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Representation of Physical Fields as Einstein Manifold

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Abstract

In this work we investigate the possibility to represent physical fields as Einstein manifold. Based on the Einstein field equations in general relativity, we establish a general formulation for determining the metric tensor of the Einstein manifold that represents a physical field in terms of the energy-momentum tensor that characterises the physical field. As illustrations, we first apply the general formulation to represent the perfect fluid as Einstein manifold. However, from the established relation between the metric tensor and the energy-momentum tensor, we show that if the trace of the energy-momentum tensor associated with a physical field is equal to zero then the corresponding physical field cannot be represented as an Einstein manifold. This situation applies to the electromagnetic field since the trace of the energy-momentum of the electromagnetic field vanishes. Nevertheless, we show that a system that consists of the electromagnetic field and non-interacting charged particles can be represented as an Einstein manifold since the trace of the corresponding energy-momentum of the system no longer vanishes. As a further investigation, we show that it is also possible to represent physical fields as maximally symmetric spaces of constant scalar curvature.

Keywords

General Relativity, Einstein Manifold, Energy-Momentum Tensor, Electromagnetic Field, Perfect Fluid, Maximally Symmetric Spaces

1. Introduction

In physics, Einstein theory of general relativity is basically a theory of gravity that describes the gravitational field in terms of differential geometry. The fundamental formulation of the theory is expressed in a mathematical form through Einstein field equations that identify mathematical objects with physical entities which are, respectively, the Ricci curvature tensor and the energy-momentum tensor. With this type of physical formulation of the gravitational field, spacetime continuum is assumed to be a four-dimensional differentiable manifold, which is normally identified as a Riemann differentiable manifold, and solutions to Einstein field equations are represented by metrics of the spacetime continuum. It should be mentioned here that we always refer to differentiable manifolds endowed with a metric as Riemann differentiable manifolds regardless of the signature of their metric tensors. Except for a few well-known solutions to Einstein field equations, which are obtained by requiring symmetric conditions, the complex system of Einstein field equations cannot generally be integrated [1].

One particular class of Riemann differentiable manifolds that have been studied mathematically and applied physically in mathematical physics is the class of Einstein manifolds where the Ricci curvature tensor is proportional to the metric tensor [2] [3]. An Einstein manifold can be a Lorentzian manifold, as applied to the spacetime continuum in general relativity, or a Euclidean manifold, as being studied in gravitational instantons.

On physical attributes to these developments, in this work we discuss the possibility to represent physical fields as Einstein manifold. By using the Einstein field equations in general relativity, first in Section 2 we establish a general formulation for determining the metric tensor of the Einstein manifold that represents a physical field in terms of the energy-momentum tensor that characterises the physical field. As an illustration, in Section 3 we apply the general formulation to represent the perfect fluid as Einstein manifold. However, from the established relationship between the metric tensor and the energy-momentum tensor we can show that if the trace of the energy-momentum tensor associated with a physical field is equal to zero then the corresponding physical field cannot be represented as an Einstein manifold. A particular example is the electromagnetic field, since the trace of the energy-momentum of the electromagnetic field vanishes therefore the electromagnetic field cannot be represented as an Einstein manifold. However, as shown in Section 4, a system that consists of the electromagnetic field and non-interacting charged particles can be represented as an Einstein manifold since the corresponding trace of such system no longer vanishes. As a further discussion, in Section 5 we investigate the possibility to represent physical fields as maximally symmetric spaces of constant scalar curvature.

