(u),u\rangle=\left(1+\int_{\mathbb{R}^2}|\nabla u|^2 dx\right)\int_{\mathbb{R}^2}|\nabla u|^2 dx+\int_{\mathbb{R}^2}V(x)|u|^2 dx-\int_{\mathbb{R}^2}f(u)u dx.$$

In order to study the problem (1.5) under the assumptions (V_1) and (V_2), we introduce the following limiting equation;

$$-\left(a+b\int_{\mathbb{R}^2}|\nabla u|^2 dx\right)\Delta u+\gamma u=f(u), \tag{2.1}$$

where we recall from (V_2) that

$$\sup_{x\in\mathbb{R}^2}V(x)=\lim_{|x|\to\infty}V(x)=\gamma>0,$$

The corresponding functional and Nehari manifold associated with (2.1) are

$$I_\infty(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u|^2+\gamma|u|^2\right)dx+\frac{1}{4}\left(\int_{\mathbb{R}^2}|\nabla u|^2 dx\right)^2-\int_{\mathbb{R}^2}F(u)dx.$$

and

$$\mathcal{N}_\infty=\left\{u\in H^1(\mathbb{R}^2)\,|\,u\neq 0,\langle I_\infty'(u),u\rangle=0\right\},$$

where

$$\langle I_\infty'(u),u\rangle=\left(1+\int_{\mathbb{R}^2}|\nabla u|^2 dx\right)\int_{\mathbb{R}^2}|\nabla u|^2 dx+\int_{\mathbb{R}^2}\gamma|u|^2 dx-\int_{\mathbb{R}^2}f(u)u dx.$$

We can easily verify that if $u\in\mathcal{N}_N$, then

$$I_V(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left(f(u)u-2F(u)\right)dx-\frac{1}{4}\left(\int_{\mathbb{R}^2}|\nabla u|^2 dx\right)^2,$$

and if $u\in\mathcal{N}_\infty$, then

$$I_\infty(u)=\frac{1}{2}\int_{\mathbb{R}^2}\left(f(u)u-2F(u)\right)dx-\frac{1}{4}\left(\int_{\mathbb{R}^2}|\nabla u|^2 dx\right)^2.$$

3. The Proof of Theorem 1.3

In this section, we want to show that (1.5) has the existence of ground state solutions.

Lemma 3.1. *$\mathcal{N}_V$ and $\mathcal{N}_\infty$ are not empty.*

Proof. we only prove that $\mathcal{N}_V$ is not empty since the proof of $\mathcal{N}_\infty$ is similar. Let $u_0\in H^1(\mathbb{R}^2)$ be positive and compactly supported in a bounded domain Ω. Define

$$\begin{aligned} h(t)&:=\langle I_V'(tu_0),tu_0\rangle\\ &=t^2\int_{\mathbb{R}^2}\left(|\nabla u_0|^2+V(x)|u_0|^2\right)dx+t^4\left(\int_{\mathbb{R}^2}|\nabla u_0|^2 dx\right)^2\\ &\quad-2\int_{\mathbb{R}^2}f(tu_0)tu_0 dx,\quad \forall t>0. \end{aligned}$$

$\mathcal{N}_V$ being not empty is a direct result of the fact; $h(t)>0$ for $t>0$ small enough and $h(t)<0$ for $t>0$ sufficiently large.

We first prove that $h(t)>0$ for $t>0$ small enough. From Remark 1.1, we conclude that for any $\varepsilon>0$, there exist C_ε such that

$$\left|f(t)\right|\le\varepsilon|t|+C_\varepsilon|t|^\mu\left(\mathrm{e}^{\beta_0t^2}-1\right)\tag{3.1}$$

for any $t\in\mathbb{R}$. Using this estimate, we can write

$$\begin{aligned}h(t)&=t^2\int_{\mathbb{R}^2}\left(|\nabla u_0|^2+V(x)|u_0|^2\right)\mathrm{d}x+t^4\left(\int_{\mathbb{R}^2}|\nabla u_0|^2\,\mathrm{d}x\right)^2-2\int_{\mathbb{R}^2}f(tu_0)tu_0\mathrm{d}x\\&\ge t^2\int_{\mathbb{R}^2}\left(|\nabla u_0|^2+V(x)|u_0|^2\right)\mathrm{d}x+t^4\left(\int_{\mathbb{R}^2}|\nabla u_0|^2\,\mathrm{d}x\right)^2\\&\quad-2\varepsilon t^2\int_{\mathbb{R}^2}u_0^2\mathrm{d}x-2|t|^{\mu+1}C_\varepsilon\int_{\mathbb{R}^2}u_0^{\mu+1}\left(\mathrm{e}^{\beta_0(tu_0)^2}-1\right)\mathrm{d}x,\end{aligned}\tag{3.2}$$

Which implies that $h(t)>0$ for small $t>0$ since $\mu>4$.

