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A Vector Tensor Calculus Description of a Euclidean Space

Pavel Grinfeld

Drexel University, Philadelphia, PA, USA
Email: pavelgrinfeld@gmail.com

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Abstract

We present a tensor description of Euclidean spaces that emphasizes the use of geometric vectors which leads to greater geometric insight and a higher degree of organization in analytical expressions. We demonstrate the effectiveness of the approach by proving a number of integral identities with vector integrands. The presented approach may be aptly described as *absolute vector calculus* or as *vector tensor calculus*.

Keywords

Tensor Calculus, Differential Geometry, Embedded Surfaces and Curves, Scalar Curvature, Gaussian Curvature

1. Introduction

Since the invention of coordinate systems in the middle of the 17th century, the subject of geometry has followed the steady path of algebraization. This is not surprising: the very idea of a coordinate system is to represent geometric objects by their coordinates, or, in the case of vector quantities, by their components with respect to the coordinate basis, thus opening up the problem to analytical methods. This is the distinct advantage of the method of coordinates over the geometric approach: while geometric arguments usually require unique insights and therefore a certain degree of ingenuity, analytical methods lean towards universality and robustness. The advent of computing has further cemented this advantage.

That being said, the use of coordinates in general theoretical investigations comes with significant costs. Chief among them is the loss of geometric insight. This shortcoming can be illustrated by examining some of the classical results of Leonhard Euler and Louis Lagrange found in their celebrated works on the calculus of variations.

In 1744, in his search for a minimal shape of revolution, Euler introduced what we would now call a *cylindrical coordinate system* (see **Figure 1**), and described the profile of the minimal surface by an unknown function $r(z)$. By making arguments that were largely *geometric* in their nature, Euler demonstrated that $r(z)$ must satisfy the equation

$$r''(z)r(z)-r'(z)^2-1=0 \tag{1}$$

whose solution

$$r(z)=a\cosh\frac{z}{a} \tag{2}$$

reveals that the desired shape is a *catenary* for which the closest distance between the surface and the axis of revolution is *a*.

In 1755, Lagrange took an even more unapologetically coordinate approach to the problem of minimal surfaces when he represented the unknown surface by the graph of a function $F(x,y)$ in Cartesian coordinates, as illustrated in **Figure 2**. Reasoning *analytically*, Lagrange demonstrated that $F(x,y)$ must satisfy the partial differential equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial x^2}\left(\frac{\partial F}{\partial y}\right)^2+\frac{\partial^2 F}{\partial y^2}\left(\frac{\partial F}{\partial x}\right)^2-2\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}\frac{\partial^2 F}{\partial x\partial y}=0, \tag{3}$$

for which a closed form solution exists only in the most special cases.

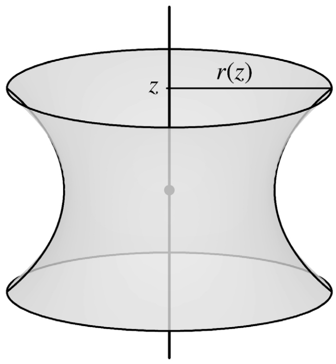

Figure 1. Euler's approach to minimal surfaces, where a surface of revolution is represented in cylindrical coordinates by $r=r(z)$.

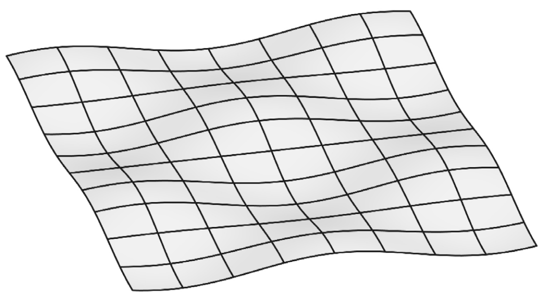

Figure 2. Lagrange's approach to minimal surfaces where a surface represented by the graph of a function $F(x,y)$ in Cartesian coordinates.

These works of Euler and Lagrange have been rightfully revered by later mathematicians for the seminal nature of the techniques developed in them as well as for the flight of creativity that their development required. To us, however, as we contemplate the relative advantages and disadvantages of the method of coordinates, these works offer an additional insight that speaks to the potential loss of geometric interpretation that comes with the use of coordinate systems. It appears that neither Euler nor Lagrange was aware of the *geometric meaning* of their equations. It was only in 1774 that the French mathematician Jean Baptiste Meusnier discovered that a minimal surface is characterized by *zero mean curvature*, a quantity of which Euler and Lagrange had complete mastery.

Crucially, loss of geometric insight is not merely an aesthetic lament: geometric insight serves as a guide for most analytical explorations and, in particular, for organizing analytical expressions into meaningful and manageable combinations. With the loss of geometric insight, we lose control over our calculations, and the resulting growth in the complexity of analytical expressions inevitably forces us to retreat in the face of computational challenges.

In response to the loss of geometric insight, two distinct approaches have been developed: the *tensor calculus* approach and the *dyadic* approach favored by modern differential geometers. Tensor calculus, which relies on the use of coordinates, preserves the geometric insight by providing a framework of *invariance* that dictates precise rules for combing analytical expressions into geometrically meaningful combinations. By contrast, the dyadic approach eschews components altogether and operates strictly in terms of geometric objects. Naturally, both approaches have their uses and misuses, their advantages and disadvantages, as well as their adherents and detractors - note the orgies-of-indices vs. the orgies-of-formalism debate based on the quotes from Elie Cartan and Hermann Weyl The truth, of course, as Cartan's and Weyl's teach us, is that elements of both approaches are essential, and the two schools of thoughts complement, rather than conflict, each other.

The goal of this paper is to describe a *vector tensor calculus*, *i.e.* a particular style of the tensor treatment of a Euclidean space that combines elements of both the tensor and dyadic approaches by emphasizing the use of geometric vectors, by which we mean a directed segment, in a tensor calculus setting. Some elements of this approach can be found in V. F. Kagan's *Foundations of the Theory of Surfaces in Tensor Terms* [1]. However, Kagan typically uses geometric vector quantities only at the outset of any particular discussion and quickly abandons them in favor of working with their components. By contrast, we will take the analysis of vector quantities much further and discover their tremendous utility in simplifying the description of Euclidean spaces and in revealing new and insightful relationships.

