

# Quantum Codes Do Not Increase Fidelity against Isotropic Errors

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## Abstract

In this article, we study the ability of error-correcting quantum codes to increase the fidelity of quantum states throughout a quantum computation. We analyze arbitrary quantum codes that encode all qubits involved in the computation, and we study the evolution of  $n$ -qubit fidelity from the end of one application of the correcting circuit to the end of the next application. We assume that the correcting circuit does not introduce new errors, that it does not increase the execution time (*i.e.* its application takes zero seconds) and that quantum errors are isotropic. We show that the quantum code increases the fidelity of the states perturbed by quantum errors but that this improvement is not enough to justify the use of quantum codes. Namely, we prove that, taking into account that the time interval between the application of the two corrections is multiplied (at least) by the number of qubits  $n$  (due to the coding), the best option is not to use quantum codes, since the fidelity of the uncoded state over a time interval  $n$  times smaller is greater than that of the state resulting from the quantum code correction.

## Keywords

Quantum Error Correcting Codes, Isotropic Quantum Computing Errors, Quantum Computing Error Fidelity, Quantum Computing Error Variance

## 1. Introduction

Currently, the largest obstacle to the development of quantum computing is still the control of quantum errors. Since the beginnings of quantum compu-

ting in the 90 s of the last century, one of the main research goals was to solve this stumbling block. To address the problem, two fundamental tools were developed: quantum error correction codes [1]-[6] in combination with fault tolerant quantum computing [7]-[13]. These studies culminated in the proof of the quantum threshold theorem, which reads as follows: a quantum computer with a physical error rate below a certain threshold can, through application of quantum error correction schemes, suppress the logical error rate to arbitrarily low levels. However, the proof of this theorem depends on the discretized treatment of quantum errors, inherited from the construction of quantum codes.

We believe that the quantum error model used for the proof of the quantum threshold theorem is not general and that the techniques developed to control quantum errors do not verify the golden rule of error control: correct all small errors exactly. For example, in the case of the coding of a qubit by means of the 5-qubit code [14] [15], it is argued, using error discretization and the fact that this code exactly corrects errors in any of the qubits, that the error probability goes from  $p$  to  $p^2$  once the correction circuit has been applied. But what is actually happening is that the probability of an error (small with high probability) in all qubits is 1 and that the code cannot correct these simultaneous errors. Then, an error occurs with probability 1 and, once the correction circuit is applied, it becomes undetectable.

Therefore, it is necessary to perform an analysis of quantum errors regardless of their discretization. The procedure we suggest is to consider quantum errors as continuous random variables and characterize them by their corresponding density functions. In this article, we analyze a specific type of error: isotropic quantum errors. An isotropic error of an  $n$ -qubit  $\Phi$  is one in which the probability of the state after the error,  $\Psi$ , only depends on the distance between the two states,  $\|\Psi - \Phi\|$ , and not on the direction in which the imprecision  $\Psi$  occurs with respect to  $\Phi$ . Isotropic errors are easy to analyze due to their central symmetry with respect to  $\Phi$ .

In [16] we have studied the ability of an arbitrary quantum code to correct these errors, using the variance as the error measure. If  $\Phi$  is the  $n$ -qubit without error state,  $\Psi$  the state resulting from a disturbance modeled by an isotropic quantum error and  $\tilde{\Phi}$  the result of applying the quantum code correction circuit, assuming that it does not introduce new errors, the result that we proved in [16] is the following:

$$V(\tilde{\Phi}) \geq V(\Psi),$$

where  $V(\tilde{\Phi}) = E[\|\tilde{\Phi} - \Phi\|^2]$  and  $V(\Psi) = E[\|\Psi - \Phi\|^2]$  are the variances of the corrected state  $\tilde{\Phi}$  and the disturbed state  $\Psi$  respectively. This means that no quantum code can handle isotropic errors, or even reduce their variance.

In the present work we are interested in analyzing the ability of quantum codes to increase fidelity against isotropic errors. Fidelity allows a better measure of quantum errors, since it takes into account that quantum states do not change if they are multiplied by a phase factor, while the variance used in [16] does not

consider phase invariance.

