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Well-Posedness and Infinite Propagation Speed for the Fifth-Order Camassa-Holm Equation with Weakly Dissipative Term

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Abstract

In this paper, we study the fifth-order Camassa-Holm equation with weakly dissipative term. We first give the local well-posedness result and the blow up criterion. Then, we establish sufficient conditions to guarantee that the solution exists globally in time. Finally, the infinite propagation speed of this equation is also investigated.

Keywords

Blow up Criterion, Global Existence, Infinite Propagation Speed

1. Introduction

In 1993, Camassa and Holm [1] derived an integrable shallow water equation taking the form as

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + u_x u_{xxx}, \tag{1.1}$$

where $u(x,t)$ denotes the height of the water above the flat bottom. Equation (1.1) describes wave motion in shallow water regime. Due to its mathematical importance and wide applications in physics, such equation has been intensively studied in the past decades and many interesting results have been obtained. Local well-posedness for the initial datum in H^s with $s > 3/2$ was proved in [2] [3]. The blow-up phenomenon was studied in [2]-[7]. McKean [7] (see also [6] for a simple proof) proved that the Camassa-Holm equation breaks if and only if some portion of the positive part of y_0 lies to the left of some portion of its negative part, here $y_0 = (1-\partial_x^2)u_0$. The hierarchy properties and algebro-geometric solutions of the Camassa-Holm equation were proposed in [8]. Global weak solution was studied in [9] [10] [11].

Later, researchers are concerned with the Camassa-Holm equation with weakly dissipative term

$$y_t + uy_x + 2u_x y + \lambda y = 0, \quad y = \left(1-\partial_x^2\right)u. \tag{1.2}$$

Wu and Yin proved in [12] [13] that the solution of (1.2) decays to zero as time goes to infinity if the initial momentum density y_0 does not change its sign. Unlike the Camassa-Holm equation, there are no traveling wave solutions of (1.2) (see [12]). Global existence and blow-up phenomena were studied in [12]-[17].

In recent years, the following fifth-order Camassa-Holm equation was studied by Liu and Qiao [18]:

$$y_t + uy_x + bu_x y = 0, \quad y = \left(1-\alpha^2\partial_x^2\right)\left(1-\beta^2\partial_x^2\right)u, \quad \alpha\beta \neq 0, \tag{1.3}$$

where b, α, β are real constants. In [18], the authors studied peakon solutions including single pseudo-peakons, two-peakons and three-peakons interactional solutions of (1.3). In [19], Tang and Liu proved that the Cauchy problem of this equation is locally well-posed in the critical Besov space $B_{2,1}^{7/2}$ or in $B_{p,r}^{s}$ with $1 \le p, r \le +\infty$, $s > \max\left\{3+1/p, 7/2\right\}$ when $\alpha = \beta = 1$. Zhu, Cao, Jiang and Qiao [20] studied the global existence of the solution of (1.3) with $\alpha \neq \beta \neq 0$. For more mathematical studies of (1.3), we refer to [21] [22] [23] [24] [25].

So far, there have been many researches on Camassa-Holm equation. However, the results on the fifth-order Camassa-Holm equation are few. In this paper, we consider the Cauchy problem of the fifth-order Camassa-Holm equation with weakly dissipative term

$$y_t + uy_x + bu_x y + \lambda y = 0, \quad t > 0, x \in \mathbb{R}, \tag{1.4}$$

$$y = \left(1-\alpha^2\partial_x^2\right)\left(1-\beta^2\partial_x^2\right)u, \quad t > 0, x \in \mathbb{R}, \tag{1.5}$$

$$u(x,0) = u_0(x), \tag{1.6}$$

where $b \in \mathbb{R}$ is a constant and $\alpha, \beta \in \mathbb{R}$ are two parameters satisfying $\alpha\beta \neq 0$. Our main purpose is to study the existence of global solution for this problem and investigate the properties of the solution.