Next, we prove that $h(t)<0$ for $t>0$ sufficiently large. The condition (f_2) implies that $F(t)\ge t^\mu F(1)$ when $t>1$. Then it follows that there exists $C_3>0$ such that $F(t)\ge t^\mu F(1)-C_3$ for all $t>0$. this together with the condition (f_2) yields that there exists $C,C_4>0$ such that $f(t)\ge Ct^\mu F(1)-C_4$ for all $t>0$. Noticing that u_0 is compactly supported in the bounded domain Ω, we can write

$$\begin{aligned}h(t)&=t^2\int_{\mathbb{R}^2}\left(|\nabla u_0|^2+V(x)|u_0|^2\right)\mathrm{d}x+t^4\left(\int_{\mathbb{R}^2}|\nabla u_0|^2\right)^2-2\int_\Omega f(tu_0)tu_0\mathrm{d}x\\&\le t^2\int_{\mathbb{R}^2}\left(|\nabla u_0|^2+V(x)|u_0|^2\right)\mathrm{d}x+t^4\left(\int_{\mathbb{R}^2}|\nabla u_0|^2\right)^2\\&\quad-2F(1)Ct^{\mu+1}\int_\Omega u_0^{\mu+1}\mathrm{d}x+2C_4\int_{\mathbb{R}^2}tu_0\mathrm{d}x,\end{aligned}\tag{3.3}$$

which implies that $h(t)$ is negative for sufficiently large $t>0$.

Now, we set

$$m_\infty=\inf\left\{I_\infty(u),u\in\mathcal{N}_\infty\right\}\text{ and }m_V=\inf\left\{I_V(u),u\in\mathcal{N}_V\right\},$$

and we claim the following lemma.

Lemma 3.2. *It holds that*

$$0<m_V<m_\infty.\tag{3.4}$$

Proof. To show that $m_V<m_\infty$, it is enough to find u satisfying $u\in\mathcal{N}_V$ such that $I_V(u)<m_\infty$. From [18], we know that if

$$\lim_{t\to+\infty}\frac{F(t)t^2}{\mathrm{e}^{\alpha_0t^2}}=\infty,$$

then m_∞ is attained by some $w\in\mathcal{N}_\infty$. By the definition of $V(x)$, it is easy to check that

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(|\nabla w|^2+V(x)|w|^2\right)\mathrm{d}x+\left(\int_{\mathbb{R}^2}|\nabla w|^2\,\mathrm{d}x\right)^2\\&<\int_{\mathbb{R}^2}\left(|\nabla w|^2+\gamma|w|^2\right)\mathrm{d}x+\left(\int_{\mathbb{R}^2}|\nabla w|^2\,\mathrm{d}x\right)^2=\int_{\mathbb{R}^2}f(w)w\mathrm{d}x.\end{aligned}$$

Hence there exists $t\in(0,1)$ such that

$$\int_{\mathbb{R}^2}\left(\left|\nabla(tw)\right|^2+V(x)\left|tw\right|^2\right)\mathrm{d}x+\left(\int_{\mathbb{R}^2}\left|\nabla(tw)\right|^2\mathrm{d}x\right)^2=\int_{\mathbb{R}^2}f(tw)tw\mathrm{d}x,$$

which implies that $u=tw\in\mathcal{N}_V$. then it follows that

$$\begin{aligned}m_V\le I_V(tw)&=\frac{1}{2}\int_{\mathbb{R}^2}\left(\left|\nabla(tw)\right|^2+V(x)\left|tw\right|^2\right)\mathrm{d}x+\frac{1}{4}\left(\int_{\mathbb{R}^2}\left|\nabla(tw)\right|^2\mathrm{d}x\right)^2-\int_{\mathbb{R}^2}F(tw)\mathrm{d}x\\&\le\frac{1}{2}\int_{\mathbb{R}^2}\left(\left|\nabla(tw)\right|^2+\gamma\left|tw\right|^2\right)\mathrm{d}x+\frac{1}{4}\left(\int_{\mathbb{R}^2}\left|\nabla(tw)\right|^2\mathrm{d}x\right)^2-\int_{\mathbb{R}^2}F(tw)\mathrm{d}x\\&=I_\infty(tw)\le\max_{t\ge0}I_\infty(tw)=I_\infty(w)=m_\infty\end{aligned}$$