2. General Formulation

In this section we investigate the possibility to represent physical fields as Einstein manifold. Based on the Einstein field equations in general relativity, we establish a relation to determine the metric tensor of the Einstein manifold that represents a physical field in terms of the energy-momentum tensor that characterises the field. In Einstein general theory of relativity, gravitational fields are represented as Riemann differentiable manifolds whose geometric structures are determined by the Einstein field equations

$$R_{\alpha\beta} = k\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) \tag{1}$$

where $R_{\alpha\beta}$ is the Ricci curvature tensor, $T_{\alpha\beta}$ an energy-momentum tensor, $g_{\alpha\beta}$ a metric tensor, the trace $T = g^{\alpha\beta}T_{\alpha\beta}$, and k a dimensional constant [4]. On the other hand, in differential geometry, Einstein manifolds are Riemann differentiable manifolds whose Ricci curvature tensor is proportional to the metric tensor expressed in the form

$$R_{\alpha\beta} = \lambda g_{\alpha\beta} \tag{2}$$

Now, if we assume that the Riemann differentiable manifolds that satisfy the Einstein field equations given in Equation (1) also possess the geometric structure of Einstein manifold given in Equation (2) then we obtain the following relation

$$\lambda g_{\alpha\beta} = k\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right) \tag{3}$$

For an n-dimensional Riemann differentiable manifold, by contracting this relation with the inverse metric tensor $g^{\alpha\beta}$, we obtain

$$\lambda = \left(\frac{1}{n} - \frac{1}{2}\right)kT \tag{4}$$

In this work we will discuss the representation of physical fields for spacetime continuum of dimension four, *i.e.*, $n = 4$, hence we have

$$\lambda = -\frac{kT}{4} \tag{5}$$

Then, the relation given in Equation (3) is simplified and reduced to the required relation which establishes the metric tensor for an Einstein manifold in terms of the energy-momentum tensor

$$g_{\alpha\beta} = \frac{4}{T}T_{\alpha\beta} \tag{6}$$

It is seen from Equation (6) that physical fields associated with energy-momentum tensors which have vanishing trace, *i.e.*, $T = 0$, cannot be represented as Einstein manifold. For example, as shown in classical electrodynamics, the energy-momentum tensor for the electromagnetic field without charged particles can be presented in the form [5]

$$T_{\alpha\beta} = \frac{1}{\mu_0}\left(-F_{\alpha\gamma}F_{\beta}^{\gamma} + \frac{1}{4}g_{\alpha\beta}F_{\lambda\sigma}F^{\lambda\sigma}\right) \tag{7}$$

And it can be shown that the trace T of this energy-momentum tensor is equal to zero. However, as we will demonstrate in Section 4, if we consider instead the energy-momentum tensor of a physical system that includes the electromagnetic field and non-interacting charged particles then we can have $T \neq 0$, and such a system can be represented as Einstein manifold.

3. Representation of Perfect Fluid as Einstein Manifold

In this section we discuss the possibility to represent perfect fluid as Einstein manifold. It is shown in fluid dynamics that in the absence of the gravitational field the energy-momentum tensor $T_{\alpha\beta}$ of a perfect fluid takes the form [6] [7] [8]

$$T_{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) u_\alpha u_\beta - p g_{\alpha\beta} \tag{8}$$

where $u_\alpha = g_{\alpha\beta} u^\beta$ with the four velocity $u^\beta = \mathrm{d}x^\beta / \mathrm{d}\tau$ being defined for a comoving fluid element. Using the result $g^{\alpha\beta} u_\alpha u_\beta = c^2$ we then obtain

$$T = \rho c^2 - 3p \tag{9}$$

In general, we have $T = \rho c^2 - 3p \neq 0$, and by applying the relation given in Equation (6) we obtain the following relation

$$g_{\alpha\beta} = \frac{4}{\rho c^2 - 3p}\left(\left(\rho + \frac{p}{c^2}\right) u_\alpha u_\beta - p g_{\alpha\beta}\right) \tag{10}$$

By rearranging Equation (10) for the metric tensor $g_{\alpha\beta}$ we then obtain the metric tensor of the Einstein manifold that represents the perfect fluid as

$$g_{\alpha\beta} = \frac{4}{c^2} u_\alpha u_\beta \tag{11}$$

The components of this metric tensor can be written in a matrix form as

$$g_{\alpha\beta} = \frac{4}{c^2}\begin{pmatrix} c^2 & cu_x & cu_y & cu_z \\ cu_x & u_x^2 & u_x u_y & u_x u_z \\ cu_y & u_x u_y & u_y^2 & u_y u_z \\ cu_z & u_x u_z & u_y u_z & u_z^2 \end{pmatrix} \tag{12}$$