Another noteworthy aspect of Kagan's work that deserves to be mentioned is its determination to communicate the essential ideas in a way that favors transparency over the technical details. Interesting, and relevant to the goals of this paper, is the fact that this element of Kagan's style displeased some of his contemporaries. In an otherwise positive review [2], A. D. Alexandrov criticized Kagan's lack of formalism:

… Other shortcomings of this book have to do with the prevalence of the tensor framework. They manifest themselves in insufficient attention to the precise definitions of concepts as well as to the specification of assumptions required for validity of theorems.

It is not my goal to criticize the work of V. F. Kagan. Such deficiencies are characteristic of an entire direction in differential geometry and can be found in the majority of books devoted to this field. They have become a matter of style that I find anachronistic, as our present notion of rigor is different from that of, say, the middle of the nineteenth century.

With Alexandrov's remarks duly noted, we will follow the spirit of Kagan's classic and favor clarity over formalism. Our main goal is to show that an emphasis on geometric vectors can provide greater insight into the structure of a Euclidean space, offer more elegant demonstrations of known results, and open doors to new results. For the more traditional approach to the tensor description of a Euclidean space, see the classical textbooks [3] [4] [5].

2. Summary of Demonstrated Identities

As an illustration of the effectiveness of the proposed approach, we will demonstrate a family of integral relationships on a smooth hypersurface S in an n-dimensional Euclidean space, where S is characterized by the *unit normal field* $\boldsymbol{N}$, *mean curvature* B_α^α, and *scalar curvature* R. A natural approach to extending the concept of geometric vectors to higher-dimensional Euclidean spaces is described in Chapter 20 of [6]. Also note that for a two-dimensional hypersurface, the scalar curvature R reduces to twice the Gaussian curvature K.

As with most integral identities in vector calculus, the presented formulas are direct corollaries of the divergence theorem. However, not all frameworks make the application of the divergence theorem straightforward or the results of its application subject to a geometric interpretation. This is almost surely the reason why some of the presented identities appear to have heretofore eluded discovery.

Assuming that S is sufficiently smooth, we will first demonstrate the well-known fact that the surface integral of the unit normal over a closed surface vanishes, *i.e.*

$$\int_S \boldsymbol{N}\,\mathrm{d}S = \boldsymbol{0}. \tag{4}$$

Similarly, we will show that the surface integral of the *curvature normal* $\boldsymbol{N}B_\alpha^\alpha$ also vanishes, *i.e.*

$$\int_S \boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = \boldsymbol{0}. \tag{5}$$

Finally, we will demonstrate that the surface integral of the invariant $\boldsymbol{N}R$ vanishes as well, *i.e.*

$$\int_S \boldsymbol{N}R\,\mathrm{d}S = \boldsymbol{0}, \tag{6}$$

which, in the case of a two-dimensional hypersurface, becomes

$$\int_S NK\,\mathrm{d}S = \mathbf{0}. \tag{7}$$

Each of the above identities will be demonstrated by expressing the integrand as the surface divergence of a first-order tensor (with vector elements) and thus making it subject to a straightforward application of the surface divergence theorem.

For the next set of integral identities, introduce the position vector $\boldsymbol{R}$ emanating from an arbitrary origin *O*. (Note that the scalar curvature R is entirely unrelated to the magnitude of the position vector $\boldsymbol{R}$.) Observe that for any vector field $\boldsymbol{U}$ whose integral over a closed surface S vanishes, *i.e.*

$$\int_S \boldsymbol{U}\,\mathrm{d}S = \mathbf{0}, \tag{8}$$

it is natural to inquire as to the value of the integral

$$\int_S \boldsymbol{R}\cdot\boldsymbol{U}\,\mathrm{d}S \tag{9}$$

since it is independent of the location of *O*. Indeed, if $\boldsymbol{R}'$ is the position vector field emanating from an alternative origin *O*', then

$$\boldsymbol{R}' = \boldsymbol{R} + \boldsymbol{d}, \tag{10}$$

where $\boldsymbol{d}$ points from *O*' to *O*, and we have

$$\int_S \boldsymbol{R}'\cdot\boldsymbol{U}\mathrm{d}S = \int_S (\boldsymbol{R}+\boldsymbol{d})\cdot\boldsymbol{U}\,\mathrm{d}S \tag{11}$$

$$= \int_S \boldsymbol{R}\cdot\boldsymbol{U}\,\mathrm{d}S + \boldsymbol{d}\cdot\int_S \boldsymbol{U}\,\mathrm{d}S \tag{12}$$

$$= \int_S \boldsymbol{R}\cdot\boldsymbol{U}\,\mathrm{d}S. \tag{13}$$

Independence from the location of *O* suggests that the integral

$$\int_S \boldsymbol{R}\cdot\boldsymbol{U}\,\mathrm{d}S \tag{14}$$

represents a geometric characteristic of the surface *S*. Indeed, for each of the vector fields $\boldsymbol{N}$, $\boldsymbol{N}B_\alpha^\alpha$, and $\boldsymbol{N}R$, the surface integral of the dot product with the position vector yields an interesting geometric quantity. Namely,

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}\,\mathrm{d}S = nV \tag{15}$$

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = -(n-1)A \tag{16}$$

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}K\,\mathrm{d}S = -\frac{1}{2}\int_S B_\alpha^\alpha\,\mathrm{d}S, \tag{17}$$

where V is the volume of the domain enclosed by *S*, A is the surface area of *S*, and B_α^α is, again, the *mean curvature*.

Note that the same logic applies to the cross product integral

$$\int_S \boldsymbol{R}\times\boldsymbol{U}\,\mathrm{d}S. \tag{18}$$

However, somewhat disappointingly, we will find that

$$\int_S \boldsymbol{R}\times\boldsymbol{N}\,\mathrm{d}S = \int_S \boldsymbol{R}\times\boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = \int_S \boldsymbol{R}\times\boldsymbol{N}K\,\mathrm{d}S = \mathbf{0}. \tag{19}$$

3. A Vector Tensor Calculus Description of a Euclidean Space and of Hypersurfaces within It

We will now describe the fundamental elements of a vector tensor calculus - a flavor of conventional tensor calculus with an emphasis on geometric vectors. Since it is not possible to present a full account in the limited available space, we will only give the definitions of the key objects, state their fundamental properties, and list the essential identities relevant to our narrative. A detailed description can be found in [6].