Throughout the paper, we denote by $L^p(\mathbb{R})$ the Lebesgue space equipped with the norm

$$\|u\|_{L^p} = \left(\int_{\mathbb{R}} |u(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \text{ if } 1 \le p < +\infty$$

and

$$\|u\|_{L^\infty} = \operatorname{esssup}\left\{|u(x)|; x \in \mathbb{R}\right\}.$$

For $s \in \mathbb{R}$, $H^s(\mathbb{R})$ denotes the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}) = \left\{u \in \mathcal{S}'(\mathbb{R}) \mid \|u\|_{H^s(\mathbb{R})}^2 = \int_{\mathbb{R}} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi < +\infty\right\},$$

where $\hat{u}(\xi)$ is the Fourier transform of u.

This paper is organized as follows. In Section 2, we present the local well-posedness result and the blow up criterion. In Section 3, we discuss the problem of global existence. Finally, we consider the infinite propagation speed in Section 4.

2. Blow up Criterion

In this section, we discuss the blow up criterion for the solution to (1.4) - (1.6). To this aim, we first give the following existence theorem.

Theorem 2.1. *Let* $u_0 \in H^s(\mathbb{R})$ *with* $s > \frac{7}{2}$. *Then the Cauchy problem* (*1.4*) - (*1.6*) *admits a unique solution* $u(x,t)$ *such that*

$$u \in C\left([0,T); H^s(\mathbb{R})\right) \cap C^1\left([0,T); H^{s-1}(\mathbb{R})\right),$$

where $T = T\left(\|u_0\|_{H^s}\right) > 0$.

Theorem 2.1 can be proved by applying Kato's method [26]. Since the argument is standard, we omit the details for simplicity. If $T < \infty$ and

$$\lim_{t \to T^-} \|u\|_{H^s} = \infty,$$

we say the solution blows up in finite time. If the norm $\|u\|_{H^s}$ is bounded at any large time, we say the solution exists globally.

Define

$$G(x) = \begin{cases} \frac{\alpha}{2(\alpha^2 - \beta^2)} e^{-\frac{|x|}{\alpha}} - \frac{\beta}{2(\alpha^2 - \beta^2)} e^{-\frac{|x|}{\beta}}, & \alpha \neq \beta, \\ \frac{1}{4\alpha}\left(1 + \frac{|x|}{\alpha}\right) e^{-\frac{|x|}{\alpha}}, & \alpha = \beta, \end{cases} \tag{2.1}$$

then the function $\left[\left(1-\alpha^2\partial_x^2\right)\left(1-\beta^2\partial_x^2\right)\right]^{-1} y$ can be expressed as

$$u(t,x) = \left[\left(1-\alpha^2\partial_x^2\right)\left(1-\beta^2\partial_x^2\right)\right]^{-1} y(t,x) = G * y.$$

Now we state the main result of this section.

Theorem 2.2. *Assume that* $u_0 \in H^4(\mathbb{R})$, *and T is the maximal existence time of the solution obtained by Theorem 2.1.*

1) If $b > \frac{1}{2}$, then the solution blows up in finite time if and only if

$$\liminf_{t \to T} \inf_{x \in \mathbb{R}} u_x = -\infty. \tag{2.2}$$

2) If $b < \frac{1}{2}$, then the solution blows up in finite time if and only if

$$\lim_{t \to T} \sup_{x \in \mathbb{R}} u_x = +\infty. \tag{2.3}$$

Proof: Using (1.5), we obtain

$$\begin{aligned} \|y\|_{L^2}^2 &= \int_{\mathbb{R}} \left[u - \left(\alpha^2 + \beta^2\right) u_{xx} + \alpha^2\beta^2 u_{xxxx}\right]^2 \mathrm{d}x \\ &= \int_{\mathbb{R}} \Big[u^2 + \left(\alpha^2+\beta^2\right)^2 u_{xx}^2 - 2\left(\alpha^2+\beta^2\right) u u_{xx} + \alpha^4\beta^4 u_{xxxx}^2 + 2\alpha^2\beta^2 u u_{xxxx} \\ &\quad - 2\left(\alpha^2+\beta^2\right)\alpha^2\beta^2 u_{xx} u_{xxxx}\Big] \mathrm{d}x \end{aligned}$$

