Journal of Applied Mathematics and Physics, 2023, 11, 780-789
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

# Existence of Supercritical Hopf Bifurcation on a Type-Lorenz System

**Evodio Muñoz-Aguirre[1], Jorge Alvarez-Mena[1], Pablo Emilio Calderón-Saavedra[2], Josué Ramírez-Ortega[1], Francisco Gabriel Hernández-Zamora[1]**

[1]Universidad Veracruzana, Xalapa, México
[2]Universidad del Tolima, Ibagué, Colombia
Email: evmunoz@uv.mx, joalvarez@uv.mx, pecalderon@ut.edu.co, josramirez@uv.mx, francischernandez@uv.mx

**How to cite this paper:** Muñoz-Aguirre, E., Alvarez-Mena, J., Calderón-Saavedra, P.E., Ramírez-Ortega, J. and Hernández-Zamora, F.G. (2023) Existence of Supercritical Hopf Bifurcation on a Type-Lorenz System. *Journal of Applied Mathematics and Physics*, **11**, 780-789.
https://doi.org/10.4236/jamp.2023.113052

**Received:** February 22, 2023
**Accepted:** March 25, 2023
**Published:** March 28, 2023



## Abstract

In this paper, a system of Lorenz-type ordinary differential equations is considered and, under some assumptions about the parameter space, the presence of the supercritical non-degenerate Hopf bifurcation is demonstrated. The technical tool used consists of the Central Manifold theorem, a well-known formula to calculate the Lyapunov coefficient and Hopf's Theorem. For particular values of the parameters in the parameter space established in the main result of this work, a graph is presented that describes the evolution of the trajectories, obtained by means of numerical simulation.

## Keywords

Lorenz-Type System, Subcritical Hopf Bifurcation, Supercritical Hopf Bifurcation, Hopf Theorem

## 1. Introduction

Lorenz-Type systems present a great variety of dynamic behaviors such as the presence of chaotic orbits [1] [2], existence of homoclinic and heteroclinic orbits [3] [4] [5], presence of bifurcaciones de Pitch-fork o Hopf [6] [7] [8], as well as the Lorenz attractor [7] [9], among others. An interesting problem is determining the geometric structure of the Lorenz attractor for specific Lorenz-type systems.

Hopf Bifurcation corresponds to the following situation: when the system parameter is varied and it crosses a critical value, the Jacobian, at equilibrium, has a pair of conjugate complex eigenvalues moving from the left half-plane to the right or vice versa, while the other eigenvalues remain fixed; at the time of crossing, the real parts of the two eigenvalues become zero, the stability of the equilibrium changes from stable to unstable, or from unstable to stable, thus giving rise to a limit cycle. When the limit cycle is stable, the Hopf Bifurcation is supercritical. When the limit cycle is unstable, the Hopf bifurcation is said to be *subcritical* [5]. The Hopf Bifurcation concept is very important and has been studied in various mathematical models of interest, such as the Homogeneous Diffusive Predator-Prey System with Holling Type II and Predator-Prey Model with Mutual Interference, see [10] and [11].

In this paper, it is proposed the study of the Hopf Bifurcation for the Lorenz-type system

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= dy - xz, \\ \dot{z} &= -bz + fx^2 + gxy, \end{aligned} \tag{1}$$

*when* $a > 0$, $f \geq 0$, $g \geq 0$, $f + g > 0$, $b, d \in \mathbb{R}$, and $(x, y, z)^{\mathrm{T}} \in \mathbb{R}^3$ represents the system state variable.

The system (1) is presented by Li and Ou in their article [8] of the year 2011; for this reason it is called *Li-Ou system*. Precisely, in the cited reference the authors demonstrate the existence of the Hopf Bifurcation, leaving open the problem of determining whether the Hopf Bifurcation is nondegenerate, as well as the problem of distinguishing whether such a bifurcation is supercritical or subcritical. In this regard, in 2013 Li and Wang claim that the issue is still open due to its complexity [7], while in 2018, Calderón-Saavedra *et al.*, address the problem for the case where the parameter $f$ is zero, proving the existence of subcritical Hopf Bifurcation [12]. In this paper, the problem is addressed for the case $f > 0$ and it is proved the existence of the supercritical Hopf Bifurcation for this system, and a concrete example is modeled showing the Hopf bifurcation with the behavior of trajectories for a particular system. This is the main result of the work presented here and the methodology developed to solve it is the same as that used by [4] and [13] to calculate the Lyapunov coefficient.