The presented approach works in Euclidean spaces of arbitrary dimension *n*. However, for the sake of specificity and simplicity, we will mostly limit our description to the three-dimensional space since the generalization to arbitrary dimension is completely straightforward.

Refer the ambient Euclidean space to arbitrary curvilinear coordinates Z^1, Z^2, Z^3 or, collectively, Z^i, and treat the position vector $\boldsymbol{R}$ as a function of Z^i, *i.e.*

$$\boldsymbol{R} = \boldsymbol{R}(Z), \tag{20}$$

where the shorthand symbol $\boldsymbol{R}(Z)$ represents the function $\boldsymbol{R}\left(Z^1, Z^2, Z^3\right)$. Then the *covariant basis* $\boldsymbol{Z}_i$, the *contravariant basis* $\boldsymbol{Z}^i$, the *covariant metric tensor* Z_{ij}, and the *contravariant metric tensor* Z^{ij} are given by the identities

$$\boldsymbol{Z}_i = \frac{\partial \boldsymbol{R}(Z)}{\partial Z^i} \tag{21}$$

$$\boldsymbol{Z}^i \cdot \boldsymbol{Z}_j = \delta^i_j \tag{22}$$

$$Z_{ij} = \boldsymbol{Z}_i \cdot \boldsymbol{Z}_j \tag{23}$$

$$Z^{ij} Z_{jk} = \delta^i_k, \tag{24}$$

where δ^i_k is the familiar *Kronecker delta symbol*. Note that in an *n*-dimensional space,

$$\delta^i_i = n. \tag{25}$$

Collectively, the objects $\boldsymbol{Z}_i$, $\boldsymbol{Z}^i$, Z_{ij}, Z^{ij}, δ^i_j, along with the *Levi-Civita symbols* ε_{ijk} and ε^{ijk}, are referred to as the *metrics* or, more specifically, the *ambient metrics*.

Suppose that a hypersurface *S* is referred to the surface coordinates S^1, S^2 or, collectively S^α. Treat the surface restriction of the position vector $\boldsymbol{R}$ as a function of S^α, *i.e.*

$$\boldsymbol{R} = \boldsymbol{R}(S), \tag{26}$$

where, again, the shorthand symbol $\boldsymbol{R}(S)$ represents the function $\boldsymbol{R}\left(S^1, S^2\right)$.

Following the ambient footprint, the *covariant basis* $\boldsymbol{S}_\alpha$, the *contravariant basis* $\boldsymbol{S}^\alpha$, the *covariant metric tensor* $S_{\alpha\beta}$, and the *contravariant metric tensor* $S^{\alpha\beta}$ are given by the identities

$$\boldsymbol{S}_\alpha = \frac{\partial \boldsymbol{R}(S)}{\partial S^\alpha} \tag{27}$$

$$\boldsymbol{S}^{\alpha} \cdot \boldsymbol{S}_{\beta} = \delta^{\alpha}_{\beta} \tag{28}$$

$$S_{\alpha\beta} = \boldsymbol{S}_{\alpha} \cdot \boldsymbol{S}_{\beta} \tag{29}$$

$$S^{\alpha\beta} S_{\beta\gamma} = \delta^{\alpha}_{\gamma}, \tag{30}$$

where δ^{α}_{β} is again the familiar Kronecker delta symbol but this time defined on the surface. Note that on a hypersurface in an n-dimensional space, we have

$$\delta^{\alpha}_{\alpha} = n-1. \tag{31}$$

Collectively, the objects $\boldsymbol{S}_{\alpha}$, $\boldsymbol{S}^{\alpha}$, $S_{\alpha\beta}$, $S^{\alpha\beta}$, δ^{α}_{β}, along with the Levi-Civita symbols $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$, are also referred to as the *metrics* or, more specifically, the *surface metrics*.

The ambient components U^{i} of a vector $\boldsymbol{U}$ are given by the dot product of $\boldsymbol{U}$ with the contravariant basis vectors $\boldsymbol{Z}^{i}$, *i.e.*

$$U^{i} = \boldsymbol{Z}^{i} \cdot \boldsymbol{U}. \tag{32}$$

Similarly, the surface components U^{α} of a vector $\boldsymbol{U}$ in the plane tangential to surface S are given by the dot product of $\boldsymbol{U}$ with the surface contravariant basis vectors $\boldsymbol{S}^{\alpha}$, *i.e.*

$$U^{\alpha} = \boldsymbol{S}^{\alpha} \cdot \boldsymbol{U}. \tag{33}$$

The shift tensor Z^{i}_{α} represents the ambient coordinates of the surface covariant basis $\boldsymbol{S}_{\alpha}$, *i.e.*

$$Z^{i}_{\alpha} = \boldsymbol{Z}^{i} \cdot \boldsymbol{S}_{\alpha}. \tag{34}$$

The shift tensor is a critical object in the traditional approach to tensor calculus, but will not figure in our analysis.

The unit normal $\boldsymbol{N}$ is given by the identity

$$\boldsymbol{N} = \frac{1}{2} \varepsilon^{\alpha\beta} \boldsymbol{S}_{\alpha} \times \boldsymbol{S}_{\beta}, \tag{35}$$

where $\varepsilon^{\alpha\beta}$ is the surface *Levi-Civita symbol*. Of the two available normal directions, the above formula chooses the one such that the set $\boldsymbol{S}_1, \boldsymbol{S}_2, \boldsymbol{N}$ is positively oriented. Clearly, the above definition of the unit normal $\boldsymbol{N}$ is valid only for a two-dimensional hypersurface and it is, indeed, a straightforward matter to generalize it to higher (and lower) dimensions.