Next, we show $m_V>0$. We prove this by contradiction. Assume that there exists some sequence $u_k\in\mathcal{N}_V$ such that $I_V(u_k)\to0$, then we have

$$\begin{aligned}I_V(u_k)&=I_V(u_k)-\frac{1}{4}\left\langle I_V'(u_k),u_k\right\rangle\\&=\frac{1}{4}\|u_k\|_{H_V}^2+\frac{1}{4}\int_{\mathbb{R}^2}\left(f(u_k)u_k-4F(u_k)\right)\mathrm{d}x\\&\ge\frac{1}{4}\|u_k\|_{H_V}^2.\end{aligned}$$

which implies that $\|u_k\|_{H_V}^2\to0$. From $u_k\in\mathcal{N}_V$ and (3.1), we know that

$$\int_{\mathbb{R}^2}f(u_k)u_k\mathrm{d}x\le\int_{\mathbb{R}^2}\varepsilon\left|u_k\right|^2+C_\varepsilon\left|u_k\right|^\mu\left(\mathrm{e}^{\beta_0u_k^2}-1\right)\mathrm{d}x.\tag{3.5}$$

By the Trudinger-Morse inequality (1.11) and $\mu>4$, we get for any $p>1$,

$$\int_{\mathbb{R}^2}\left|u_k\right|^\mu\left(\mathrm{e}^{\beta_0u_k^2}-1\right)\mathrm{d}x\le\left(\int_{\mathbb{R}^2}\left|u_k\right|^{\mu p}\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^2}\left(\mathrm{e}^{p'\beta_0u_k^2}-1\right)\mathrm{d}x\right)^{\frac{1}{p'}}\le C\|u_k\|_{H_V}^\mu.\tag{3.6}$$

From (3.5) and (3.6), there exist $C_1,C_2>0$, such that

$$\int_{\mathbb{R}^2}f(u_k)u_k\mathrm{d}x\le C_1\varepsilon\|u_k\|_{H_V}^2+C_2\|u_k\|_{H_V}^\mu.$$

Therefore

$$\|u_k\|_{H_V}^2\le\int_{\mathbb{R}^2}f(u_k)u_k\le C_1\varepsilon\|u_k\|_{H_V}^2+C_2\|u_k\|_{H_V}^\mu.$$

Since $\mu>4$, there exist $\rho>0$ such that

$$\|u_k\|_{H_V}^2\ge\rho>0.$$

which is a contradicion to $\|u_k\|_{H_V}^2\to0$.

We now consider a minimizing sequence $\{u_k\}_k\subset\mathcal{N}_V$ for m_V. Since

$$\int_{\mathbb{R}^2}\left|\nabla(|u_k|)\right|^2\mathrm{d}x\le\int_{\mathbb{R}^2}\left|\nabla(u_k)\right|^2\mathrm{d}x,$$

we can assume that $u_k\ge0$. The (A-R) condition (f_2), $I_V(u_k)\to m_V>0$ and Remark 2.2 give that $\{u_k\}_k$ is bounded in $H^1(\mathbb{R}^2)$, and then up to a subsequence, there exists $u\in H^1(\mathbb{R}^2)$, such that

$$\begin{aligned}&u_k\rightharpoonup u\quad\text{in } H^1(\mathbb{R}^2),\text{ and in } L^p(\mathbb{R}^2)\text{ for any } p>1,\\&u_k\to u\quad\text{in } L^p_{loc}(\mathbb{R}^2),\\&u_k\to u\quad\text{a.e.}\end{aligned}$$

By extracting a subsequence, if necessary, we define $\beta,l\ge0$ as

$$\beta=\lim_k\int_{\mathbb{R}^2}f(u_k)u_k\mathrm{d}x\ \text{ and }\ l=\int_{\mathbb{R}^2}f(u)u\mathrm{d}x.$$

By the weak convergence, it is obvious that $l \in [0, \beta]$.

Lemma 3.3. *It holds that* $\beta > 0$.

Proof. We proof this by contradiction. Assume that $\beta = 0$. Then we have

$$I_V(u_k) = \frac{1}{2}\int_{\mathbb{R}^2}\left(f(u_k)u_k - 2F(u_k)\right)\mathrm{d}x - \frac{1}{4}\left(\int_{\mathbb{R}^2}|\nabla u_k|^2\,\mathrm{d}x\right)^2\mathrm{d}x \to 0,$$

which contradicts (3.4).