$$= \int_{\mathbb{R}} \Big[u^2 + \left(\alpha^2+\beta^2\right)^2 u_{xx}^2 + 2\left(\alpha^2+\beta^2\right) u_x^2 + \alpha^4\beta^4 u_{xxxx}^2 + 2\alpha^2\beta^2 u_{xx}^2$$
$$+ 2\left(\alpha^2+\beta^2\right)\alpha^2\beta^2 u_{xxx}^2 \Big] \mathrm{d}x,$$

then

$$C_0 \|u\|_{H^4}^2 \le \|y\|_{L^2}^2 \le C \|u\|_{H^4}^2 . \tag{2.4}$$

In case of $b > \frac{1}{2}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} y^2 \mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}} u\left(y^2\right)_x \mathrm{d}x - b\int_{\mathbb{R}} y^2 u_x \mathrm{d}x - \int_{\mathbb{R}} \lambda y^2 \mathrm{d}x$$
$$= \left(\frac{1}{2} - b\right)\int_{\mathbb{R}} y^2 u_x \mathrm{d}x - \int_{\mathbb{R}} \lambda y^2 \mathrm{d}x,$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y\|_{L^2}^2 + 2\lambda\|y\|_{L^2}^2 = (1-2b)\int_{\mathbb{R}} y^2 u_x \mathrm{d}x \le (1-2b)\inf_{x\in\mathbb{R}} u_x \int_{\mathbb{R}} y^2 \mathrm{d}x.$$

If $\inf_{x\in\mathbb{R}} u_x \ge -M$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y\|_{L^2}^2 \le \left[-(1-2b)M - 2\lambda\right]\|y\|_{L^2}^2 . \tag{2.5}$$

By Gronwall's inequality, there holds

$$\|y\|_{L^2}^2 \le \mathrm{e}^{\left[-(1-2b)M-2\lambda\right]t}\|y_0\|_{L^2}^2 . \tag{2.6}$$

Hence, $\|u(x,t)\|_{H^4}$ is bounded for all $t \in [0,T)$, which contradicts the maximal property of *T*.

In case of $b < \frac{1}{2}$, we can obtain similar result with the same arguments as above. This completes the proof of Theorem 2.2. □

3. Global Existence Results

In this section, we will show two global existence results. To begin with, motivated by Mckean's deep observation for the Camassa-Holm equation [7], we define the particle trajectory by

$$\begin{cases} q_t = u(q,t), & 0<t<T, \quad x\in\mathbb{R}, \\ q(x,0) = x, & x\in\mathbb{R}, \end{cases} \tag{3.1}$$

where T is the lifespan of the solution. From (3.1), we get

$$\frac{\mathrm{d}q_t}{\mathrm{d}x} = q_{tx} = u_x(q,t)q_x, \quad t\in(0,T),$$

which yields

$$\begin{cases} q_x = \exp\left\{\int_0^t u_x(q,s)\mathrm{d}s\right\}, & 0<t<T, x\in\mathbb{R}, \\ q_x(x,0) = 1, & x\in\mathbb{R}. \end{cases} \tag{3.2}$$

The relation (3.2) implies that q_x is always positive, hence $q(x,t)$ is an increasing function. A direct calculation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y(q)q_x^b\right)=\left[y_t(q)+u(q,t)y_x(q)+bu_x(q,t)y(q)\right]q_x^b=-\lambda y(q)q_x^b,$$

so we have

$$y(q)q_x^b=\mathrm{e}^{-\lambda t}y_0(x). \tag{3.3}$$

From (3.3), we also get

$$\mathrm{e}^{\frac{\lambda t}{b}}\int_{\mathbb{R}}y^{\frac{1}{b}}\mathrm{d}x=\int_{\mathbb{R}}y_0^{\frac{1}{b}}\mathrm{d}x. \tag{3.4}$$

Theorem 3.1. *Assume* $b=\frac{1}{2}$ *or* $b=2$, *and* $u_0\in H^s(\mathbb{R})$ *with* $s>\frac{7}{2}$, *then the solution of (1.4) - (1.6) exists globally in time.*

Proof: We first consider the case $b=\frac{1}{2}$. Multiplying both sides of Equation (1.4) by y and integrating with respect to x, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y\|_{L^2}^2+2\lambda\|y\|_{L^2}^2=0,$$

hence, we obtain $\|y\|_{L^2}=\mathrm{e}^{-\lambda t}\|y_0\|_{L^2}$. By (2.4) and Sobolev's inequality, we see that $\|u_x\|_{L^\infty}$ is bounded by $\|y\|_{L^2}$, so we can extend the local solution to be a global solution.