The (ambient) *covariant derivative* ∇_k is a differential operator that preserves the tensor property of its inputs. It satisfies the product rule, the sum rule, and the metrinilic property with respect to all the ambient metrics, *i.e.*

$$\nabla_k Z_{ij}, \nabla_k Z^{ij}, \nabla_k \delta^{i}_{j}, \nabla_k \varepsilon_{ijk}, \nabla_k \varepsilon^{ijk} = 0 \tag{36}$$

$$\nabla_k \boldsymbol{Z}_i, \nabla_k \boldsymbol{Z}^i = \boldsymbol{0} \tag{37}$$

In affine coordinates, the covariant derivative ∇_k coincides with the partial derivative $\partial / \partial Z^k$. The same is true of ∇_k with respect to tensors of order zero, *i.e.* invariants, in arbitrary coordinates. In particular, the covariant basis $\boldsymbol{Z}_i$ can be expressed in terms of the covariant derivative, *i.e.*

$$\boldsymbol{Z}_i = \nabla_i \boldsymbol{R}. \tag{38}$$

The *surface covariant derivative* ∇_γ is a differential operator that applies to objects defined on the surface. It, too, is distinguished by the property that it preserves the tensor property of its inputs. It satisfies the sum rule, the product rule, and the metrinilic property with respect to ambient metrics, *i.e.*

$$\nabla_\gamma Z_{ij}, \nabla_\gamma Z^{ij}, \nabla_\gamma \delta^i_j, \nabla_\gamma \varepsilon_{ijk}, \nabla_\gamma \varepsilon^{ijk} = 0, \tag{39}$$

and most of the surface metrics, *i.e.*

$$\nabla_\gamma S_{\alpha\beta}, \nabla_\gamma S^{\alpha\beta}, \nabla_\gamma \delta^\alpha_\beta, \nabla_\gamma \varepsilon_{\alpha\beta}, \nabla_\gamma \varepsilon^{\alpha\beta} = 0, \tag{40}$$

with the notable exception of the surface bases $\boldsymbol{S}_\alpha$ and $\boldsymbol{S}^\alpha$. It coincides with the partial derivative $\partial/\partial S^\gamma$ in affine coordinates, provided that the surface admits such coordinates. It also coincides with the partial derivative for tensors of order zero in arbitrary coordinates. In particular,

$$\boldsymbol{S}_\alpha = \nabla_\alpha \boldsymbol{R}. \tag{41}$$

The ambient version of the *divergence theorem* states that the volume integral of an invariant quantity $\nabla_i T^i$, known as the *divergence* of T^i, equals the surface integral of the invariant quantity $N_i T^i$, *i.e.*

$$\int_\Omega \nabla_i T^i \mathrm{d}\omega = \int_S N_i T^i \mathrm{d}S, \tag{42}$$

where Ω is a domain enclosed by the surface *S*, and N_i are the ambient components of the *exterior* normal $\boldsymbol{N}$. Note that the divergence theorem remains valid for tensors $\boldsymbol{T}^i$ with vector elements.

The *surface* divergence theorem applies to a patch *S* with a boundary *L* and states that the surface integral of the invariant quantity $\nabla_\alpha T^\alpha$, known as the *surface divergence* of T^α, equals the boundary integral of the invariant $n_\alpha T^\alpha$, *i.e.*

$$\int_S \nabla_\alpha T^\alpha \mathrm{d}S = \int_L n_\alpha T^\alpha \mathrm{d}L, \tag{43}$$

where n_α are the surface components of the exterior unit normal $\boldsymbol{n}$ to the boundary *L* that lies in the plane tangent to the surface. Once again, the theorem remains valid for a tensor $\boldsymbol{T}^\alpha$ with vector elements.

The concept of curvature arises in the analysis of the vectors $\nabla_\alpha \boldsymbol{S}_\beta$. While $\nabla_\alpha \boldsymbol{S}_\beta$ do not vanish, they are orthogonal to the surface and can therefore be written in the form

$$\nabla_\alpha \boldsymbol{S}_\beta = \boldsymbol{N} B_{\alpha\beta}, \tag{44}$$

where $B_{\alpha\beta}$ is known as the *curvature tensor*. Note that the sign of $B_{\alpha\beta}$ depends on the choice of the orientation of $\boldsymbol{N}$. Thanks to Equation (41), the combination $\boldsymbol{N} B_{\alpha\beta}$, which we will refer to as the *vector curvature tensor*, is given in terms of the position vector $\boldsymbol{R}$ by the identity

$$\boldsymbol{N} B_{\alpha\beta} = \nabla_\alpha \nabla_\beta \boldsymbol{R}. \tag{45}$$

From this equation, it immediately follows that the curvature tensor $B_{\alpha\beta}$ is symmetric, *i.e.*

$$B_{\alpha\beta} = B_{\beta\alpha}. \tag{46}$$

The invariant B_α^α is known as the *mean curvature*.

Raising the subscript β in Equation (45) and contracting with α yields

$$\boldsymbol{N}B_\alpha^\alpha = \nabla_\alpha \nabla^\alpha \boldsymbol{R}. \tag{47}$$

The vector quantity $\boldsymbol{N}B_\alpha^\alpha$ is known as the *curvature normal* by analogy with the like-named characteristic of a curve. In fact, for any smooth curve in an n-dimensional Euclidean space, the quantity $\nabla_\alpha \nabla^\alpha \boldsymbol{R}$ represents the curve's curvature normal conventionally defined as $\mathrm{d}^2\boldsymbol{R}(s)/\mathrm{d}s^2$. Also note that the orientation of the curvature normal does not depend on which of the two available orientations for $\boldsymbol{N}$ is chosen.

In words, the above identity states that the curvature normal of a surface is the surface Laplacian of the position vector. Alternatively, it may be described as the surface divergence of the contravariant basis vector $\boldsymbol{S}^\alpha$, *i.e.*

$$\boldsymbol{N}B_\alpha^\alpha = \nabla_\alpha \boldsymbol{S}^\alpha. \tag{48}$$

By expressing the curvature normal as the surface divergence of another quantity, Equation (48) virtually anticipates the fact that the integral of the curvature normal over a closed surface S vanishes.