Lemma 3.4 *The case* $l = 0$ *cannot occur.*

Proof. We prove this by contradiction. If $l = 0$, then $u = 0$, and $u_k \to 0$ in $L^2_{loc}(\mathbb{R}^2)$. we first claim that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2}(\gamma - V(x))|u_k|^2\,\mathrm{d}x = 0. \tag{3.7}$$

For any fixed $\varepsilon > 0$, we take $R_\varepsilon > 0$ such that

$$|\gamma - V(x)| \le \varepsilon \quad \text{for any } |x| > R_e.$$

combining this and the boundedness of u_k in $H^1(\mathbb{R}^2)$, we derive that

$$\begin{aligned}\int_{\mathbb{R}^2}(\gamma - V(x))|u_k|^2\,\mathrm{d}x &= \int_{B_{R_\varepsilon}}(\gamma - V(x))|u_k|^2\,\mathrm{d}x + \int_{B^c_{R_\varepsilon}}(\gamma - V(x))|u_k|^2\,\mathrm{d}x \\ &\le c\int_{B_{R_\varepsilon}}|u_k|^2\,\mathrm{d}x + K\varepsilon,\end{aligned}$$

where $K = \sup_k \int_{\mathbb{R}^2}|u_k|^2\,\mathrm{d}x$. This together with $u_k \to 0$ in $L^2_{loc}(\mathbb{R}^2)$ as $k \to +\infty$ yields that

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2}(\gamma - V(x))|u_k|^2\,\mathrm{d}x \le c\varepsilon,$$

which implies that (3.7) hold.

Similarly to the proof of ([19]. Proposition 6.1), we can get there exists some sequence $t_k \ge 1$ such that $t_k u_k \in \mathcal{N}_\infty$ and $\{t_k\}_k$ converges to 1 as $k \to +\infty$.

Now, by (3.7), we can write

$$\begin{aligned} m_\infty &\le \lim_{k\to+\infty} I_\infty(t_k u_k) \\ &= \lim_{k\to+\infty}\left(I_V(t_k u_k) + \frac{t_k^2}{2}\int_{\mathbb{R}^2}(\gamma - V(x))|u_k|^2\,\mathrm{d}x\right) \\ &= \lim_{k\to+\infty} I_V(t_k u_k) \\ &= \lim_{k\to+\infty} t_k^2\left(\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u_k|^2 + V(x)|u_k|^2\right)\mathrm{d}x - \int_{\mathbb{R}^2}\frac{F(t_k u_k)}{t_k^2 u_k^2}u_k^2\,\mathrm{d}x + \frac{t_k^2}{4}\left(\int_{\mathbb{R}^2}|\nabla u_k|^2\right)^2\right). \end{aligned}$$

This together with the monotonicity of $\frac{F(t)}{t^2}$ and $\lim_{k\to+\infty} t_k = 1$ gives

$$m_\infty \le \lim_{k\to+\infty} I_V(u_k) = m_V,$$

which contradicts (3.4). This accomplishes the proof of Lemma 3.4.

Note that (f_5) implies the following inequality;

$$\begin{aligned} &\frac{1-t^4}{4}f(u)u + F(tu) - F(u) \\ &= \int_t^1\left[\frac{f(u)}{u^3} - \frac{f(su)}{(su)^3}\right]s^3u^4\,\mathrm{d}s \ge 0. \quad \forall u \ne 0, 0 \le t \le 1. \end{aligned} \tag{3.8}$$

Lemma 3.5. *If* $l=\beta$, *then* $u\in\mathcal{N}_V$ *and* $I_V(u)=m_V$.

Proof. If $l=\beta$, then

$$\lim_{k\to+\infty}\int_{\mathbb{R}^2} f(u_k)u_k\mathrm{d}x \to \int_{\mathbb{R}^2} f(u)u\mathrm{d}x.$$

Then we can get

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x+\left(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^2\\ &\le \lim_{k\to+\infty}\int_{\mathbb{R}^2}\left(|\nabla u_k|^2+V(x)|u_k|^2\right)\mathrm{d}x+\left(\int_{\mathbb{R}^2}|\nabla u_k|^2\,\mathrm{d}x\right)^2\\ &=\lim_{k\to+\infty}\int_{\mathbb{R}^2} f(u_k)u_k\mathrm{d}x\\ &=\int_{\mathbb{R}^2} f(u)u\mathrm{d}x.\end{aligned}$$