By using the coordinate system (ct, x, y, z), and with the metric tensor given in Equation (11) in terms of the four velocity, the line element $\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$ for the Einstein manifold that represents the perfect fluid can be specified and written as follows

$$\begin{aligned}\mathrm{d}s^2 = \frac{4}{c^2}\Big[& c^2 (c\mathrm{d}t)^2 + u_x^2 \mathrm{d}x^2 + u_x^2 \mathrm{d}y^2 + u_x^2 \mathrm{d}z^2 + 2cu_x (c\mathrm{d}t)\mathrm{d}x + 2cu_y (c\mathrm{d}t)\mathrm{d}y \\ & + 2cu_z (c\mathrm{d}t)\mathrm{d}z + 2u_x u_y \mathrm{d}x\mathrm{d}y + 2u_x u_z \mathrm{d}x\mathrm{d}z + 2u_y u_z \mathrm{d}y\mathrm{d}z \Big]\end{aligned} \tag{13}$$

4. Representation of System of Electromagnetic Field and Non-Interacting Charged Particles as Einstein Manifold

In this section we show that it is possible to represent a system consisting of the electromagnetic field and non-interacting charged particles as an Einstein manifold. As mentioned above in Section 2, due to the fact that the trace T of the energy-momentum tensor of the electromagnetic field given in Equation (7) is equal to zero therefore the relation given in Equation (6) for the metric tensor in terms of the energy-momentum tensor cannot be used to represent the electromagnetic field as Einstein manifold. However, when a system consisting of the electromagnetic field and non-interacting charged particles is considered then the total energy-momentum tensor $T_{\alpha\beta}$ of the whole system can be shown to take the form [5]

$$T_{\alpha\beta} = \frac{1}{\mu_0}\left(-F_{\alpha\lambda}F_{\beta}^{\ \lambda} + \frac{1}{4}g_{\alpha\beta}F_{\lambda\sigma}F^{\lambda\sigma}\right) + \rho u_\alpha u_\beta \frac{\mathrm{ds}}{\mathrm{dt}} \tag{14}$$

where $T_{\alpha\beta}(p) = \rho u_\alpha u_\beta\, \mathrm{ds}/\mathrm{dt}$ is the energy-momentum tensor for non-interacting charged particles with $\rho = \sum_i m_i(\boldsymbol{r} - \boldsymbol{r}_i)$ being the mass density, and the trace T of the total energy-momentum tensor $T_{\alpha\beta}$ in this case can be shown as

$$T = \gamma\rho c^2 \tag{15}$$

where $\gamma = \mathrm{ds}/\mathrm{dt} = \sqrt{1 - v^2/c^2}$. Furthermore, since the energy-momentum tensor $T_{\alpha\beta}$ given in Equation (14) for the system of the electromagnetic field and non-interacting charged particles also satisfies the conservation law, therefore we can apply the Einstein field equations of general relativity. In this case, by using Equation (6), we obtain

$$g_{\alpha\beta} = \frac{4}{F_{\lambda\sigma}F^{\lambda\sigma} - \mu_0\gamma\rho c^2}\left(F_{\alpha\lambda}F_{\beta}^{\ \lambda} - \mu_0\gamma\rho u_\alpha u_\beta\right) \tag{16}$$

The components $\mu_0\gamma\rho u_\alpha u_\beta$ of this metric tensor can be written in a matrix form as

$$\mu_0\gamma\rho u_\alpha u_\beta = \mu_0\gamma\rho\begin{pmatrix} c^2 & cu_x & cu_y & cu_z \\ cu_x & u_x^2 & u_xu_y & u_xu_z \\ cu_y & u_xu_y & u_y^2 & u_yu_z \\ cu_z & u_xu_z & u_yu_z & u_z^2 \end{pmatrix} \tag{17}$$