Then, we give the proof for $b=2$. Let

$$E(t)=\int_{\mathbb{R}}\left[u^2+\left(\alpha^2+\beta^2\right)u_x^2+\alpha^2\beta^2u_{xx}^2\right]\mathrm{d}x,$$

then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t)&=\int_{\mathbb{R}}\left[2uu_t+2\left(\alpha^2+\beta^2\right)u_xu_{xt}+2\alpha^2\beta^2u_{xx}u_{xxt}\right]\mathrm{d}x\\&=\int_{\mathbb{R}}\left[2uu_t-2\left(\alpha^2+\beta^2\right)uu_{xxt}+2\alpha^2\beta^2uu_{xxxxt}\right]\mathrm{d}x\\&=2\int_{\mathbb{R}}uy_t\mathrm{d}x\\&=-2\int_{\mathbb{R}}u\left(uy_x+2u_xy+\lambda y\right)\mathrm{d}x\\&=-2\lambda E(t).\end{aligned}$$

The above identity gives $E(t)=\mathrm{e}^{-2\lambda t}E(0)$, so once again we can see $\|u_x\|_{L^\infty}$ is bounded. This ends the proof of Theorem 3.1. □

Theorem 3.2. *Let* $\alpha\neq\beta$ *and* $u_0\in H^s(\mathbb{R})$ *with* $s>\frac{7}{2}$. *Assume that* $y_0=\left(1-\alpha^2\partial_x^2\right)\left(1-\beta^2\partial_x^2\right)u_0$ *does not change sign, then the solution of (1.4) - (1.6) exists globally in time.*

Proof: Differentiating $\int_{\mathbb{R}}y\mathrm{d}x$ with respect to t and using (1.5), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}y\mathrm{d}x&=-\int_{\mathbb{R}}\left(uy_x+bu_xy+\lambda y\right)\mathrm{d}x\\&=-(b-1)\int_{\mathbb{R}}u_xy\mathrm{d}x-\lambda\int_{\mathbb{R}}y\mathrm{d}x\\&=-(b-1)\int_{\mathbb{R}}u_x\left(u-\left(\alpha^2+\beta^2\right)u_{xx}+\alpha^2\beta^2u_{xxxx}\right)\mathrm{d}x-\lambda\int_{\mathbb{R}}y\mathrm{d}x\\&=-\lambda\int_{\mathbb{R}}y\mathrm{d}x,\end{aligned}$$

which yields

$$\int_{\mathbb{R}} y\mathrm{d}x = \mathrm{e}^{-\lambda t}\int_{\mathbb{R}} y_0\mathrm{d}x. \tag{3.5}$$

When $\alpha \neq \beta$, using the expression of G (see (2.1)), $u(x,t)$ and $u_x(x,t)$ can be represented as

$$\begin{aligned} u =& \frac{\alpha}{2(\alpha^2-\beta^2)}\left(\mathrm{e}^{-\frac{x}{\alpha}}\int_{-\infty}^{x}\mathrm{e}^{\frac{\xi}{\alpha}}y(\xi,t)\mathrm{d}\xi + \mathrm{e}^{\frac{x}{\alpha}}\int_{x}^{+\infty}\mathrm{e}^{-\frac{\xi}{\alpha}}y(\xi,t)\mathrm{d}\xi\right) \\ &-\frac{\beta}{2(\alpha^2-\beta^2)}\left(\mathrm{e}^{-\frac{x}{\beta}}\int_{-\infty}^{x}\mathrm{e}^{\frac{\xi}{\beta}}y(\xi,t)\mathrm{d}\xi + \mathrm{e}^{\frac{x}{\beta}}\int_{x}^{+\infty}\mathrm{e}^{-\frac{\xi}{\beta}}y(\xi,t)\mathrm{d}\xi\right), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} u_x =& \frac{1}{2(\alpha^2-\beta^2)}\left(-\mathrm{e}^{-\frac{x}{\alpha}}\int_{-\infty}^{x}\mathrm{e}^{\frac{\xi}{\alpha}}y(\xi,t)\mathrm{d}\xi + \mathrm{e}^{\frac{x}{\alpha}}\int_{x}^{+\infty}\mathrm{e}^{-\frac{\xi}{\alpha}}y(\xi,t)\mathrm{d}\xi\right) \\ &+\frac{1}{2(\alpha^2-\beta^2)}\left(\mathrm{e}^{-\frac{x}{\beta}}\int_{-\infty}^{x}\mathrm{e}^{\frac{\xi}{\beta}}y(\xi,t)\mathrm{d}\xi - \mathrm{e}^{\frac{x}{\beta}}\int_{x}^{+\infty}\mathrm{e}^{-\frac{\xi}{\beta}}y(\xi,t)\mathrm{d}\xi\right). \end{aligned} \tag{3.7}$$

