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A Nonexistence Result for Choquard-Type Hamiltonian System

Zexi Wang

School of Mathematics and Statistics, Southwest University, Chongqing, China
Email: 1099519293@qq.com

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Abstract

In this article, we establish a nonexistence result of nontrivial non-negative solutions for the following Choquard-type Hamiltonian system by the Pohožaev identity $\begin{cases} -\Delta u+u=\left(I_{\mu_1}*\frac{|v|^p}{|x|^\alpha}\right)\frac{|v|^{p-2}v}{|x|^\alpha}, & \text{in } \mathbb{R}^N\setminus\{0\},\\ -\Delta v+v=\left(I_{\mu_2}*\frac{|u|^q}{|x|^\beta}\right)\frac{|u|^{q-2}u}{|x|^\beta}, & \text{in } \mathbb{R}^N\setminus\{0\},\\ u(x),v(x)\to 0, & \text{when } |x|\to\infty, \end{cases}$, when $N\ge 3$, $0<\mu_1,\mu_2<N$, $0\le\alpha\le\frac{\mu_1}{2}$, $0\le\beta\le\frac{\mu_2}{2}$, $p,q>1$, and $\frac{N+\mu_1-2\alpha}{p}+\frac{N+\mu_2-2\beta}{q}\le 2(N-2)$, where $I_{\mu_i}=\frac{1}{|x|^{N-\mu_i}}$ and $*$ denotes the convolution in $\mathbb{R}^N$, $i=1,2$.

Keywords

Nonexistence, Choquard-Type Hamiltonian System, Pohožaev Identity

1. Introduction and Statement of Main Result

Recently, a lot of attention has been focused on the study of the following Choquard-type Hamiltonian system

$$\begin{cases} -\Delta u+u=\left(I_{\mu_1}*\frac{|v|^p}{|x|^\alpha}\right)\frac{|v|^{p-2}v}{|x|^\alpha}, & \text{in } \mathbb{R}^N\setminus\{0\},\\ -\Delta v+v=\left(I_{\mu_2}*\frac{|u|^q}{|x|^\beta}\right)\frac{|u|^{q-2}u}{|x|^\beta}, & \text{in } \mathbb{R}^N\setminus\{0\},\\ u(x),v(x)\to 0, & \text{when } |x|\to\infty, \end{cases} \tag{1.1}$$

where $N \geq 3$, $0 < \mu_1, \mu_2 < N$, $0 \leq \alpha \leq \frac{\mu_1}{2}$, $0 \leq \beta \leq \frac{\mu_2}{2}$, $p, q > 1$, $I_{\mu_i} = \frac{1}{|x|^{N-\mu_i}}$ for $i = 1, 2$, $*$ is the convolution in $\mathbb{R}^N$.

When $\alpha = \beta = 0$, $\mu_1 = \mu_2$, $p = q$, $u = v$, (1.1) reduces to the following classic Choquard equation

$$-\Delta u + u = \left(I_{\mu_1} * |u|^p\right)|u|^{p-2} u, \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Equation (1.2) has a physical prototype, as pointed out by Lieb [1], Choquard introduced the equation

$$i\psi_t = -\Delta\psi - \left(I_2 * |\psi|^2\right)\psi, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+,$$

to describe an electron trapped in its own hole as an approximation to Hartree-Fock theory for a one component plasma. It also arises in multiple particle systems and Quantum Mechanics [2]. In a pioneering work, set $\psi(x,t) = \mathrm{e}^{it} u(x)$, then

$$-\Delta u + u = \left(I_2 * |u|^2\right)u, \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

and Lieb [1] first obtained the existence and uniqueness of a ground state solution to (1.3) via variational methods. Lions [3] [4] considered the same problem and proved the existence and multiplicity of normalized solutions. The classification of positive solutions was first studied by Ma and Zhao [5].