If the above equality holds, then $u\in\mathcal{N}_V$, and the lemma is proved. Therefore, it remains to show that the case where

$$\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x+\left(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^2<\int_{\mathbb{R}^2} f(u)u\mathrm{d}x \tag{3.9}$$

cannot occur. In fact, if (3.9) holds, we can take some $t\in(0,1)$ such that $tu\in\mathcal{N}_V$. Indeed, let

$$g(t)=\int_{\mathbb{R}^2}\left(|\nabla(tu)|^2+V(x)|tu|^2\right)\mathrm{d}x+\left(\int_{\mathbb{R}^2}|\nabla(tu)|^2\,\mathrm{d}x\right)^2-\int_{\mathbb{R}^2} f(tu)tu\mathrm{d}x,$$

Obviously, $g(t)$ is positive for small *t*. This together with $g(1)<0$ implies that there exists $t\in(0,1)$ such that $g(t)=0$, *i.e.*, $tu\in\mathcal{N}_V$.

Using (f_5), we can obtain

$$f(tu)tu<t^4 f(u)u. \tag{3.10}$$

From (3.8) we know that

$$F(tu)>\frac{t^4-1}{4}f(u)u+F(u). \tag{3.11}$$

Combining (3.10) and (3.11), we derive

$$\frac{1}{4}f(tu)tu-F(tu)<\frac{1}{4}f(u)u-F(u). \tag{3.12}$$

Since $t\in(0,1)$, by the define of $\mathcal{N}_V(u)$, (3.12) and Fatou's lemma, we deduce that

$$\begin{aligned}m_V&\le I_V(tu)-\frac{1}{4}\langle I_V'(tu),tu\rangle\\ &=\frac{t^2}{4}\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x+\int_{\mathbb{R}^2}\left(\frac{1}{4}f(tu)tu-F(tu)\right)\mathrm{d}x\\ &<\frac{1}{4}\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x+\int_{\mathbb{R}^2}\left(\frac{1}{4}f(u)u-F(u)\right)\mathrm{d}x\\ &=I_V(u)-\frac{1}{4}\langle I_V'(u),u\rangle\\ &\le\lim_{k\to\infty}\left[I_V(u_k)-\frac{1}{4}\langle I_V'(u_k),u_k\rangle\right]=m_V.\end{aligned}$$

Which is a contradiction.

In the following, we consider the case $0<l<\beta$. If $0<l<\beta$, then $u_k \rightharpoonup u \neq 0$ in $H^1(\mathbb{R}^2)$. We can choose an increasing sequence $\{R_j\}_j \to +\infty$ such that $R_{j+1} > R_j + 1$,

$$\int_{B_{R_j}} f(u)u\mathrm{d}x = l + o_j(1) \tag{3.13}$$

and

$$\int_{B_{R_j}^c} |u|^p \mathrm{d}x = o_j(1)$$

for any $2 \le p < \infty$. We define

$$C_j = B_{R_{j+1}} \setminus B_{R_j} = \{x \in \mathbb{R}^2 \mid R_j \le |x| < R_j + 1\}.$$

Lemma 3.6. *For the* C_j *given above, we have*

$$\int_{C_j} f(u_k)u_k \mathrm{d}x = o_j(1) \tag{3.14}$$

and

$$\int_{C_j} |\nabla u_k|^2 \mathrm{d}x = o_j(1) \tag{3.15}$$

Proof. We prove (3.14) by contradiction. If there exists some subsequence $\{j_i\}_i$ of $\{j\}$ such that (3.14) fails, then we must have

$$\sum_{i=1}^{\infty} \int_{C_{j_i}} f(u_k)u_k \mathrm{d}x = \infty.$$

However, we have

$$\begin{aligned}\sum_{i=1}^{\infty} \int_{C_{j_i}} f(u_k)u_k \mathrm{d}x &\le \int_{\mathbb{R}^2} f(u_k)u_k \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \left(|\nabla u_k|^2 + V(x)|u_k|^2\right)\mathrm{d}x + \left(\int_{\mathbb{R}^2} |\nabla u_k|^2 \mathrm{d}x\right)^2 < \infty,\end{aligned}$$

which arrives at a contradiction. Similarly, we can also prove (3.15).

Lemma 3.7. ([17]) *It holds that*

$$\lim_{k\to+\infty} \int_{\mathbb{R}^2} F(u_k)\mathrm{d}x = \int_{\mathbb{R}^2} F(u)\mathrm{d}x.$$

which was proved in [17].