The covariant derivative of the unit normal $\boldsymbol{N}$ is given by the *Weingarten equation*

$$\nabla^\alpha \boldsymbol{N} = -\boldsymbol{S}^\beta B_\beta^\alpha. \tag{49}$$

One of the most elegant identities involving the curvature tensor is the *Gauss equations of the surface* which read

$$B_{\alpha\gamma}B_{\beta\delta} - B_{\alpha\delta}B_{\beta\gamma} = R_{\alpha\beta\gamma\delta}, \tag{50}$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann-Christoffel tensor. For a two-dimensional hypersurface in a three-dimensional Euclidean space, the Gauss equations reduce to the form

$$B_{\alpha\gamma}B_{\beta\delta} - B_{\alpha\delta}B_{\beta\gamma} = K\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}, \tag{51}$$

where K is the Gaussian curvature and $\varepsilon_{\alpha\beta}$ is again the Levi-Civita symbol. Raising the subscripts α and β yields

$$B_\gamma^\alpha B_\delta^\beta - B_\delta^\alpha B_\gamma^\beta = K\delta_{\gamma\delta}^{\alpha\beta}, \tag{52}$$

where $\delta_{\gamma\delta}^{\alpha\beta} = \varepsilon^{\alpha\beta}\varepsilon_{\gamma\delta}$ is known as the *second-order delta system*.

Note that for *any* second-order system A_β^α in two dimensions we have

$$A_\gamma^\alpha A_\delta^\beta - A_\delta^\alpha A_\gamma^\beta = A\delta_{\gamma\delta}^{\alpha\beta}, \tag{53}$$

where A is the determinant of A_β^α. Therefore, on a two-dimensional hypersurface, the Gauss equations are equivalent to the statement that the Gaussian curvature equals the determinant B of B_β^α, *i.e.*

$$K = B. \tag{54}$$

In fact, an interesting generalization of the Gauss-Bonnet theorem to a closed hypersurface in an arbitrary-dimensional Euclidean space is the statement that the surface integral

$$\int_S B\mathrm{d}S \tag{55}$$

depends on the topology of S but not its shape.

In n dimensions, write the Gauss equations

$$B_{\alpha\gamma}B_{\beta\delta} - B_{\alpha\delta}B_{\beta\gamma} = R_{\alpha\beta\gamma\delta} \tag{50}$$

with α and β as superscripts, *i.e.*

$$B^{\alpha}_{\gamma}B^{\beta}_{\delta} - B^{\alpha}_{\delta}B^{\beta}_{\gamma} = R^{\alpha\beta}_{\cdot\cdot\gamma\delta}, \tag{56}$$

and contract α with γ and β with δ, *i.e.*

$$B^{\alpha}_{\alpha}B^{\beta}_{\beta} - B^{\alpha}_{\beta}B^{\beta}_{\alpha} = R^{\alpha\beta}_{\cdot\cdot\alpha\beta}. \tag{57}$$

The invariant

$$R = R^{\alpha\beta}_{\cdot\cdot\alpha\beta} \tag{58}$$

is known as the *scalar curvature* and thus we have

$$B^{\alpha}_{\alpha}B^{\beta}_{\beta} - B^{\alpha}_{\beta}B^{\beta}_{\alpha} = R. \tag{59}$$

In words, the difference between the square of the trace of the curvature tensor B^{α}_{β} and the trace of the *third fundamental form* $B^{\alpha}_{\beta}B^{\beta}_{\gamma}$ equals the scalar curvature.

Closely related to the Gauss equations, are the *Codazzi equations*

$$\nabla_{\alpha}B_{\beta\gamma} = \nabla_{\beta}B_{\alpha\gamma}, \tag{60}$$

which, in combination with the symmetry of $B_{\beta\gamma}$, imply that the tensor $\nabla_{\alpha}B_{\beta\gamma}$ is symmetric in all of its subscripts. Below, we will use the immediate consequence of the Codazzi equations obtained by raising the index γ and contracting it with α, *i.e.*

$$\nabla_{\alpha}B^{\alpha}_{\beta} = \nabla_{\beta}B^{\alpha}_{\alpha}. \tag{61}$$

4. Demonstrations of the Integral Identities

4.1. The Integral $\int_S \boldsymbol{N}\mathrm{d}S$

With the help of the ambient and the surface versions of the divergence theorems we will now prove, in two different ways, the fact that the integral of the unit normal $\boldsymbol{N}$ vanishes, *i.e.*

$$\int_S \boldsymbol{N}\,\mathrm{d}S = \boldsymbol{0}. \tag{4}$$

Indeed, since $\boldsymbol{N} = N^i\boldsymbol{Z}_i$, the ambient version of the divergence theorem tells that

$$\int_S \boldsymbol{N}\,\mathrm{d}S = \int_S N^i\boldsymbol{Z}_i\,\mathrm{d}S = \int_\Omega \nabla^i\boldsymbol{Z}_i\,\mathrm{d}Z. \tag{62}$$

We now observe that the integrand in the last integral vanishes by the metrinilic property (36) and thus the proof is complete.

One dissatisfying aspect of this proof is the fact that it engages elements from the ambient space while the identity

$$\int_S \boldsymbol{N}\,\mathrm{d}S = \boldsymbol{0} \tag{4}$$

itself includes only fields defined on the surface *S*. Therefore, it will behoove us to construct a proof that likewise only involves fields defined on *S*. In its literal form, the following proof applies only to a two-dimensional hypersurface but can be easily generalized to arbitrary dimension.