It is clear that there is still a lot to analyze in the system, for example, it would be necessary to distinguish regions where the subcritical Hopf Bifurcation exists, to determine the extension of the regions where the Supercritical Hopf Bifucation is still present. In the same way, it would be necessary to address the Hopf bifurcation control for the system.

Lorenz-type systems are of interest in various topics of physics and engineering, such as synchronization [9], control and Hopf bifurcation control ([13]-[18]), to name a few.

This paper is organized as follows. In Section 2, some preliminary results are described regarding the existence of the Hopf bifurcation in system (1). In Section 3, it is shown that the Hopf bifurcation is nondegenerate and supercritical for some constraints on the parameters. It is illustrated geometrically the behavior of trajectories for a particular system. Finally, in Section 4, the conclusions of this paper are presented.

## 2. Dynamics of the Li-Ou System

### 2.1. Symmetry

The following lemma shows that the Li-Ou system is symmetric with respect to the $Z$ axis.

**Lemma 1** *The system* (*1*) *is invariant under the linear transformation* $T:\mathbb{R}^3 \to \mathbb{R}^3$ *defined by* $T(x,y,z) = (-x,-y,z)$ *for all* $(x,y,z) \in \mathbb{R}^3$.

Proof follows from $F \circ T = T \circ F$ for the vector field $F$ associated with system (1). For a demonstration of this result see [13].

### 2.2. Equilibriums

Analysis of the system begins by determining the equilibrium points, for which the following system of equations is solved:

$$a(y-x) = 0,$$
$$dy - xz = 0,$$
$$gxy - bz + fx^2 = 0.$$

Equilibrium points of (1) are classified according to the following cases.

Case $bd < 0$. The origin is the only equilibrium point of the system and is denoted by $P_0$.

Case $bd = 0$. There are two possibilities, $d \neq 0$ or $d = 0$. When $d \neq 0$, it is obtained that $b = 0$, and every point of the form $(0,0,z)$ is an equilibrium of the system. When $d = 0$, the only equilibrium point of the system is $P_0$.

Case $bd > 0$. The system has three equilibrium points:

$$P_0,\ P_1 = \left(\sqrt{\frac{bd}{f+g}}, \sqrt{\frac{bd}{f+g}}, d\right) \text{ y } P_2 = \left(-\sqrt{\frac{bd}{f+g}}, -\sqrt{\frac{bd}{f+g}}, d\right).$$

First two cases are not of interest, the analysis focuses on the case $bd > 0$. In 2011, Li and Ou [8] showed that in $P_0$ the system (1) presents a dynamic without bifurcation, in contrast, at the equilibrium points $P_1$ and $P_2$ the system presents Hopf bifurcation. With the purpose of offering a self-contained work, the result from [8] for the system (1), with $f = 0$, is stated in Theorem 1.

**Theorem 1** *For the system* (*1*) *with parameters* $a > 0$, $b > 0$, $g > 0$ *and* $f = 0$, *the following statements hold*:

*1*) *For* $0 < d < \frac{a+b}{3}$, *the equilibrium points* $P_1$ *and* $P_2$ *are stable.*

*2*) *When* $\frac{a+b}{3} < d$, *the equilibriums* $P_1$ *and* $P_2$ *are unstable.*

*3*) *When* $d = \frac{a+b}{3}$, *in each equilibrium* $P_1$ *and* $P_2$ *arises a periodic orbit with period* $T = \frac{2\pi}{\sqrt{ab}}$.

*Therefore, the system presents a Hopf bifurcation at the equilibrium points* $P_1$ *and* $P_2$ *with bifurcation critical value* $d_-(0) := \frac{a+b}{3}$.

In addition, the result of [8] for the system (1), with $f > 0$, is stated in Theorem 2.

**Theorem 2** *The system* (*1*) *is considered with parameters* $a > 0$, $b > 0$, $f > 0$, $g \geq 0$ *and* $f + g > 0$. *The following statements hold*:

*1*) *When* $0 < d < d_-(f)$, *the equilibriums* $P_1$ *and* $P_2$ *are stable*.

*2*) *When* $d > d_-(f)$, *the equilibriums* $P_1$ *and* $P_2$ *are unstable*.