On the other hand, the components $F_{\lambda\sigma}F^{\lambda\sigma}$ and $F_{\alpha\lambda}F_{\beta}^{\ \lambda}$ can be obtained by using the electromagnetic field tensor given by

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{18}$$

With the electromagnetic field tensor given in Equation (18), by applying the Minkowski metric tensor with signature $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$ we can derive the following results

$$F_{\mu}^{\ \nu} = F_{\mu\sigma}\eta^{\sigma\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{19}$$

$$F_{\lambda\sigma}F^{\lambda\sigma} = -E^2/c^2 + B^2 \tag{20}$$

$$F_{\alpha\lambda}F_{\beta}^{\ \lambda} = \begin{pmatrix} -E^2/c^2 & E_yB_z/c - E_zB_y/c & -E_xB_z/c + E_zB_x/c & E_xB_y/c - E_yB_x/c \\ E_yB_z/c - E_zB_y/c & E_x^2/c^2 - B_y^2 - B_z^2 & E_xE_y/c^2 + B_xB_y & E_xE_z/c^2 + B_xB_z \\ -E_xB_z/c + E_zB_x/c & E_xE_y/c^2 + B_xB_y & E_y^2/c^2 - B_x^2 - B_z^2 & E_yE_z/c^2 + B_yB_z \\ E_xB_y/c - E_yB_x/c & E_xE_z/c^2 + B_xB_z & E_yE_z/c^2 + B_yB_z & E_z^2/c^2 - B_x^2 - B_y^2 \end{pmatrix} \tag{21}$$

where $E^2 = E_x^2 + E_y^2 + E_z^2$ and $B^2 = B_x^2 + B_y^2 + B_z^2$. It is important to emphasise here that all relationships between the physical quantities related to physical fields under consideration are still obtained by applying the Minkowski metric tensor as in special relativity. We only use the final form of an energy-momentum tensor associated with a physical field as a form for the metric tensor $g_{\alpha\beta}$ to represent the corresponding physical fields as Einstein manifold.

Also, by using the coordinate system (ct, x, y, z) with the metric tensor given in Equation (16), together with the above results obtained for the components $\mu_0\gamma\rho u_\alpha u_\beta$, $F_{\lambda\sigma}F^{\lambda\sigma}$ and $F_{\alpha\lambda}F_\beta^{\ \lambda}$, the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ for the Einstein manifold that represents the system of the electromagnetic field and non-interacting charged particles can be written as

$$\begin{aligned}\mathrm{d}s^2 = \frac{4}{-E^2/c^2 + B^2 - \mu_0\gamma\rho c^2}\Big[&\left(-E^2/c^2 - \mu_0\gamma\rho c^2\right)(c\mathrm{d}t)^2 \\ &+\left(E_x^2/c^2 - B_y^2 - B_z^2 - \mu_0\gamma\rho u_x^2\right)\mathrm{d}x^2 \\ &+\left(E_y^2/c^2 - B_x^2 - B_z^2 - \mu_0\gamma\rho u_y^2\right)\mathrm{d}y^2 \\ &+\left(E_z^2/c^2 - B_x^2 - B_y^2 - \mu_0\gamma\rho u_z^2\right)\mathrm{d}z^2 \\ &+2\left(E_y B_z/c - E_z B_y/c - \mu_0\gamma\rho c u_x\right)c\mathrm{d}t\mathrm{d}x \\ &+2\left(-E_x B_z/c + E_z B_x/c - \mu_0\gamma\rho c u_y\right)c\mathrm{d}t\mathrm{d}y \\ &+2\left(E_x B_y/c - E_y B_x/c - \mu_0\gamma\rho c u_z\right)c\mathrm{d}t\mathrm{d}z \\ &+2\left(E_x E_y/c^2 + B_x B_y - \mu_0\gamma\rho u_x u_y\right)\mathrm{d}x\mathrm{d}y \\ &+2\left(E_x E_z/c^2 + B_x B_z - \mu_0\gamma\rho u_x u_z\right)\mathrm{d}x\mathrm{d}z \\ &+2\left(E_y E_z/c^2 + B_y B_z - \mu_0\gamma\rho u_y u_z\right)\mathrm{d}y\mathrm{d}z\Big]\end{aligned} \tag{22}$$