Lemma 3.8. ([20]) *Let* Ω *be a domain in* $\mathbb{R}^N$. *Suppose* $\{g_n\},\{h_n\} \subset L^1(\Omega)$ *and* $h \in L^1(\Omega)$. *If*

$$0 \le g_n \le h_n, \quad g_n(x) \to 0, \quad h_n \to h \text{ a.e. } x \in \Omega$$

and

$$\lim_{n\to\infty} \int_\Omega h_n = \int_\Omega h$$

then $\lim_{n\to\infty} \int_\Omega g_n = 0$.

Lemma 3.9. *It hold that*

$$\lim_{k\to+\infty} \int_{B_{R_j}} f(u_k)u_k \mathrm{d}x = \int_{B_{R_j}} f(u)u\mathrm{d}x$$

provided j is large enough.

Proof. Since u_k is bounded in $H^1(\mathbb{R}^2)$, there is $C>0$, and for any $2\le p<+\infty$, we have

$$\|u_k\|_{L^p(B_{R_j})}\le\|u_k\|_{L^p(\mathbb{R}^2)}\le\|u_k\|_{H^1(\mathbb{R}^2)}\le C. \tag{3.16}$$

According to (ii) in definition 6.1 in reference [21], when $m(E)<\infty$, it can be obtained

$$\lim_{p\to+\infty}\|f\|_p=\|f\|_\infty. \tag{3.17}$$

where E stands for measurable set.

From (3.16) and (3.17), we can deduce that

$$\lim_{p\to+\infty}\|f\|_p=\|f\|_\infty\le C.$$

which implies that u_k is bounded in B_{R_j}.

We can let $g_k=f(u_k)u_k-f(u)u$, using (1.4), we can derive

$$\begin{aligned}|g_k|&=|f(u_k)u_k-f(u)u|\le|f(u_k)u_k|\\&\le\varepsilon|u_k|^2+C_\varepsilon|u_k|^{\mu+1}\left(e^{\beta_0u_k^2}-1\right).\end{aligned}$$

Because u_k is bounded in B_{R_j}, then there is $M>0$, we have

$$g_k\le\varepsilon|u_k|^2+C_\varepsilon|u_k|^{\mu+1}\left(e^{\beta_0M}-1\right)=\varepsilon|u_k|^2+C_1|u_k|^{\mu+1}.$$

where $C_1=C_\varepsilon\cdot\left(e^{\beta_0M}-1\right)$.

Using $u_k\to u$ in $L^p_{loc}(\mathbb{R}^2)$ and lemma 3.8, we can get

$$\lim_{k\to+\infty}\int_{B_{R_j}}|f(u_k)u_k-f(u)u|=0.$$

which implies that

$$\lim_{k\to+\infty}\int_{B_{R_j}}f(u_k)u_k\,dx=\int_{B_{R_j}}f(u)u\,dx. \tag{3.18}$$

The proof is completed.

From (3.18) and Lemma 3.6, since $R_j+1<R_{j+1}$, we can extract a subsequence u_{k_j} such that for every $j\in\mathbb{N}$,

$$\int_{B_{R_j}}f\left(u_{k_j}\right)u_{k_j}\,dx=l+o_j(1)$$

and

$$\int_{C_j}f\left(u_{k_j}\right)u_{k_j}\,dx=o_j(1),\quad\int_{C_j}\left|\nabla u_{k_j}\right|^2dx=o_j(1),\quad\int_{C_j}u_{k_j}^2\,dx=o_j(1).$$

Now, we take $\{u_{k_j}\}$ as a new minimizing sequence renaming it $\{u_j\}_j$.

Lemma 3.10. *It cannot be*

$$\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)dx+\left(\int_{\mathbb{R}^2}|\nabla u|^2\,dx\right)^2<\int_{\mathbb{R}^2}f(u)u\,dx. \tag{3.19}$$

Proof. If (3.19) is true, then there exists some $t\in(0,1)$ such that $tu\in\mathcal{N}_V$. Since $t\in(0,1)$, by the define of $\mathcal{N}_V(u)$, (3.12) and Fatou's lemma, we deduce that

$$
\begin{aligned}
m_V &\le I_V(tu)-\frac{1}{4}\langle I_V'(tu),tu\rangle \\
&=\frac{t^2}{4}\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x+\int_{\mathbb{R}^2}\frac{1}{4}f(tu)tu-F(tu)\mathrm{d}x \\
&<\frac{1}{4}\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x+\int_{\mathbb{R}^2}\frac{1}{4}f(u)u-F(u)\mathrm{d}x \\
&=I_V(u)-\frac{1}{4}\langle I_V'(u),u\rangle \\
&\le \lim_{k\to\infty}\left[I_V(u_k)-\frac{1}{4}\langle I_V'(u_k),u_k\rangle\right]=m_V.
\end{aligned}
$$

which is a contradiction.