*3*) *For* $d = d_-(f)$, *an orbit appears at each equilibrium* $P_1$ *and* $P_2$ *with period* $T = \frac{2\pi}{\omega}$, *where* $\omega = \sqrt{\frac{2abd_-(f)}{a+b-d_-(f)}}$.

*Thus, the system presents a Hopf bifurcation at the equilibrium points* $P_1$ *and* $P_2$ *with bifurcation critical value*

$$d_-(f) = \frac{3a(f+g)+(a+b)f}{2f} - \frac{\sqrt{9a^2(f+g)^2 + 2a(f+g)(a+b)f + (a+b)^2 f^2}}{2f}. \tag{2}$$

Analysis of the Hopf bifurcation is very important in the study of the stability of the periodic orbits of a system (see [6] [19] [20]). On the other hand, when the stability of a periodic orbit is not desired, it is possible to disturb the system in order to change its stability. This process is called stability control. In the Hopf bifurcation control, the information that provides the analysis of the bifurcation is considered primary information (see [15] [18] [21] [22]). A Hopf bifurcation analysis consists of determining whether the Hopf bifurcation is nondegenerate, and whether it is the case, to distinguish if it is supercritical or subcritical, in this activity, the first Lyapunov coefficient plays a fundamental role [7] [23] [24].

## 3. Nondegenerate Hopf Bifurcation

### 3.1. Case *f* = 0

This case was studied in [12] where it is shown that the bifurcation Hopf is non-degenerate and supercritical in a specific region of parameters. The following result is for a self-contained presentation of this work.

**Theorem 3** *When the parameters satisfy* $b > 0$, $a = b$, $d = \frac{a+b}{3}$, $g > 0$ *and* $f = 0$, *the system* (*1*) *presents a Hopf bifurcation nondegenerate and supercritical at equilibrium points* $P_1$ *and* $P_2$.

### 3.2. Case *f* > 0

This section presents the main result of this work, which provides two specific regions of system parameters, where the Hopf bifurcation is nondegenerate and supercritical.

Regions in the parameters space of the system are determined (1):

$$R_I : \left\{ (a,b,d,f,g) \in \mathbb{R}^5 \mid a > 0, a = b, d = d_-(f), f = g = 1 \right\}$$

and

$$R_{II}:\left\{(a,b,d,f,g)\in\mathbb{R}^5 \mid a>0, a=b, d=d_-(f), f=1, g=2\right\}.$$

**Theorem 4** *If the system parameters* (*1*) *are in the region* $R_I$ *or the region* $R_{II}$, *the periodic orbits around* $P_1$ *and* $P_2$ *are stable. Therefore, the system* (*1*) *presents Hopf bifurcation non-degenerate and supercritical in R at equilibrium points* $P_1$ *and* $P_2$.

**Proof.** Under the conditions in the parameters, $a>0$, $a=b$, $d=d_-(f)$, $f>0$ and $g\geq 0$, Theorem 2 guarantees the existence of the Hopf bifurcation at equilibrium points $P_1$ and $P_2$. For analysis of the Hopf bifurcation at these equilibrium points, it proceeds as follows. By the symmetry of the system with respect to the $z$ axis (Lemma 1), the critical point $P_1$ is analyzed and the results are extended to the point critical $P_2$. Using the well-known formula for the first Lyapunov coefficient (see page 98 of Y. Kuznetsov [5]) it is determined that the first Lyapunov coefficient is negative at the equilibrium point $P_1$. Finally, it is concluded by the Hopf Theorem (see page 98 of Y. Kuznetsov [5]) that the Hopf bifurcation is supercritical.

Jacobian matrix A of system (1) evaluated at equilibrium $P_1$ is

$$A=\begin{pmatrix} -a & a & 0 \\ -d_-(f) & d_-(f) & -h \\ (2f+g)h & gh & -a \end{pmatrix}, \text{ with } h=\sqrt{\frac{ad_-(f)}{f+g}}.$$

Solving the system $Aq=i\omega q$, the eigenvalue $\lambda_1=i\omega$ is obtained, with eigenvector

$$q=\begin{pmatrix} ah \\ ah+\omega hi \\ \omega^2+(d_-(f)-a)\omega i \end{pmatrix}.$$