Without loss of generality, we may assume $y_0 \geq 0$ as the case $y_0 \leq 0$ can be discussed similarly. If $\alpha > \beta > 0$, by (3.5) and (3.7), we have

$$\begin{aligned} u_x &\leq \frac{1}{2(\alpha^2-\beta^2)}\left(\mathrm{e}^{\frac{x}{\alpha}}\int_{x}^{\infty}\mathrm{e}^{-\frac{\xi}{\alpha}}y(\xi,t)\mathrm{d}\xi + \mathrm{e}^{-\frac{x}{\beta}}\int_{-\infty}^{x}\mathrm{e}^{\frac{\zeta}{\beta}}y(\xi,t)\mathrm{d}\xi\right) \\ &\leq \frac{1}{2(\alpha^2-\beta^2)}\left(\int_{\mathbb{R}} y(\xi,t)\mathrm{d}\xi + \int_{\mathbb{R}} y(\xi,t)\mathrm{d}\xi\right) \\ &= \frac{1}{\alpha^2-\beta^2}\mathrm{e}^{-\lambda t}\int_{\mathbb{R}} y_0(\xi,t)\mathrm{d}\xi, \end{aligned}$$

and

$$\begin{aligned} u_x &\geq \frac{1}{2(\alpha^2-\beta^2)}\left(-\mathrm{e}^{-\frac{x}{\alpha}}\int_{-\infty}^{x}\mathrm{e}^{\frac{\xi}{\alpha}}y(\xi,t)\mathrm{d}\xi - \mathrm{e}^{\frac{x}{\beta}}\int_{x}^{+\infty}\mathrm{e}^{-\frac{\xi}{\beta}}y(\xi,t)\mathrm{d}\xi\right) \\ &= -\frac{1}{\alpha^2-\beta^2}\mathrm{e}^{-\lambda t}\int_{\mathbb{R}} y_0(\xi,t)\mathrm{d}\xi. \end{aligned}$$

Similarly, if $\beta > \alpha > 0$, we can also obtain

$$-\frac{1}{\beta^2-\alpha^2}\mathrm{e}^{-\lambda t}\int_{\mathbb{R}} y_0(\xi,t)\mathrm{d}\xi \leq u_x(x,t) \leq \frac{1}{\beta^2-\alpha^2}\mathrm{e}^{-\lambda t}\int_{\mathbb{R}} y_0(\xi,t)\mathrm{d}\xi.$$

Hence, by Theorems 2.1 - 2.2, we know that the problem (1.4) - (1.6) has a unique global solution. □

4. Infinite Propagation Speed

In this section, we study the infinite propagation phenomenon for the Equation (1.4). To this aim, we set

$$E_1(t) = \int_{\mathbb{R}} \mathrm{e}^{\frac{x}{\alpha}}y(x,t)\mathrm{d}x, \quad F_1(t) = \int_{\mathbb{R}} \mathrm{e}^{-\frac{x}{\alpha}}y(x,t)\mathrm{d}x,$$

and

$$E_2(t)=\int_{\mathbb{R}} e^{\frac{x}{\beta}} y(x,t)\mathrm{d}x, \quad F_2(t)=\int_{\mathbb{R}} e^{-\frac{x}{\beta}} y(x,t)\mathrm{d}x.$$

The main result of this section is stated as follows.