Lemma 3.11. *It cannot be*

$$
\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x+\left(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^2>\int_{\mathbb{R}^2}f(u)u\mathrm{d}x. \tag{3.20}
$$

Proof. We prove (3.20) by contradiction. If

$$
\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x+\left(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^2>\int_{\mathbb{R}^2}f(u)u\mathrm{d}x. \tag{3.21}
$$

Since $u_j \to u$ weakly in $H^1(\mathbb{R}^2)$, from Lemma 3.9, we can deduce

$$
\int_{\mathbb{R}^2}f(u_j)u_j\mathrm{d}x+o_j(1)=\int_{\mathbb{R}^2}f(u)u\mathrm{d}x.
$$

This implies that

$$
\begin{aligned}
\|u\|_{H_V}^2 &\le \lim_{j\to+\infty}\|u_j\|_{H_V}^2 \\
&=\lim_{j\to+\infty}\left[\int_{\mathbb{R}^2}f(u_j)u_j\mathrm{d}x-\left(\int_{\mathbb{R}^2}|\nabla u_j|^2\,\mathrm{d}x\right)^2\right] \\
&=\int_{\mathbb{R}^2}f(u)u\mathrm{d}x-\lim_{j\to+\infty}\left(\int_{\mathbb{R}^2}|\nabla u_j|^2\,\mathrm{d}x\right)^2 \\
&\le \int_{\mathbb{R}^2}f(u)u\mathrm{d}x-\left(\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x\right)^2 \\
&<\|u\|_{H_V}^2.
\end{aligned}
$$

which is a contradiction.

End of the proof of Theorem 1.3. Lemma 3.10 and Lemma 3.11 imply $I_V'(u)=0$. Hence

$$
\begin{aligned}
m_V &\le I_V(u)-\frac{1}{4}\langle I_V'(u),u\rangle \\
&=\frac{1}{4}\int_{\mathbb{R}^2}\left(|\nabla u|^2+V(x)|u|^2\right)\mathrm{d}x+\int_{\mathbb{R}^2}\frac{1}{4}f(u)u-F(u)\mathrm{d}x \\
&\le \lim_{k\to\infty}\frac{1}{4}\int_{\mathbb{R}^2}\left(|\nabla u_k|^2+V(x)|u_k|^2\right)\mathrm{d}x+\lim_{k\to\infty}\int_{\mathbb{R}^2}\frac{1}{4}f(u_k)u_k-F(u_k)\mathrm{d}x \\
&=\lim_{k\to\infty}\left[I_V(u_k)-\frac{1}{4}\langle I_V'(u_k),u_k\rangle\right]=m_V.
\end{aligned}
$$

which implies that u is a minimum point for I_V on $\mathcal{N}_V$ since $u\neq 0$. Therefore u is a ground state solution of the Equation (1.5) through the definition of the ground state.

4. Conclusion

In this paper, we use the Nehari manifold technique to prove the existence of ground state solutions for a class of Schrödinger-Kirchhoff equations with vanishing potential and exponential growth.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Kirchhoff, G. (1883) Vorlesungen Uber Mechanik. Teubner, Leipzig.

[2] Lions, J.L. (1978) On Some Questions in Boundary Value Problems of Mathematical Physics. *North-Holland Mathematics Studies*, **30**, 284-346. https://doi.org/10.1016/S0304-0208(08)70870-3

[3] Perera, K. and Zhang, Z. (2006) Nontrivial Solutions of Kirchhoff-Type Problems via the Yang Index. *Journal of Differential Equations*, **221**, 246-255. https://doi.org/10.1016/j.jde.2005.03.006

[4] He, X.-M. and Zou, W.-M. (2010) Multiplicity of Solutions for a Class of Kirchhoff Type Problems. *Acta Mathematicae Applicatae Sinica*, *English Series*, **26**, 387-394. https://doi.org/10.1007/s10255-010-0005-2

[5] Sun, J. and Liu, S. (2012) Nontrivial Solutions of Kirchhoff Type Problems. *Applied Mathematics Letters*, **25**, 500-504. https://doi.org/10.1016/j.aml.2011.09.045