The adjoint eigenvector $p\in C^3$, that is, the vector that satisfies the equation $A^{\mathrm{T}}p=-\omega ip$, is

$$p=\begin{pmatrix} \left(\omega^2+gh^2-ad_-(f)\right)+\left(a+d_-(f)\right)\omega i \\ ab-a\omega i \\ -ah \end{pmatrix}.$$

It is necessary to determine a vector parallel to p that satisfies the property $\langle p,q\rangle=1$, hence, vector $p$ is normalized:

$$p=\frac{1}{2ah\omega\left(d_-(f)-a\right)i}\begin{pmatrix} \left(\omega^2+gh^2-ad_-(f)\right)+\left(a+d_-(f)\right)\omega i \\ ab-a\omega i \\ -ah \end{pmatrix}.$$

In order to calculate the first Lyapunov coefficient, the equilibrium point must be transferred to the origin

$$P_1=\left(\sqrt{\frac{ad_-(f)}{f+g}},\sqrt{\frac{ad_-(f)}{f+g}},d_-(f)\right)=\left(h,h,d_-(f)\right).$$

This is done by the transformation $Y = X - P_1$, where $X$ satisfies the system (1). Then $Y$ satisfies the system

$$\dot{Y} = \begin{pmatrix} -a & a & 0 \\ -d_-(f) & d_-(f) & -h \\ (2f+g)h & gh & -a \end{pmatrix} Y + y_1 KY, \tag{3}$$

where $y_1$ is the first $Y$ coordinate and $K$ is the matrix,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ f & g & 0 \end{pmatrix}.$$

It is observed that system (3) has the form $\dot{Y} = AY + F(Y)$, where $A$ is the Jacobian matrix of the system evaluated at equilibrium $P_1$ and the nonlinear part

$$F(Y) = \begin{pmatrix} 0 \\ -y_1 y_3 \\ f y_1^2 + g y_1 y_2 \end{pmatrix}.$$

Thus, $F(Y) = O\left(\|Y\|^2\right)$. On the other hand, the Taylor expansion of $F$ in a neighborhood of $Y = 0$ is expressed by

$$F(Y) = \frac{1}{2} B(Y,Y) + \frac{1}{6} C(Y,Y,Y) + \left(O\|Y\|^4\right),$$

where $B(Y,Y)$ and $C(Y,Y,Y)$ are multilinear vector functions with $Y \in \mathbb{R}^3$. To find an expression for multilinear vector functions $B$ and $C$, the partial derivatives of the components functions $(F_1, F_2, F_3)$ of $F$ are used. The first component function is the zero function, so it does not contribute to the expressions that are searched for. The partial derivatives of $F_2$ are,

$$\frac{\partial F_2}{\partial y_1} = -y_3; \quad \frac{\partial F_2}{\partial y_2} = 0; \quad \frac{\partial F_2}{\partial y_3} = -y_1; \quad \frac{\partial^2 F_2}{\partial y_1^2} = 0;$$

$$\frac{\partial^2 F_2}{\partial y_1 y_2} = 0; \quad \frac{\partial^2 F_2}{\partial y_1 y_3} = -1; \quad \frac{\partial^2 F_2}{\partial y_2 y_1} = 0; \quad \frac{\partial^2 F_2}{\partial y_2^2} = 0;$$

$$\frac{\partial^2 F_2}{\partial y_2 y_3} = 0; \quad \frac{\partial^2 F_2}{\partial y_3 y_1} = -1; \quad \frac{\partial^2 F_2}{\partial y_3 y_2} = 0; \quad \frac{\partial^2 F_2}{\partial y_3^2} = 0.$$

Then the function $B_2$ is expressed in the form

$$B_2(q,q) = q^{\mathrm{T}} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} q = -2ah\left(\omega^2 + (d-a)\omega i\right).$$

The partial derivatives of $F_3$ are used,

$$\frac{\partial F_3}{\partial y_1} = g y_2 + 2 f y_1; \quad \frac{\partial F_3}{\partial y_2} = g y_1; \quad \frac{\partial F_3}{\partial y_3} = 0; \quad \frac{\partial^2 F_3}{\partial y_1^2} = 2f;$$

$$\frac{\partial^2 F_3}{\partial y_1 y_2} = g; \quad \frac{\partial^2 F_3}{\partial y_1 y_3} = 0; \quad \frac{\partial^2 F_3}{\partial y_2 y_1} = g; \quad \frac{\partial^2 F_3}{\partial y_2^2} = 0;$$

$$\frac{\partial^2 F_3}{\partial y_2 y_3}=0;\quad \frac{\partial^2 F_3}{\partial y_3 y_1}=0;\quad \frac{\partial^2 F_3}{\partial y_3 y_2}=0;\quad \frac{\partial^2 F_3}{\partial y_3^2}=0.$$