Theorem 4.1. *Let* $b \ge 0, \lambda \in \mathbb{R}$ *. Suppose that the initial value* $u_0(x) \not\equiv 0$ *is supported in the interval* $[a,c]$ *. Then for any* $t \in (0,T]$ *, the solution* $u(x,t)$ *of (1.4) - (1.6) can be expressed by*

$$u(x,t)=\begin{cases} \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t)-\frac{\beta}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\beta}} E_2(t), & x>q(c,t), \\ \frac{\alpha}{2(\alpha^2-\beta^2)} e^{\frac{x}{\alpha}} F_1(t)-\frac{\beta}{2(\alpha^2-\beta^2)} e^{\frac{x}{\beta}} F_2(t), & x<q(a,t). \end{cases}$$

Moreover, if $\alpha>0, 0<\beta \le \sqrt{\frac{3}{2}}\alpha, 0 \le b \le \min\left\{3-\frac{2\beta^2}{\alpha^2}, \frac{5}{3}\right\}$, then $e^{\lambda t}E_1(t)$ is a strictly increasing function and $e^{\lambda t}F_1(t)$ is a strictly decreasing function. Similarly, if $\beta>0, 0<\alpha \le \sqrt{\frac{3}{2}}\beta, 0 \le b \le \min\left\{3-\frac{2\beta^2}{\alpha^2}, \frac{5}{3}\right\}$, then $e^{\lambda t}E_2(t)$ is a strictly increasing function and $e^{\lambda t}F_2(t)$ is a strictly decreasing function.

Proof: By (3.3), we have

$$y(q(x,t),t)=0, \quad x<a \text{ or } x>c.$$

Since

$$\begin{aligned} u(x,t) &= \left(\frac{\alpha^2}{\alpha^2-\beta^2} p_1-\frac{\beta^2}{\alpha^2-\beta^2} p_2\right) * y(x,t) \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\alpha}} y(\xi)\mathrm{d}\xi-\frac{\beta}{2(\alpha^2-\beta^2)} \int_{\mathbb{R}} e^{-\frac{|x-\xi|}{\beta}} y(\xi)\mathrm{d}\xi \qquad (4.1) \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)} \int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\alpha}} y(\xi)\mathrm{d}\xi-\frac{\beta}{2(\alpha^2-\beta^2)} \int_{q(a,t)}^{q(c,t)} e^{-\frac{|x-\xi|}{\beta}} y(\xi)\mathrm{d}\xi, \end{aligned}$$

when $x>q(c,t)$, we get

$$\begin{aligned} u(x,t) &= \frac{\alpha}{2(\alpha^2-\beta^2)} \int_{q(a,t)}^{q(c,t)} e^{-\frac{x-\xi}{\alpha}} y(\xi)\mathrm{d}\xi-\frac{\beta}{2(\alpha^2-\beta^2)} \int_{q(a,t)}^{q(c,t)} e^{-\frac{x-\xi}{\beta}} y(\xi)\mathrm{d}\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} \int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\alpha}} y(\xi)\mathrm{d}\xi-\frac{\beta}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\beta}} \int_{q(a,t)}^{q(c,t)} e^{\frac{\xi}{\beta}} y(\xi)\mathrm{d}\xi \\ &= \frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} E_1(t)-\frac{\beta}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\beta}} E_2(t). \end{aligned}$$

Similarly, when $x<q(a,t)$, we have

$$u(x,t)=\frac{\alpha}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\alpha}} F_1(t)-\frac{\beta}{2(\alpha^2-\beta^2)} e^{-\frac{x}{\beta}} F_2(t).$$

Note that

$$\frac{\mathrm{d}E_1(t)}{\mathrm{d}t}=\int_{\mathbb{R}}\mathrm{e}^{\frac{\xi}{\alpha}}y_t(\xi,t)\mathrm{d}\xi. \tag{4.2}$$