As for (1.2), Moroz and Van Schaftingen [6] studied the positivity, regularity, decay asymptotics and radial symmetry of ground state solutions for $\frac{N-2}{N+\mu_1} < \frac{1}{p} < \frac{N}{N+\mu_1}$. Meanwhile, they also proved that (1.2) has no nontrivial smooth H^1 solution for either $\frac{1}{p} \leq \frac{N-2}{N+\mu_1}$ or $\frac{1}{p} \geq \frac{N}{N+\mu_1}$ by using the Pohožaev identity. The number $\frac{N+\mu_1}{N}$ and $\frac{N+\mu_1}{N-2}$ (if $N \geq 3$) are called the lower and upper critical exponents related to the Hardy-Littlewood-Sobolev inequality, respectively. Furthermore, if $\mu_1 = \mu_2$, $0 \leq \alpha = \beta \leq \frac{\mu_1}{2}$ and $p = q$, $u = v$ in (1.1), Du *et al.* [7] also established the nonexistence result for $p \geq \frac{N+\mu_1-2\alpha}{N-2}$ or $p \leq \frac{N+\mu_1-2\alpha}{N}$ when $N \geq 3$ by the Pohožaev identity. By using the method of moving planes in integral forms introduced by Chen *et al.* [8], Le [9] proved the following equation has no positive solution if $p < \frac{N+\mu_1}{N-2}$ ($p \in \mathbb{R}$ when $N \leq 2$), and every positive solution u has the form

$$u(x) = c\left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{N-2}{2}} \text{ for some } c, \lambda > 0 \text{ and } x_0 \in \mathbb{R}^N,$$

$$-\Delta u = \left(I_{\mu_1} * |u|^p\right)|u|^{p-2} u, \quad \text{in } \mathbb{R}^N.$$

As for more investigations about Choquard equations, we refer to [10].

For the Hamiltonian system, if $\alpha=\beta=0$ in (1.1), Maia and Miyagaki [11] studied the following Choquard-type Hamiltonian system

$$\begin{cases}-\Delta u+u=\left(I_{\mu_1}*|v|^p\right)|v|^{p-2}v, & \text{in } \mathbb{R}^N,\\ -\Delta v+v=\left(I_{\mu_2}*|u|^q\right)|u|^{q-2}u, & \text{in } \mathbb{R}^N,\\ u(x),v(x)\to 0, & \text{when } |x|\to\infty.\end{cases} \tag{1.4}$$

However, different from (1.2), the structure of (1.4) makes it quite difficult to obtain the Pohožaev identity for (1.4). In the spirit of the method in [12], Maia and Miyagaki used similar arguments to overcome this difficulty and obtained the Pohožaev identity, they proved that: if $N\geq 3$, (1.4) has no nontrivial non-negative C^2 solution for $p\geq\frac{N+\mu_1}{N-2}$ and $q\geq\frac{N+\mu_2}{N-2}$; if $N=2$, (1.4) has no nontrivial non-negative C^2 solution for $p\leq\frac{2+\mu_1}{2}$ and $q\leq\frac{2+\mu_2}{2}$. The key idea of [12] is that, for the Hamiltonian system of 2 equations, consider a pair of non-negative solutions (u,v), define $U:=x\cdot\nabla u(x)$ and $V:=x\cdot\nabla v(x)$. Through a straightforward calculation to obtain $v\Delta U$ and $u\Delta V$, then by using the differential knowledge and (u,v) is solution, we can get a differential form of Pohožaev identity, which will produce the Pohožaev identity. This new idea also helps Kou and An [13] to generalize the well-known results of Mitidieri [14] and discuss the nonexistence result of positive solutions for the Hamiltonian system in a non-star shaped domain. By the method of moving planes in integral forms, Le [15] also showed that the following system has no positive classical solution

$$\begin{cases}-\Delta u=\left(I_{\mu_1}*|v|^p\right)|v|^{p-2}v, & \text{in } \mathbb{R}^N,\\ -\Delta v=\left(I_{\mu_2}*|u|^q\right)|u|^{q-2}u, & \text{in } \mathbb{R}^N,\end{cases}$$

when $N\geq 3$, $1<p\leq\frac{N+\mu_1}{N-2}$, $1<q\leq\frac{N+\mu_2}{N-2}$ and $p+q<\frac{2N+\mu_1+\mu_2}{N-2}$. Other existence or nonexistence results of solutions for equations can be find, we refer readers to [16] [17] [18] [19] [20] and references therein.