Recall that the normal $\boldsymbol{N}$ is given by the identity

$$\boldsymbol{N} = \frac{1}{2}\varepsilon^{\alpha\beta}\boldsymbol{S}_\alpha \times \boldsymbol{S}_\beta. \tag{35}$$

Since

$$\boldsymbol{S}_\alpha = \nabla_\alpha \boldsymbol{R}, \tag{41}$$

we have

$$\boldsymbol{N} = \frac{1}{2}\varepsilon^{\alpha\beta}\nabla_\alpha \boldsymbol{R} \times \boldsymbol{S}_\beta. \tag{63}$$

By the combination of the product rule and the metrinilic property (40) of ∇_α with respect to $\varepsilon^{\alpha\beta}$, we have

$$\boldsymbol{N} = \frac{1}{2}\nabla_\alpha\left(\varepsilon^{\alpha\beta}\boldsymbol{R}\times\boldsymbol{S}_\beta\right) - \frac{1}{2}\varepsilon^{\alpha\beta}\boldsymbol{R}\times\nabla_\alpha\boldsymbol{S}_\beta. \tag{64}$$

Furthermore, the fact that $\nabla_\alpha \boldsymbol{S}_\beta = \boldsymbol{N}B_{\alpha\beta}$ yields

$$\boldsymbol{N} = \frac{1}{2}\nabla_\alpha\left(\varepsilon^{\alpha\beta}\boldsymbol{R}\times\boldsymbol{S}_\beta\right) - \frac{1}{2}\boldsymbol{R}\times\boldsymbol{N}\varepsilon^{\alpha\beta}B_{\alpha\beta}. \tag{65}$$

Next, note that since $B_{\alpha\beta}$ is symmetric and $\varepsilon^{\alpha\beta}$ is skew-symmetric, the double contraction $\varepsilon^{\alpha\beta}B_{\alpha\beta}$ vanishes. Thus, the normal $\boldsymbol{N}$ is given by the identity

$$\boldsymbol{N} = \frac{1}{2}\nabla_\alpha\left(\varepsilon^{\alpha\beta}\boldsymbol{R}\times\boldsymbol{S}_\beta\right). \tag{66}$$

In other words, $\boldsymbol{N}$ can be expressed as half the surface divergence of $\varepsilon^{\alpha\beta}\boldsymbol{R}\times\boldsymbol{S}_\beta$, which immediately yields the desired result by an application of the surface divergence theorem. Indeed, integrate both sides of the above identity over the surface *S*, *i.e.*

$$\int_S \boldsymbol{N}\,\mathrm{d}S = \frac{1}{2}\int_S \nabla_\alpha\left(\varepsilon^{\alpha\beta}\boldsymbol{R}\times\boldsymbol{S}_\beta\right)\mathrm{d}S. \tag{67}$$

When *S* is closed and therefore does not have a boundary, the surface divergence theorem tells us that the surface integral of the unit normal vanishes, as we set out to show. When *S* is a patch with a boundary *L*, the surface divergence theorem reads

$$\int_S \boldsymbol{N}\,\mathrm{d}S = \frac{1}{2}\int_L n_\alpha\varepsilon^{\alpha\beta}\boldsymbol{R}\times\boldsymbol{S}_\beta\,\mathrm{d}L. \tag{68}$$

Since the combination $n_\alpha\varepsilon^{\alpha\beta}$ equals the components T^β of the unit tangent vector $\boldsymbol{T}$ to the boundary *L*, we discover the formula

$$\int_S \boldsymbol{N}\,\mathrm{d}S = \frac{1}{2}\int_L \boldsymbol{R}\times\boldsymbol{T}\,\mathrm{d}L. \tag{69}$$

Note that both $\boldsymbol{N}$ and $\boldsymbol{T}$ have two possible orientations that must be coordinated according to the following rule of thumb: when the fingers of the right hand follow the direction of $\boldsymbol{T}$, the thumb must point in the direction of $\boldsymbol{N}$.

In particular, Equation (69) shows that the integral of the unit normal $\boldsymbol{N}$ over a surface patch depends only on the shape of its contour boundary and not the shape of the patch itself.

The geometric quantities found in Equation (69) are illustrated in **Figure 3**.

4.2. The Integral $\int_S \boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S$

Let us now turn our attention to the identity

$$\int_S \boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = \mathbf{0} \tag{5}$$

involving the integral of the curvature normal $\boldsymbol{N}B_\alpha^\alpha$. Recall that the curvature normal $\boldsymbol{N}B_\alpha^\alpha$ is the surface divergence of the contravariant basis $\boldsymbol{S}^\alpha$, *i.e.*

$$\boldsymbol{N}B_\alpha^\alpha = \nabla_\alpha \boldsymbol{S}^\alpha, \tag{48}$$

from which, once again, the desired identity follows immediately by an application of the surface divergence theorem. Indeed, integrating both sides over S, we find

$$\int_S \boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = \int_S \nabla_\alpha \boldsymbol{S}^\alpha\,\mathrm{d}S. \tag{70}$$

If the surface S is closed and therefore does not have a boundary L, the integral on the right vanishes, as we set out to show. Meanwhile, for a patch S with a boundary L, an application of the surface divergence theorem yields

$$\int_S \boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = \int_L n_\alpha \nabla^\alpha \boldsymbol{R}\,\mathrm{d}L. \tag{71}$$

Since

$$n_\alpha \nabla^\alpha \boldsymbol{R} = n_\alpha \boldsymbol{S}^\alpha = \boldsymbol{n}, \tag{72}$$

where $\boldsymbol{n}$ is the exterior unit normal to the boundary L that lies in the plane tangent to the surface S, we arrive at the final identity

$$\int_S \boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = \int_L \boldsymbol{n}\,\mathrm{d}L. \tag{73}$$

Two proofs of a special case of this identity can be found in [7]. The contrast between the complexity of those proofs and that of the one presented here speaks to the effectiveness of our vector tensor calculus framework.

The key elements in the above identity are illustrated in **Figure 4**.

Note that unlike the integral of the unit normal $\boldsymbol{N}$, the integral of the curvature normal $\boldsymbol{N}B_\alpha^\alpha$ *does* depend on the shape of the patch (albeit only in the immediate vicinity of L) since the shape of the patch dictates the direction of the normal $\boldsymbol{n}$.

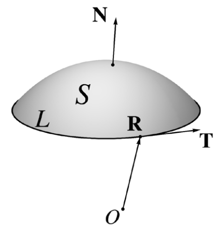

Figure 3. Geometric elements found in Equation (69).

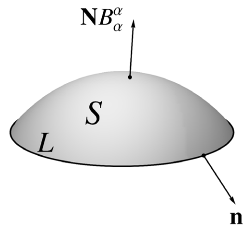

Figure 4. Geometric elements found in Equation (73).