[6] Trudinger, N.S. (1967) On Imbeddings into Orlicz Spaces and Some Applications. *Indiana University Mathematics Journal*, **17**, 473-483. https://doi.org/10.1512/iumj.1968.17.17028

[7] Moser, J. (1970) A Sharp Form of an Inequality by N. Trudinger. *Indiana University Mathematics Journal*, **20**, 1077-1092. https://doi.org/10.1512/iumj.1971.20.20101

[8] Cao, D.M. (1992) Nontrivial Solution of Semilinear Elliptic Equations with Critical Exponent in R. *Communications in Partial Differential Equations*, **17**, 407-435. https://doi.org/10.1080/03605309208820848

[9] do Ó, J.M. (1997) N-Laplacian Equations in $\mathbb{R}^N$ with Critical Growth. *Abstract and Applied Analysis*, **2**, 301-315. https://doi.org/10.1155/S1085337597000419

[10] do Ó, J.M., de Souza, M., Medeiros, E. and Severo, U. (2014) An Improvement for the Trudinger-Moser Inequality and Applications. *Journal of Differential Equations*, **256**, 1317-1329. https://doi.org/10.1016/j.jde.2013.10.016

[11] Wu, X. (2011) Existence of Nontrivial Solutions and High Energy Solutions for Schrödinger-Kirchhoff-Type Equations in $\mathbb{R}^N$. *Nonlinear Analysis*: *Real World Applications*, **12**, 1278-1287. https://doi.org/10.1016/j.nonrwa.2010.09.023

[12] Futtado, M. and Zanata, H. (2021) Kirchhoff-Schrödinger Equations in $\mathbb{R}^2$ with Critical Exponential Growth and Indefinite Potential. *Communications in Contemporary Mathematics*, **23**, Article ID: 2050030. https://doi.org/10.1142/S0219199720500303

[13] Chen, L., Lu, G. and Zhu, M. (2021) Sharp Trudinger-Moser Inequality and Ground State Solutions to Quasi-Linear Schrödinger Equations with Degenerate Potentials in $\mathbb{R}^N$. *Advanced Nonlinear Studies*, **21**, 733-749. https://doi.org/10.1515/ans-2021-2146

[14] Masmoudi, N. and Sani, F. (2015) Trudinger-Moser Inequalities with the Exact Growth Condition in $\mathbb{R}^N$ and Applications. *Communications in Partial Differential Equations*, **40**, 1408-1440. https://doi.org/10.1080/03605302.2015.1026775

[15] Wang, X. and Chen, L. (2021) Sharp Weighted Trudinger-Moser Inequalities with the L^n Norm in the Entire Space $\mathbb{R}^n$ and Existence of Their Extremal Functions. *Potential Analysis*, **54**, 153-181. https://doi.org/10.1007/s11118-019-09821-8

[16] Chen, L., Lu, G. and Zhu, M. (2020) Ground States of Bi-Harmonic Equations with Critical Exponential Growth Involving Constant and Trapping Potentials. *Calculus of Variations and Partial Differential Equations*, **59**, Article No. 185. https://doi.org/10.1007/s00526-020-01831-4

[17] Chen, L., Lu, G. and Zhu, M. (2021) A Critical Trudinger-Moser Inequality Involving a Degenerate Potential and Nonlinear Schrödinger Equations. *Science China Mathematics*, **64**, 1391-1410. https://doi.org/10.1007/s11425-020-1872-x

[18] Chen, S., Rădulescu, V.D., Tang, X. and Wen, L. (2022) Planar Kirchhoff Equations with Critical Exponential Growth and Trapping Potential. *Mathematische Zeitschrift*, **302**, 1061-1089. https://doi.org/10.1007/s00209-022-03102-8

[19] Wang, L., Zhang, B.L. and Cheng, K. (2018) Ground State Sign-Changing Solutions for the Schrödinger-Kirchhoff Equation in $\mathbb{R}^3$. *Journal of Mathematical Analysis and Applications*, **466**, 1545-1569. https://doi.org/10.1016/j.jmaa.2018.06.071

[20] Chen, S.X. and Wu, X. (2019) Existence of Positive Solutions for a Class of Quasilin-ear Schrödinger Equations of Choquard Type. *Journal of Mathematical Analysis and Applications*, **475**, 1754-1777. https://doi.org/10.1016/j.jmaa.2019.03.051

[21] Zhou, M.Q. (1995) Real Variable Function. Beijing University Press, Beijing. (In Chinese)