Motivated by the aforementioned papers, in the present paper, we give a nonexistence result of nontrivial non-negative solutions for (1.1) with $\alpha,\beta\geq 0$ by the Pohožaev identity.

The main result of this paper is the following:

Theorem 1.1. *Assume that* $N\geq 3$, $0<\mu_1,\mu_2<N$, $0\leq\alpha\leq\frac{\mu_1}{2}$, $0\leq\beta\leq\frac{\mu_2}{2}$, $p,q>1$, *and* $\frac{N+\mu_1-2\alpha}{p}+\frac{N+\mu_2-2\beta}{q}\leq 2(N-2)$, *if*

$$(u,v)\in\left(C^2\left(\mathbb{R}^N\setminus\{0\}\right)\cap H^1\left(\mathbb{R}^N\right)\cap L^{\frac{2Np}{N+\mu_1-2\alpha}}\left(\mathbb{R}^N\right)\right)\times\left(C^2\left(\mathbb{R}^N\setminus\{0\}\right)\cap H^1\left(\mathbb{R}^N\right)\cap L^{\frac{2Nq}{N+\mu_2-2\beta}}\left(\mathbb{R}^N\right)\right)$$ *is a pair of non-negative*

solutions of (*1.1*), *then* $(u,v)=(0,0)$. *In particular,* (*1.1*) *does not have a pair of nontrivial non-negative solutions for* $p\ge\frac{N+\mu_1-2\alpha}{N-2}$ *and* $q\ge\frac{N+\mu_2-2\beta}{N-2}$.

In Section 2, we will give the proof of Theorem 1.1. To facilitate reading, we use the notations:

- $C^2(\mathbb{R}^N\setminus\{0\})$ is the space of functions whose 2-th derivatives are continuous in $\mathbb{R}^N\setminus\{0\}$.
- $C_0^\infty(\mathbb{R}^N)$ is the space of functions infinitely differentiable with compact support in $\mathbb{R}^N$.
- $L^p(\mathbb{R}^N)$, $p\in[1,+\infty)$ is the usual Lebesgue space endowed with the norm $\|u\|_t=\left(\int_{\mathbb{R}^N}|u|^t\,\mathrm{d}x\right)^{\frac{1}{t}}$.
- $H^1(\mathbb{R}^N)=\left\{u\in L^2(\mathbb{R}^N):\nabla u\in L^2(\mathbb{R}^N)\right\}$ is endowed with norm $\|u\|=\left(\int_{\mathbb{R}^N}|\nabla u|^2+u^2\mathrm{d}x\right)^{\frac{1}{2}}$.

2. Proof of Theorem 1.1

To study (1.1), we need the following doubly weighted Hardy-Littlewood-Sobolev inequality proved in [21].

Proposition 2.1. (*Doubly Weighted Hardy-Littlewood-Sobolev Inequality*) *Let* $t,r>1$ *and* $0<\mu<N$ *with* $0\le\alpha+\beta\le\mu$, $\frac{1}{t}+\frac{\alpha+\beta-\mu}{N}+\frac{1}{r}=1$, $\alpha<\frac{N}{t'}$, $\beta<\frac{N}{r'}$, $f\in L^t(\mathbb{R}^N)$ *and* $h\in L^r(\mathbb{R}^N)$, *where* $\frac{1}{t}+\frac{1}{t'}=1$ *and* $\frac{1}{r}+\frac{1}{r'}=1$. *Then there exists a constant* $C(\alpha,\beta,\mu,N,t,r)>0$ *which is independent of* f,h *such that*

$$\int_{\mathbb{R}^N}\left(I_\mu*\frac{f(x)}{|x|^\alpha}\right)\frac{h(x)}{|x|^\beta}\mathrm{d}x\le C(\alpha,\beta,\mu,N,t,r)\|f\|_t\|h\|_r.$$

From Proposition 2.1, we easily get the following remark.