4.3. The Integral $\int_S \boldsymbol{N}R\mathrm{d}S$

Finally, let us demonstrate the identity

$$\int_S \boldsymbol{N}R\,\mathrm{d}S = \mathbf{0}, \tag{6}$$

where R is the scalar curvature. Recall that for a two-dimensional hypersurface, $R = 2K$, where K is the Gaussian curvature, and therefore (6) implies

$$\int_S \boldsymbol{N}K\,\mathrm{d}S = \mathbf{0}. \tag{7}$$

To begin the proof, apply the covariant derivative ∇_α to both sides of the Weingarten equation

$$\nabla^\alpha \boldsymbol{N} = -\boldsymbol{S}^\beta B^\alpha_\beta, \tag{49}$$

to obtain an expression for the surface Laplacian of the unit normal $\boldsymbol{N}$, *i.e.*

$$\nabla_\alpha \nabla^\alpha \boldsymbol{N} = -\nabla_\alpha \left(\boldsymbol{S}^\beta B^\alpha_\beta\right). \tag{74}$$

An application of the product rule on the right side yields

$$\nabla_\alpha \nabla^\alpha \boldsymbol{N} = -\nabla_\alpha \boldsymbol{S}^\beta\, B^\alpha_\beta - \boldsymbol{S}^\beta \nabla_\alpha B^\alpha_\beta. \tag{75}$$

Since

$$\nabla_\alpha \boldsymbol{S}^\beta = \boldsymbol{N}B^\beta_\alpha \tag{76}$$

and, by the Codazzi equations,

$$\nabla_\alpha B^\alpha_\beta = \nabla_\beta B^\alpha_\alpha, \tag{61}$$

we have

$$\nabla_\alpha \nabla^\alpha \boldsymbol{N} = -\boldsymbol{N}B^\beta_\alpha B^\alpha_\beta - \boldsymbol{S}^\beta \nabla_\beta B^\alpha_\alpha. \tag{77}$$

A reverse application of the product rule to the second term on the right yields

$$\nabla_\alpha \nabla^\alpha \boldsymbol{N} = -\boldsymbol{N}B^\beta_\alpha B^\alpha_\beta - \nabla_\beta \left(\boldsymbol{S}^\beta B^\alpha_\alpha\right) + \nabla_\beta \boldsymbol{S}^\beta B^\alpha_\alpha, \tag{78}$$

and, since $\nabla_\beta \boldsymbol{S}^\beta = \boldsymbol{N} B^\beta_\beta$, we have

$$\nabla_\alpha \nabla^\alpha \boldsymbol{N} = \boldsymbol{N}\left(B^\alpha_\alpha B^\beta_\beta - B^\beta_\alpha B^\alpha_\beta\right) - \nabla_\beta\left(\boldsymbol{S}^\beta B^\alpha_\alpha\right). \tag{79}$$

Since

$$B^\alpha_\alpha B^\beta_\beta - B^\alpha_\beta B^\beta_\alpha = R, \tag{59}$$

we find that the surface Laplacian of $\boldsymbol{N}$ is given by

$$\nabla_\alpha \nabla^\alpha \boldsymbol{N} = \boldsymbol{N} R - \nabla_\beta\left(\boldsymbol{S}^\beta B^\alpha_\alpha\right). \tag{80}$$

Solving for $\boldsymbol{N}R$, we find

$$\boldsymbol{N}R = \nabla_\alpha\left(\nabla^\alpha \boldsymbol{N} + \boldsymbol{S}^\alpha B^\beta_\beta\right), \tag{81}$$

and therefore, much like $\boldsymbol{N}$ and $\boldsymbol{N} B^\alpha_\alpha$, the invariant $\boldsymbol{N}R$, can be expressed as the surface divergence of a first-order tensor and thus the desired identity follows immediately by an application of the surface divergence theorem. Indeed, as before, integrate both sides over the surface S, *i.e.*

$$\int_S \boldsymbol{N} R \,\mathrm{d}S = \int_S \nabla_\alpha\left(\nabla^\alpha \boldsymbol{N} + \boldsymbol{S}^\alpha B^\beta_\beta\right)\mathrm{d}S. \tag{82}$$

When the surface S is closed and therefore does not have a boundary L, an application of the surface divergence theorem yields the desired result

$$\int_S \boldsymbol{N} R \,\mathrm{d}S = \boldsymbol{0}, \tag{6}$$

which, for a two-dimensional hypersurface, reduces to

$$\int_S \boldsymbol{N} K \,\mathrm{d}S = \boldsymbol{0}. \tag{7}$$

Meanwhile, if S is a surface patch with a boundary L, then the surface divergence theorem reads

$$\int_S \boldsymbol{N} R \,\mathrm{d}S = \int_L\left(n_\alpha \nabla^\alpha \boldsymbol{N} + n_\alpha \boldsymbol{S}^\alpha B^\beta_\beta\right)\mathrm{d}L, \tag{83}$$

or, equivalently,

$$\int_S \boldsymbol{N} R \,\mathrm{d}S = \int_L\left(n_\alpha \nabla^\alpha \boldsymbol{N} + \boldsymbol{n} B^\alpha_\alpha\right)\mathrm{d}L. \tag{84}$$

With the help of Weingarten's Equation (49), this identity can be rewritten in the form

$$\int_S \boldsymbol{N} R \,\mathrm{d}S = \int_L\left(n_\beta B^\alpha_\alpha - n_\alpha B^\alpha_\beta\right)\boldsymbol{S}^\beta \mathrm{d}L. \tag{85}$$

For a two-dimensional hypersurface, where $R = 2K$, the above formula becomes

$$\int_S \boldsymbol{N} K \,\mathrm{d}S = \frac{1}{2}\int_L\left(n_\beta B^\alpha_\alpha - n_\alpha B^\alpha_\beta\right)\boldsymbol{S}^\beta \mathrm{d}L. \tag{86}$$

4.4. The Integral $\int_S \boldsymbol{R}\cdot\boldsymbol{N}\,\mathrm{d}S$

Let us now prove the related integral identities involving dot products with the position vector $\boldsymbol{R}$. Since the following calculations rely on the very same elements that we used extensively in the foregoing discussion, we will present the derivations as staccato chains of identities with minimal references.