Then the function $B_3$ is expressed in the form

$$B_3(q,q)=q^{\mathrm{T}}\begin{pmatrix}2f & g & 0\\ g & 0 & 0\\ 0 & 0 & 0\end{pmatrix}q=2ah^2(af+ag+g\omega i).$$

Finally, the expression for the multifunction is obtained

$$B(q,q)=2ah\begin{pmatrix}0\\ -\omega^2-(d_-(f)-a)\omega i\\ h(af+ag+g\omega i)\end{pmatrix}.$$

By a similar process, the following expressions are obtained,

$$B(q,\bar{q})=2ah\begin{pmatrix}0\\ -\omega^2\\ ah(f+g)\end{pmatrix},$$

$$B(\bar{q},\bar{q})=2ah\begin{pmatrix}0\\ -\omega^2+(d_-(f)-a)\omega i\\ h(ag+af-g\omega i)\end{pmatrix},$$

$$C(q,q,\bar{q})=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

The Inverse matrix $A^{-1}$ of $A$ is given by

$$A^{-1}=\frac{1}{h}\begin{pmatrix}\dfrac{-gh^2+bd_-(f)}{2ah(f+g)} & \dfrac{-b}{2h(f+g)} & \dfrac{1}{2(f+g)}\\ \dfrac{bd_-(f)+h^2(2f+g)}{2ah(f+g)} & \dfrac{-b}{2h(f+g)} & \dfrac{1}{2(f+g)}\\ \dfrac{d}{a} & -1 & 0\end{pmatrix}.$$

While the matrix $2i\omega I_3-A$, and its inverse are written below,

$$2i\omega I_3-A=\begin{pmatrix}2i\omega+a & -a & 0\\ d_-(f) & 2i\omega-d_-(f) & h\\ -h(2f+g) & -gh & 2i\omega+b\end{pmatrix}.$$

On the other hand, the inverse matrix $(2i\omega I_3-A)^{-1}$ is given by the expression

$$\frac{1}{r}\begin{pmatrix}\dfrac{-4\omega^2-bd_-(f)+h^2g+2(b-d_-(f))\omega i}{2} & \dfrac{ab+2a\omega i}{2} & \dfrac{-ah}{2}\\ \dfrac{-\left[(bd_-(f)+h^2(2f+g))+2d_-(f)\omega i\right]}{2} & \dfrac{(ab-4\omega^2)+2(a+b)\omega i}{2} & \dfrac{-h(a+2\omega i)}{2}\\ h(-d_-(f)(f+g)+(2f+g)\omega i) & h(a(f+g)+g\omega i) & \omega\left[-2\omega+(a-d_-(f))i\right]\end{pmatrix},$$

with $r=-6abd_{-}(f)-i2\omega\left(4\omega^2+bd_{-}(f)-gh^2-ab\right)$.

With what has been done up to now, it has the elements from the formula of the First Lyapunov Coefficient of [5]. Thus, if the hypotheses $a=b$, $f=g=1$, $d_{-}(f)=\frac{53}{100}a$, $h=\frac{51}{100}a$ and $\omega=\frac{849}{1000}a$ are considered, it is found that the Lyapunov coefficient in $R_I$ is

$$\ell_1(0)=-1.7320a^2<0.$$

With the respective parameters, in the $R_{II}$ region it is found that the Lyapunov coefficient is

$$\ell_1(0)=-1.528219575a^2<0.$$

Value of the coefficient was obtained with a program carried out with the Maple software.

Since $\ell_1(0)$ is negative in the regions $R_I$ and $R_{II}$, as a consequence of Hopf Theorem, the periodic orbit that emerges in the point $P_1$ is stable. Therefore, in both parameter regions, the system (1) presents non-degenerate and supercritical Hopf bifurcation at equilibrium points $P_1$ and $P_2$.