Remark 2.1. Let $0<\mu<N$, $0\le\alpha\le\frac{\mu}{2}$, and $p>1$. Assume that $u\in L^{\frac{2Np}{N+\mu-2\alpha}}(\mathbb{R}^N)$, then there exists $C(\alpha,\mu,N)>0$ such that

$$\int_{\mathbb{R}^N}\left(I_\mu*\frac{|u|^p}{|x|^\alpha}\right)\frac{|u|^p}{|x|^\alpha}\mathrm{d}x\le C(\alpha,\mu,N)\|u\|^{2p}_{\frac{2Np}{N+\mu-2\alpha}}.\tag{2.1}$$

Consider a cut-off function $\varphi\in C_0^\infty(\mathbb{R}^N,[0,1])$ such that $\varphi(x)=1$ if $|x|\le1$, $\varphi(x)=0$ if $|x|\ge2$. For any fixed $\lambda>0$, set

$$\tilde{u}_\lambda(x)=\varphi(\lambda x)x\cdot\nabla u(x)\ \text{and}\ \tilde{v}_\lambda(x)=\varphi(\lambda x)x\cdot\nabla v(x).$$

Lemma 2.1. *Let* $u,v\ge0$ *and*

$(u,v)\in\left(C^2(\mathbb{R}^N\setminus\{0\})\cap L^{\frac{2Np}{N+\mu_1-2\alpha}}(\mathbb{R}^N)\right)\times\left(C^2(\mathbb{R}^N\setminus\{0\})\cap L^{\frac{2Nq}{N+\mu_2-2\beta}}(\mathbb{R}^N)\right)$, *then*

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^{\alpha}}\right)\frac{v^{p-1}\tilde{v}_{\lambda}}{|x|^{\alpha}}\mathrm{d}x=\frac{-N-\mu_1+2\alpha}{2p}\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^{\alpha}}\right)\frac{v^p}{|x|^{\alpha}}\mathrm{d}x,$$

and

$$\lim_{\lambda\to 0}\int_{\mathbb{R}^N}\left(I_{\mu_2}*\frac{u^q}{|x|^{\beta}}\right)\frac{u^{q-1}\tilde{u}_{\lambda}}{|x|^{\beta}}\mathrm{d}x=\frac{-N-\mu_2+2\beta}{2q}\int_{\mathbb{R}^N}\left(I_{\mu_2}*\frac{u^q}{|x|^{\beta}}\right)\frac{u^q}{|x|^{\beta}}\mathrm{d}x.$$