For the integral

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}\,\mathrm{d}S \tag{87}$$

we have

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}\,\mathrm{d}S = \int_S \boldsymbol{R}\cdot N^i \boldsymbol{Z}_i\,\mathrm{d}S \tag{88}$$

$$\text{by Equation (42)} \quad = \int_\Omega \nabla^i\left(\boldsymbol{R}\cdot\boldsymbol{Z}_i\right)\mathrm{d}\Omega \tag{89}$$

$$\text{by Equation (37)} \quad = \int_\Omega \nabla^i\boldsymbol{R}\cdot\boldsymbol{Z}_i\,\mathrm{d}\Omega \tag{90}$$

$$\text{by Equation (38)} \quad = \int_\Omega \boldsymbol{Z}^i\cdot\boldsymbol{Z}_i\,\mathrm{d}\Omega \tag{91}$$

$$\text{by Equation (22)} \quad = \int_\Omega \delta_i^i\,\mathrm{d}\Omega \tag{92}$$

$$\text{by Equation (25)} \quad = n\int_\Omega \mathrm{d}\Omega \tag{93}$$

$$= nV. \tag{94}$$

In summary,

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}\,\mathrm{d}S = nV. \tag{15}$$

This is a well-known elegant expression for the volume of the enclosed domain as a surface integral.

4.5. The Integral $\int_S \boldsymbol{R}\cdot\boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S$

Let us now turn our attention to the integral

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S. \tag{95}$$

Since

$$\boldsymbol{N}B_\alpha^\alpha = \nabla_\alpha\nabla^\alpha\boldsymbol{R}, \tag{47}$$

a reverse application of the product rule yields

$$\boldsymbol{R}\cdot\boldsymbol{N}B_\alpha^\alpha = \nabla_\alpha\left(\boldsymbol{R}\cdot\nabla^\alpha\boldsymbol{R}\right) - \nabla_\alpha\boldsymbol{R}\cdot\nabla^\alpha\boldsymbol{R}. \tag{96}$$

Thus, by the surface divergence theorem,

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = -\int_S \nabla_\alpha\boldsymbol{R}\cdot\nabla^\alpha\boldsymbol{R}\,\mathrm{d}S. \tag{97}$$

Continuing, we find

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = -\int_S \nabla_\alpha\boldsymbol{R}\cdot\nabla^\alpha\boldsymbol{R}\,\mathrm{d}S \tag{98}$$

$$\text{by Equation (41)} \quad = -\int_S \boldsymbol{S}_\alpha\cdot\boldsymbol{S}^\alpha\,\mathrm{d}S \tag{99}$$

$$\text{by Equation (28)} \quad = -\int_S \delta_\alpha^\alpha\,\mathrm{d}S \tag{100}$$

$$\text{by Equation (31)} \quad = -\left(n-1\right)\int_S \mathrm{d}S \tag{101}$$

$$= -\left(n-1\right)A. \tag{102}$$

In summary,

$$\int_S \boldsymbol{R}\cdot\boldsymbol{N}B_\alpha^\alpha\,\mathrm{d}S = -\left(n-1\right)A \tag{16}$$

4.6. The Integral $\int_S \boldsymbol{R} \cdot \boldsymbol{N} R \mathrm{d}S$

Recall that the expression for the invariant $\boldsymbol{N}R$ in the divergence form reads

$$\boldsymbol{N}R = \nabla_\alpha \left(\nabla^\alpha \boldsymbol{N} + \boldsymbol{S}^\alpha B_\beta^\beta \right). \tag{81}$$

Thus, by a reverse application of the product rule,

$$\boldsymbol{R} \cdot \boldsymbol{N}R = \nabla_\alpha \left(\boldsymbol{R} \cdot \left(\nabla^\alpha \boldsymbol{N} + \boldsymbol{S}^\alpha B_\beta^\beta \right) \right) - \nabla_\alpha \boldsymbol{R} \cdot \left(\nabla^\alpha \boldsymbol{N} + \boldsymbol{S}^\alpha B_\beta^\beta \right), \tag{103}$$

and thus, by the surface divergence theorem,

$$\int_S \boldsymbol{R} \cdot \boldsymbol{N} R \mathrm{d}S = -\int_S \nabla_\alpha \boldsymbol{R} \cdot \left(\nabla^\alpha \boldsymbol{N} + \boldsymbol{S}^\alpha B_\beta^\beta \right) \mathrm{d}S. \tag{104}$$

Continuing, we find

$$\int_S \boldsymbol{R} \cdot \boldsymbol{N} R \mathrm{d}S = -\int_S \nabla_\alpha \boldsymbol{R} \cdot \left(\nabla^\alpha \boldsymbol{N} + \boldsymbol{S}^\alpha B_\beta^\beta \right) \mathrm{d}S \tag{105}$$

$$\text{by Equation (27)} \quad = -\int_S \boldsymbol{S}_\alpha \cdot \left(\nabla^\alpha \boldsymbol{N} + \boldsymbol{S}^\alpha B_\beta^\beta \right) \mathrm{d}S \tag{106}$$

$$\text{by Equation (49), (28)} \quad = -\int_S \left(-\delta_\alpha^\beta B_\beta^\alpha + \delta_\alpha^\alpha B_\beta^\beta \right) \mathrm{d}S \tag{107}$$

$$= -\int_S \left(\delta_\alpha^\alpha - 1 \right) B_\beta^\beta \, \mathrm{d}S \tag{108}$$

$$\text{by Equation (31)} \quad = -(n-2) \int_S B_\alpha^\alpha \, \mathrm{d}S. \tag{109}$$

In summary,

$$\int_S \boldsymbol{R} \cdot \boldsymbol{N} R \mathrm{d}S = -(n-2) \int_S B_\alpha^\alpha \, \mathrm{d}S, \tag{110}$$

For the special case of a two-dimensional hypersurface, where $n = 3$ and $R = 2K$, we have

$$\int_S \boldsymbol{R} \cdot \boldsymbol{N} K \, \mathrm{d}S = -\frac{1}{2} \int_S B_\alpha^\alpha \, \mathrm{d}S. \tag{17}$$

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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