For the particular case in which there is only the electric field, *i.e.*, $\boldsymbol{B} = 0$, then the above line element reduces to

$$\begin{aligned}\mathrm{d}s^2 &= \frac{4}{-E^2/c^2 - \mu_0\gamma\rho c^2}\left[\left(-E^2/c^2\right)(c\mathrm{d}t)^2 + \left(E_x\mathrm{d}x + E_y\mathrm{d}y + E_z\mathrm{d}z\right)^2/c^2\right] \\ &+ \frac{-4\mu_0\gamma\rho}{-E^2/c^2 - \mu_0\gamma\rho c^2}\left[\left(c(c\mathrm{d}t) + u_x\mathrm{d}x + u_y\mathrm{d}y + u_z\mathrm{d}z\right)^2\right]\end{aligned} \tag{23}$$

On the other hand, for the case of magnetic field only with $\boldsymbol{E} = 0$, the line element given in Equation (22) reduces to

$$\begin{aligned}\mathrm{d}s^2 &= \frac{4}{B^2 - \mu_0\gamma\rho c^2}\left[\left(B_x\mathrm{d}x + B_y\mathrm{d}y + B_z\mathrm{d}z\right)^2 - B^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\right] \\ &+ \frac{-4\mu_0\gamma\rho}{B^2 - \mu_0\gamma\rho c^2}\left[\left(c(c\mathrm{d}t) + u_x\mathrm{d}x + u_y\mathrm{d}y + u_z\mathrm{d}z\right)^2\right]\end{aligned} \tag{24}$$

5. Representation of Physical Fields as Spaces of Constant Scalar Curvature

In this section we investigate the possibility to represent physical fields as spaces of constant scalar curvature. As shown in our previous work, by utilizing Einstein field equations given in Equation (1), a system of general relativistic field equations for the Riemann curvature tensor can be obtained. In differential geometry, it is shown that the covariant Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ satisfies the Bianchi identities [9] [10] [11]

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0 \tag{25}$$

The Bianchi identities given in Equation (25) can be contracted to give the following relation between the Riemann curvature tensor and the Ricci curvature tensor

$$\nabla_\mu R^\mu_{\beta\nu\lambda} = \nabla_\nu R_{\beta\lambda} - \nabla_\lambda R_{\beta\nu} \tag{26}$$

Employing Equation (26), we then suggest that a system of physical field equations for the Riemann curvature tensor can be written in the form [12]

$$\nabla_\mu R^\mu_{\beta\nu\lambda} = J_{\beta\nu\lambda} \tag{27}$$

where the quantity $J_{\beta\nu\lambda}$ is identified as a third rank current as

$$J_{\beta\nu\lambda} = \nabla_\nu R_{\beta\lambda} - \nabla_\lambda R_{\beta\nu} \tag{28}$$

The third rank current $J_{\beta\nu\lambda}$ defined in terms of the covariant derivatives of the Ricci curvature tensor in Equation (28) is purely geometrical, therefore we would need to identify the mathematical object $\nabla_\mu R_{\alpha\beta}$ with a physical entity. For this purpose, we again employ the Einstein field equations of general relativity given in Equation (1) to write the current $J_{\beta\nu\lambda}$ in terms of the energy-momentum tensor $T_{\alpha\beta}$ as

$$J_{\beta\nu\lambda} = k\left(\nabla_\nu\left(T_{\beta\lambda} - \frac{1}{2}g_{\beta\lambda}T\right) - \nabla_\lambda\left(T_{\beta\nu} - \frac{1}{2}g_{\beta\nu}T\right)\right) \tag{29}$$

Then the field equations for the Riemann curvature tensor given in Equation (27) can take the form of physical field equations

$$\nabla_\mu R^\mu_{\beta\nu\lambda} = k\left(\nabla_\nu\left(T_{\beta\lambda} - \frac{1}{2}g_{\beta\lambda}T\right) - \nabla_\lambda\left(T_{\beta\nu} - \frac{1}{2}g_{\beta\nu}T\right)\right) \tag{30}$$