Proof. A direct computation, one has

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^{\alpha}}\right)\frac{v^{p-1}\tilde{v}_{\lambda}}{|x|^{\alpha}}\mathrm{d}x\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|x-y|^{\mu_1-N}|y|^{-\alpha}v^p(y)|x|^{-\alpha}v^{p-1}(x)\varphi(\lambda x)\left[x\cdot\nabla v(x)\right]\mathrm{d}x\mathrm{d}y\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|x-y|^{\mu_1-N}|y|^{-\alpha}v^p(y)|x|^{-\alpha}\varphi(\lambda x)\left[x\cdot\nabla\left(\frac{v^p(x)}{p}\right)\right]\mathrm{d}x\mathrm{d}y\\
&=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(y)\varphi(\lambda x)\left[x\cdot\nabla\left(\frac{v^p(x)}{p}\right)\right]\\
&\quad+|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(x)\varphi(\lambda y)\left[y\cdot\nabla\left(\frac{v^p(y)}{p}\right)\right]\mathrm{d}x\mathrm{d}y\\
&=-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Bigg[(\mu_1-N)|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(y)\varphi(\lambda x)\frac{v^p(x)}{p}\frac{x(x-y)}{|x-y|^2}\\
&\quad+(\mu_1-N)|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(x)\varphi(\lambda y)\frac{v^p(y)}{p}\frac{-y(x-y)}{|x-y|^2}\\
&\quad-\alpha|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(y)\varphi(\lambda x)\frac{v^p(x)}{p}\\
&\quad-\alpha|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(x)\varphi(\lambda y)\frac{v^p(y)}{p}\\
&\quad+\lambda|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(y)\left[x\cdot\nabla\varphi(\lambda x)\right]\frac{v^p(x)}{p}\\
&\quad+\lambda|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(x)\left[y\cdot\nabla\varphi(\lambda y)\right]\frac{v^p(y)}{p}\\
&\quad+N|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(y)\varphi(\lambda x)\frac{v^p(x)}{p}\\
&\quad+N|x-y|^{\mu_1-N}|x|^{-\alpha}|y|^{-\alpha}v^p(x)\varphi(\lambda y)\frac{v^p(y)}{p}\Bigg]\mathrm{d}x\mathrm{d}y\\
&=\frac{N-\mu_1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-\mu_1}}\frac{v^p(x)}{|x|^{\alpha}}\frac{v^p(y)}{|y|^{\alpha}}\frac{(x-y)\cdot(x\varphi(\lambda x)-y\varphi(\lambda y))}{|x-y|^2}\mathrm{d}x\mathrm{d}y\\
&\quad+\frac{\alpha}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-\mu_1}}\frac{v^p(x)}{|x|^{\alpha}}\frac{v^p(y)}{|y|^{\alpha}}\varphi(\lambda x)\mathrm{d}x\mathrm{d}y\\
&\quad-\frac{N}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-\mu_1}}\frac{v^p(x)}{|x|^{\alpha}}\frac{v^p(y)}{|y|^{\alpha}}\varphi(\lambda x)\mathrm{d}x\mathrm{d}y\\
&\quad-\frac{\lambda}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N-\mu_1}}\frac{v^p(x)}{|x|^{\alpha}}\frac{v^p(y)}{|y|^{\alpha}}\left[x\cdot\nabla\varphi(\lambda x)\right]\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Using the Lebesgue dominated convergence theorem, we get the first equality. Similarly, we obtain the second equality.

Lemma 2.2. *Under the assumptions of Theorem 1.1, the following identity holds*

$$4\int_{\mathbb{R}^N} uv\mathrm{d}x = \left[\frac{N+\mu_1-2\alpha}{p}-(N-2)\right]\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\mathrm{d}x$$
$$+\left[\frac{N+\mu_2-2\beta}{q}-(N-2)\right]\int_{\mathbb{R}^N}\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}\mathrm{d}x.$$

Proof. A direct computation, we have

$$v\Delta\tilde{u}_\lambda = \lambda^2(x\cdot\nabla u)v\Delta\varphi(\lambda x)+2\lambda v\nabla\varphi(\lambda x)\cdot\nabla(x\cdot\nabla u)+\varphi(\lambda x)v\Delta(x\cdot\nabla u),$$
$$\varphi(\lambda x)v\Delta(x\cdot\nabla u)=\varphi(\lambda x)v\left[2\Delta u+x\cdot\nabla(\Delta u)\right],$$

and

$$\begin{aligned}&\frac{\partial}{\partial_{x_i}}\left[\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}x_i\right]\\&=(\mu_1-N)\int_{\mathbb{R}^N}\frac{I_{\mu_1}(x-y)v^p(y)v^p(x)x_i(x_i-y_i)}{|x-y|^2|x|^\alpha|y|^\alpha}\mathrm{d}y\\&\quad+p\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^{p-1}}{|x|^\alpha}v_{x_i}x_i-\alpha\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\frac{x_i^2}{|x|^2}+\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}.\end{aligned}\tag{2.2}$$

Since (u,v) solves (1.1), we have $-\Delta u+u=\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^{p-1}}{|x|^\alpha}$, combing this with (2.2), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} v\Delta\tilde{u}_\lambda\mathrm{d}x\\&=\lambda^2\int_{\mathbb{R}^N}(x\cdot\nabla u)v\Delta\varphi(\lambda x)\mathrm{d}x+2\lambda\int_{\mathbb{R}^N}v\nabla\varphi(\lambda x)\cdot\nabla(x\cdot\nabla u)\mathrm{d}x\\&\quad+2\int_{\mathbb{R}^N}\varphi(\lambda x)uv\mathrm{d}x-2\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}v\tilde{u}_\lambda(x)\mathrm{d}x-\sum_{i=1}^N\int_{\mathbb{R}^N}\varphi(\lambda x)\frac{\partial}{\partial_{x_i}}\left[\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}x_i\right]\mathrm{d}x\\&\quad+\sum_{i=1}^N\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^{p-1}}{|x|^\alpha}v_{x_i}x_i\mathrm{d}x+N\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\mathrm{d}x.\end{aligned}\tag{2.3}$$