□

**Example 1** *To illustrate the Theorem 4, two particular trajectories are presented together with their graphs of the Li-Ou system, with values of the parameters,* $(a,b,d,f,g)=\left(1,1,\frac{1439}{2500},1,2\right)$. *The chosen values verify the hypotheses,* $b>0$, $b=a$, $f>0$, $g>0$ *and* $d=d_{-}(f)$ *and clearly belong to the* $R_{II}$ *region, therefore, it is verified what ensures the Theorem 4, the presence of a stable periodic orbit.*

*In* **Figure 1**, *it is represented in red the graph of the orbit with initial condition*

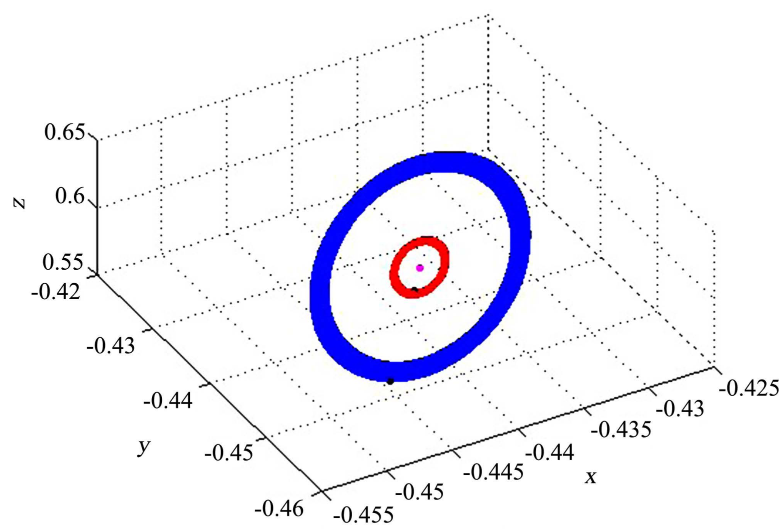


**Figure 1.** Stable limit cycle $(a,b,d,f,g)=\left(1,1,\frac{1439}{2500},1,2\right)$.

$$(x_0, y_0, z_0) = (-0.4473, -0.4544, 0.5863),$$

*and it is represented in blue the graph of the orbit with initial condition*

$$(x_0, y_0, z_0) = (-0.4398, -0.4413, 0.5777).$$

*Graphs of the orbits in* **Figure 1** *give evidence of the existence of a stable periodic orbit.*

## 4. Conclusion

Under some hypotheses in the parameters of the Li-Ou system, it is showed that the Hopf bifurcation, the existence of which has been known since the year 2011 [8], is non-degenerate and supercritical, Theorem 3. For this purpose, the symmetry of the system with respect to the $z$ axis was used to reduce the analysis to only one critical point and the well-known formula for the first Lyapunov coefficient. Theorem 3 is illustrated geometrically, graphically showing the evolution of two trajectories for an instance of the Li-Ou system.

## Acknowledgements

To CONACYT-Mexico for the scholarship granted to Pablo Emilio Calderón Saavedra.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

## References

[1] Lorenz, E. (1963) Deterministic Nonperiodic Flow. *Journal of the Atmospheric Sciences*, **20**, 130-141. https://doi.org/10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2

[2] Lorenz, E.N. (1993) The Essence of Chaos. University of Washington Press, Washington DC. https://doi.org/10.4324/9780203214589

[3] Chen, G., Moiola, J. and Wang, H. (2000) Bifurcation Control: Theories, Methods and Applications. *International Journal of Bifurcation and Chaos*, **10**, 511-548. https://doi.org/10.1142/S0218127400000360

[4] Guckenheimer, J. and Holemes, P. (1990) Nonlinear Oscillations Dynamical Systems and Bifurcations of Vectors Fields. Spriger-Verlag, Berlin.