For physical fields that can be represented as Einstein manifold, we can apply the relation given in Equation (6), *i.e.*, $g_{\alpha\beta} = (4/T)T_{\alpha\beta}$, and with the metric condition $\nabla_\mu g_{\alpha\beta} = 0$, then the equation for the third rank current $J_{\beta\nu\lambda}$ given in Equation (29) can be rewritten in the form

$$J_{\beta\nu\lambda} = -\frac{k}{4}\left(g_{\beta\lambda}\nabla_\nu T - g_{\beta\nu}\nabla_\lambda T\right) \tag{31}$$

Since from the Einstein field equations given in Equation (1), we obtain $R = -kT$, therefore, if the trace T of the energy-momentum tensor associated with a physical field is constant then the Ricci scalar curvature R of the corresponding Einstein manifold that represents the physical field is also constant, or inversely. With the condition of constancy of the trace T of an energy-momentum tensor, from Equation (31) we obtain $J_{\beta\nu\lambda} = 0$, and the field equations for the Riemann curvature tensor given in Equation (30) are reduced to

$$\nabla_\mu R^\mu_{\beta\nu\lambda} = 0 \tag{32}$$

Again, using the metric identity $\nabla_\mu g_{\alpha\beta} = 0$ and the symmetric properties associated with the Riemann curvature tensor, particular solutions to the field equations for the Riemann curvature tensor given in Equation (32) can be written in the form

$$R_{\alpha\beta\mu\nu} = K\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{33}$$

where K is a dimensional constant. Using the relationship given in Equation (6) between the metric tensor and the energy-momentum tensor, Equation (33) for the Riemann curvature tensor can be expressed entirely in terms of the energy-momentum tensor as

$$R_{\alpha\beta\mu\nu} = \frac{16K}{T^2}\left(T_{\alpha\mu}T_{\beta\nu} - T_{\alpha\nu}T_{\beta\mu}\right) \tag{34}$$

In differential geometry, a differentiable manifold of dimension n whose Riemann curvature tensor satisfies Equation (33) with K identified with scalar curvature is called a maximally symmetric space, because the metric of such space admits the maximal number of Killing vectors, which is $n(n+1)/2$. In fact, it can be shown that the Ricci curvature tensor $R_{\alpha\beta}$ and the Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ for maximally symmetric spaces take the forms [6]

$$R_{\alpha\beta} = \frac{R}{n}g_{\alpha\beta} \tag{35}$$

$$R_{\alpha\beta\mu\nu} = \frac{R}{n(n-1)}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \tag{36}$$

And it can also be shown that for spaces of dimension three or higher the Ricci scalar curvature R is constant. In this case, for our formulation we can identify the constant K in Equation (33) with $R/n(n-1)$. In cosmology, spaces that are characterised by the property given in Equation (33) are isotropic and homogeneous, and from which the Robertson-Walker metric can be established

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - S^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right) \tag{37}$$

where $k = K/|K|$ and the function $S(t)$ is the scale factor [13] [14] [15].

6. Conclusion

In this work we have investigated the possibility to represent physical fields as Einstein manifold. Based on the Einstein field equations, the objective of our work is to introduce and establish a general framework to determine the metric tensor in terms of the energy-momentum tensor of a physical field by assuming that the differentiable manifold that is characterised by the physical field is also an Einstein manifold. We then apply the general formulation to represent the perfect fluid as an Einstein manifold. The general formulation can also be applied for a system that consists of the electromagnetic field and non-interacting charged particles. However, the general formulation cannot be applied to the electromagnetic field since the trace of the energy-momentum tensor associated with the electromagnetic field is equal to zero. We have also shown that physical fields that satisfy certain conditions for their associated energy-momentum tensors can also be represented as spaces of constant scalar curvature. These spaces are maximally symmetric with the maximal number of Killing vectors.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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