Analogously,

$$\begin{aligned}&\int_{\mathbb{R}^N} u\Delta\tilde{v}_\lambda\mathrm{d}x\\&=\lambda^2\int_{\mathbb{R}^N}(x\cdot\nabla v)u\Delta\varphi(\lambda x)\mathrm{d}x+2\lambda\int_{\mathbb{R}^N}u\nabla\varphi(\lambda x)\cdot\nabla(x\cdot\nabla v)\mathrm{d}x\\&\quad+2\int_{\mathbb{R}^N}\varphi(\lambda x)uv\mathrm{d}x-2\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}\mathrm{d}x\end{aligned}$$

$$\begin{aligned}&+\int_{\mathbb{R}^N} u\tilde{v}_\lambda(x)\mathrm{d}x-\sum_{i=1}^N\int_{\mathbb{R}^N}\varphi(\lambda x)\frac{\partial}{\partial_{x_i}}\left[\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}x_i\right]\mathrm{d}x\\&+\sum_{i=1}^N\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^{q-1}}{|x|^\beta}u_{x_i}x_i\mathrm{d}x+N\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}\mathrm{d}x.\end{aligned}\tag{2.4}$$

We multiply the first equation in (1.1) by $\tilde{v}_\lambda$, and integrate over $\mathbb{R}^N$, subtract from (2.3) to obtain

$$\begin{aligned}&2\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\mathrm{d}x-2\int_{\mathbb{R}^N}\varphi(\lambda x)uv\mathrm{d}x\\&=-\int_{\mathbb{R}^N}v\Delta\tilde{u}_\lambda\mathrm{d}x+\lambda^2\int_{\mathbb{R}^N}(x\cdot\nabla u)v\Delta\varphi(\lambda x)\mathrm{d}x\\&\quad+2\lambda\int_{\mathbb{R}^N}v\nabla\varphi(\lambda x)\cdot\nabla(x\cdot\nabla u)\mathrm{d}x\\&\quad-\sum_{i=1}^N\int_{\mathbb{R}^N}\varphi(\lambda x)\frac{\partial}{\partial_{x_i}}\left[\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}x_i\right]\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}v\tilde{u}_\lambda(x)\mathrm{d}x+N\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}\tilde{v}_\lambda\Delta u\mathrm{d}x-\int_{\mathbb{R}^N}u\tilde{v}_\lambda\mathrm{d}x+2\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^{p-1}}{|x|^\alpha}\tilde{v}_\lambda\mathrm{d}x.\end{aligned}\tag{2.5}$$

Similarly,

$$\begin{aligned}&2\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}\mathrm{d}x-2\int_{\mathbb{R}^N}\varphi(\lambda x)uv\mathrm{d}x\\&=-\int_{\mathbb{R}^N}u\Delta\tilde{v}_\lambda\mathrm{d}x+\lambda^2\int_{\mathbb{R}^N}(x\cdot\nabla v)u\Delta\varphi(\lambda x)\mathrm{d}x\\&\quad+2\lambda\int_{\mathbb{R}^N}u\nabla\varphi(\lambda x)\cdot\nabla(x\cdot\nabla v)\mathrm{d}x\\&\quad-\sum_{i=1}^N\int_{\mathbb{R}^N}\varphi(\lambda x)\frac{\partial}{\partial_{x_i}}\left[\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}x_i\right]\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}u\tilde{v}_\lambda(x)\mathrm{d}x+N\int_{\mathbb{R}^N}\varphi(\lambda x)\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}\mathrm{d}x\\&\quad+\int_{\mathbb{R}^N}\tilde{u}_\lambda\Delta v\mathrm{d}x-\int_{\mathbb{R}^N}v\tilde{u}_\lambda\mathrm{d}x+2\int_{\mathbb{R}^N}\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^{q-1}}{|x|^\beta}\tilde{u}_\lambda\mathrm{d}x.\end{aligned}\tag{2.6}$$