[5] Kuznetsov, Y.A. (2004) Elements of Applied Bifurcation Theory. 3rd Edition, Springer, New York. https://doi.org/10.1007/978-1-4757-3978-7

[6] Algaba, A., Domínguez, M., Merino, M. and Rodríguez, A.L. (2015) Study of the Hopf Bifurcation in the Lorenz, Chen and Lu Systems. *Nonlinear Dynamics*, **79**, 885-902. https://doi.org/10.1007/s11071-014-1709-2

[7] Li, H. and Wang, M. (2013) Hopf Bifurcation Analysis in a Lorenz-Type Systems. *Nonlinear Dynamical*, **71**, 235-240. https://doi.org/10.1007/s11071-012-0655-0

[8] Li, X. and Ou, Q.J. (2011) Dynamical Properties and Simulation of a New Lorenz-Like Chaotic System. *Nonlinear Dynamical*, **65**, 255-270. https://doi.org/10.1007/s11071-010-9887-z

[9] Chen, G. and Lü, J. (2003) Dynamics of the Lorenz Family: Analysis, Control and Synchronization. Chinese Science Press, Beijing.

[10] Yu, H., Dai, C., Wang, S. and Zhao, M. (2020) Global Asymptotic Stability and Hopf Bifurcation in a Homogeneous Diffusive Predator-Prey System with Holling Type II Functional Response. *Applied Mathematics*, **11**, 389-406. https://doi.org/10.4236/am.2020.115028

[11] Yang, Y., Mohamed, A., Xia, S., Abkar, M., Abbakar, K. and Hassan, O. (2021) Stability Analysis and Hopf Bifurcation for Ode System of Predator-Prey Model with Mutual Interference. *Applied Mathematics*, **12**, 793-802. https://doi.org/10.4236/am.2021.129053

[12] Calderón-Saavedra, P.E., Alvarez-Mena, J. and Muñoz-Aguirre, E. (2018) Tratamiento analítico de la bifurcación de hopf en una extensión del sistema de lü. *Revista de Matemática*: *Teoría y Aplicaciones*, **25**, 29. https://doi.org/10.15517/rmta.v1i25.32230

[13] Calderon-Saavedra, P.E., Muñoz-Aguirre, E., Alvarez-Mena, J. and G'omez-Pérez, S. (2018) Hopf Bifurcation Control in a Lorenz Type System. *Journal of Applied Mathematics and Physics*, **6**, 1704-1719. https://doi.org/10.4236/jamp.2018.68146

[14] Cruz Villar, C.A. (2007) Optimal Stabilization of Unstable Periodic Orbits Embedded Chaotic Systems. *Revista Mexicana de Física*, **53**, 415-420.

[15] Pyragas, K. (2006) Delayed Feedback Control of Chaos. *Philosophical Transactions of the Royal Society A*, **364**, 2309-2334. https://doi.org/10.1098/rsta.2006.1827

[16] Saito, T., Itoh, S. and Yagi, M. (2004) Control of Chaos by Linear and Nonlinear Feedback Methods. *Journal of Plasma and Fusion Research Series*, **6**, 263-266.

[17] Yu, P. and Chen, G. (2004) Hopf Bifurcation Control Using Nonlinear Feedback with Polynomial Functions. *International Journal of Bifurcation and Chaos*, **14**, 1683-1704. https://doi.org/10.1142/S0218127404010291

[18] Yu, P. and Lü, J. (2011) Bifurcation Control for a Class of Lorenz-Like Systems. *International Journal of Bifurcation and Chaos*, **21**, 2647-2664. https://doi.org/10.1142/S0218127411030003

[19] Castellanos, V., Blé, G. and Llibre, J. (2016) Existence of Limit Cycles in a Tritrophic Food Chain Model with Holling Functional Responses of Type II and III. *Mathematical Methods in the Applied Sciences*, **39**, 3996-4006. https://doi.org/10.1002/mma.3842

[20] Sotomayor, J., Melo, L. and Braga, D. (2006) Stability and Hopf Bifurcation in the Watt Governor Systems. Cornell University Library, Ithaca.

[21] Liao, X., Wang, L. and Yu, P. (2007) Stability of Dynamical Systems. Elsevier, London. https://doi.org/10.1016/S1574-6917(07)05001-5

[22] Tian, Y.-P. (2012) Frequency-Domain Analysis and Design of Distributed Control Systems. Wiley, Hoboken. https://doi.org/10.1002/9780470828229

[23] Dutta, T.K., Prajapati, P. and Haloi, S. (2016) Supercritical and Subcritical Hopf Bifurcations in Non-Linear Maps. *International Journal of Innovative Research in Technology & Science*, **4**, 14-20.

[24] García, J. (2005) Análisis de Bifurcaciones en Modelos de Calidad Ambiental. Tesis Doctoral, Universidad de las Palmas de Gran Canaria, Las Palmas.