By

$$y=\left(1-\alpha^2\partial_x^2\right)\left(1-\beta^2\partial_x^2\right)u=u-\left(\alpha^2+\beta^2\right)u_{xx}+\alpha^2\beta^2u_{xxxx},$$

then we get

$$\begin{aligned} y_t&=-uy_x-byu_x-\lambda y\\ &=-(b+1)uu_x+\left(\alpha^2+\beta^2\right)u_{xxx}u-\alpha^2\beta^2u_{xxxxx}u\\ &\quad+b\left(\alpha^2+\beta^2\right)u_{xx}u_x-b\alpha^2\beta^2u_{xxxx}u_x-\lambda y. \end{aligned} \tag{4.3}$$

Inserting (4.3) into (4.2), we have

$$\begin{aligned}\frac{\mathrm{d}E_1(t)}{\mathrm{d}t}&=-(b+1)\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}uu_x\mathrm{d}x+\left(\alpha^2+\beta^2\right)\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_{xxx}u\mathrm{d}x-\alpha^2\beta^2\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_{xxxxx}u\mathrm{d}x\\ &\quad+b\left(\alpha^2+\beta^2\right)\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_{xx}u_x\mathrm{d}x-b\alpha^2\beta^2\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_{xxxx}u_x\mathrm{d}x-\lambda E_1(t),\end{aligned}$$

then

$$\begin{aligned}\frac{\mathrm{d}\left(\mathrm{e}^{\lambda t}E_1(t)\right)}{\mathrm{d}t}&=\mathrm{e}^{\lambda t}\Bigg(\frac{b+1}{2\alpha}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u^2\mathrm{d}x-\frac{\alpha^2+\beta^2}{2\alpha^3}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u^2\mathrm{d}x+\frac{3\left(\alpha^2+\beta^2\right)}{2\alpha}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_x^2\mathrm{d}x\\ &\quad+\frac{\beta^2}{2\alpha^3}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u^2\mathrm{d}x-\frac{5\beta^2}{2\alpha}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_x^2\mathrm{d}x+\frac{5\alpha\beta^2}{2}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_{xx}^2\mathrm{d}x\\ &\quad-\frac{b\left(\alpha^2+\beta^2\right)}{2\alpha}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_x^2\mathrm{d}x+\frac{b\beta^2}{2\alpha}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_x^2\mathrm{d}x-\frac{3b\alpha\beta^2}{2}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_{xx}^2\mathrm{d}x\Bigg)\\ &=\mathrm{e}^{\lambda t}\Bigg(\frac{b}{2\alpha}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u^2\mathrm{d}x-\frac{(b-3)\alpha^2+2\beta^2}{2\alpha}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_x^2\mathrm{d}x\\ &\quad+\frac{(5-3b)\alpha\beta^2}{2}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_{xx}^2\mathrm{d}x\Bigg).\end{aligned} \tag{4.4}$$

Applying similar arguments as above, we can obtain

$$\begin{aligned}\frac{\mathrm{d}\left(\mathrm{e}^{\lambda t}F_1(t)\right)}{\mathrm{d}t}&=\mathrm{e}^{\lambda t}\Bigg(-\frac{b}{2\alpha}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u^2\mathrm{d}x+\frac{(b-3)\alpha^2+2\beta^2}{2\alpha}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_x^2\mathrm{d}x\\ &\quad-\frac{(5-3b)\alpha\beta^2}{2}\int_{\mathbb{R}}\mathrm{e}^{\frac{x}{\alpha}}u_{xx}^2\mathrm{d}x\Bigg).\end{aligned} \tag{4.5}$$

According to (4.4) and (4.5), it is easy to see that if $\alpha>0$, $0<\beta\le\sqrt{\frac{3}{2}}\alpha$ and $0\le b\le\min\left\{3-\frac{2\beta^2}{\alpha^2},\frac{5}{3}\right\}$, the function $\mathrm{e}^{\lambda t}E_1(t)$ is strictly increasing and the function $\mathrm{e}^{\lambda t}F_1(t)$ is strictly decreasing. In the same way, we can obtain the desired result when $\beta>0$, $0<\alpha\le\sqrt{\frac{3}{2}}\beta$ and $0\le b\le\min\left\{3-\frac{2\beta^2}{\alpha^2},\frac{5}{3}\right\}$. □

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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