Recalling that $u(x),v(x)\to 0$ when $|x|\to\infty$ and $\varphi\in C_0^\infty(\mathbb{R}^N,[0,1])$, by divergence theorem and Fubini theorem, we obtain

$$-\int_{\mathbb{R}^N}v\Delta\tilde{u}_\lambda\mathrm{d}x+\int_{\mathbb{R}^N}\tilde{v}_\lambda\Delta u\mathrm{d}x-\int_{\mathbb{R}^N}u\Delta\tilde{v}_\lambda\mathrm{d}x+\int_{\mathbb{R}^N}\tilde{u}_\lambda\Delta v\mathrm{d}x=0$$

and

$$\begin{aligned}&-\sum_{i=1}^N\int_{\mathbb{R}^N}\varphi(\lambda x)\frac{\partial}{\partial_{x_i}}\left[\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}x_i\right]\mathrm{d}x-\sum_{i=1}^N\int_{\mathbb{R}^N}\varphi(\lambda x)\frac{\partial}{\partial_{x_i}}\left[\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}x_i\right]\mathrm{d}x\\&=\lambda\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\left[x\cdot\nabla\varphi(\lambda x)\right]\mathrm{d}x+\lambda\int_{\mathbb{R}^N}\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}\left[x\cdot\nabla\varphi(\lambda x)\right]\mathrm{d}x.\end{aligned}$$

Consequently, adding (2.5) and (2.6), by Lemma 2.2, it follows that

$$2\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\mathrm{d}x+2\int_{\mathbb{R}^N}\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}\mathrm{d}x-4\int_{\mathbb{R}^N}uv\mathrm{d}x$$
$$=N\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\mathrm{d}x+\frac{-N-\mu_1+2\alpha}{p}\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}$$
$$+N\int_{\mathbb{R}^N}\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}\mathrm{d}x+\frac{-N-\mu_2+2\beta}{q}\int_{\mathbb{R}^N}\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta},$$

which is equivalent to

$$4\int_{\mathbb{R}^N}uv\mathrm{d}x=\left[\frac{N+\mu_1-2\alpha}{p}-(N-2)\right]\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\mathrm{d}x$$
$$+\left[\frac{N+\mu_2-2\beta}{q}-(N-2)\right]\int_{\mathbb{R}^N}\left(I_{\mu_2}*\frac{u^q}{|x|^\beta}\right)\frac{u^q}{|x|^\beta}\mathrm{d}x.$$

The proof of Lemma 2.2 is complete.

Proof of Theorem 1.1: If

$$(u,v)\in\left(C^2\left(\mathbb{R}^N\setminus\{0\}\right)\cap H^1\left(\mathbb{R}^N\right)\cap L^{\frac{2Np}{N+\mu_1-2\alpha}}\left(\mathbb{R}^N\right)\right)\times\left(C^2\left(\mathbb{R}^N\setminus\{0\}\right)\cap H^1\left(\mathbb{R}^N\right)\cap L^{\frac{2Nq}{N+\mu_2-2\beta}}\left(\mathbb{R}^N\right)\right)$$

is a pair of non-negative solutions of (1.1), suppose that $\frac{N+\mu_1-2\alpha}{p}+\frac{N+\mu_2-2\beta}{q}\le 2(N-2)$, by Lemma 2.2, there holds

$$0\le 4\int_{\mathbb{R}^N}uv\mathrm{d}x=\left(\frac{N+\mu_1-2\alpha}{p}+\frac{N+\mu_2-2\beta}{q}-2N+4\right)\int_{\mathbb{R}^N}\left(I_{\mu_1}*\frac{v^p}{|x|^\alpha}\right)\frac{v^p}{|x|^\alpha}\mathrm{d}x\le 0,$$

which indicates $v=0$. Similarly, we can prove that $u=0$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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