We represent  $n$ -qubits as points of the unit real sphere of dimension  $2d - 1$  (where  $d = 2^n$  [17])  $S^{2d-1} = \{x \in \mathbb{R}^{2d} \mid \|x\| = 1\}$ , taking coordinates with respect to the computational basis  $\left[|0\rangle, |1\rangle, \dots, |2^n - 1\rangle\right]$ ,

$$\Psi = (x_0 + ix_1, x_2 + ix_3, \dots, x_{2d-2} + ix_{2d-1}). \tag{1}$$

We consider quantum computing errors as random variables with density function defined on  $S^{2d-1}$ . In [16] we mention that it is easy to relate this representation to the usual representation in quantum computing by density matrices and that the representation through random variables is in fact more accurate.

We define the variance of a random variable  $X$  as the mean of the quadratic deviation from the mean value  $\mu$  of  $X$ ,  $V(X) = E\left[\|X - \mu\|^2\right]$ . In our case, since the random variable  $X$  represents a quantum computing error, the mean value of  $X$  is the  $n$ -qubit  $\Phi$  resulting from an errorless computation. Without loss of generality, we will assume that the mean value of every quantum computing error will always be  $\Phi = |0\rangle$ . To achieve this, it suffices to move  $\Phi$  into  $|0\rangle$  through a unitary transformation. Therefore, using the pure quantum states given by Formula (1), the variance of  $X$  will be

$$V(X) = E\left[\|\Psi - \Phi\|^2\right] = E\left[2 - 2x_0\right] = 2 - 2\int_{S^{2d-1}} x_0 f(x) dx. \tag{2}$$

Obviously the variance satisfies  $V(X) \in [0, 4]$ . In [18] an expression for the variance of the sum of two independent errors on  $S^{2d-1}$  is given. This expression is proved for isotropic errors and conjectured in general:

$$V(X_1 + X_2) = V(X_1) + V(X_2) - \frac{V(X_1)V(X_2)}{2}. \tag{3}$$

Considering the representation of errors through random variables, the definition of fidelity is just

$$F^2(X) = E\left[|\langle\Psi|\Phi\rangle|^2\right] = E\left[x_0^2 + x_1^2\right] = \int_{S^{2d-1}} (x_0^2 + x_1^2) f(x) dx. \tag{4}$$

Then, the problem we want to address is the following: Let  $\Phi_0$  be an  $m$ -qubit and  $\Phi$  the corresponding  $n$ -qubit encoded by an  $(n, m)$ -quantum code  $\mathcal{C}$ . Suppose that the coded state  $\Phi$  undergoes an error, becoming the state  $\Psi$ . To fix the error, the code correction circuit is applied and the final state  $\tilde{\Phi}$  is obtained. While  $\Phi$  is a pure state,  $\Psi$  and  $\tilde{\Phi}$  are random variables (mixed states).

We also want to study the alternative of not using quantum codes. In this case, we suppose that the initial state  $\Phi_0$  suffers an error, becoming the state  $\Psi_0$ . State  $\Psi_0$  is also a random variable. Then our goal is to compare the fidelities of  $\Psi$ ,  $\tilde{\Phi}$  and  $\Psi_0$ :

$$F(\Psi) = E\left[|\langle\Psi|\Phi\rangle|^2\right], F(\tilde{\Phi}) = E\left[|\langle\tilde{\Phi}|\Phi\rangle|^2\right] \text{ and } F(\Psi_0) = E\left[|\langle\Psi_0|\Phi_0\rangle|^2\right].$$

In order to compare the fidelities we will assume that the corrector circuit of  $\mathcal{C}$  does not introduce new errors and it does not increase the execution time. In

other words, what we are doing is estimate the theoretical capacity of the code to correct quantum computing errors.

In the case of isotropic errors we shall prove that

$$F(\Psi_0) \geq F(\tilde{\Phi}) \geq F(\Psi). \quad (5)$$

This result leads us to the conclusion that the best option to optimize fidelity against isotropic errors is not to use quantum codes. This result goes in the same direction as that obtained in [16], which indicates that quantum codes do not reduce the variance against isotropic errors.

However, the most widely used model of errors in quantum computing is qubit-independent errors. The study of this type of quantum error is much more complex than that of isotropic errors, because it lacks the symmetry. Despite this technical difficulty, we have proved in [19] that the 5-qubit code [14] [15] is not able to reduce the variance against qubit independent errors. This result, together with those obtained in [16] and in this article, clearly reveals the difficulty of the quantum error control challenge and strongly points out that the continuous nature of quantum errors cannot be ignored.

There are many works related to the control of quantum computing errors, in addition to those already mentioned above. General studies and surveys on the subject [20]-[27], about the quantum computation threshold theorem [28] [29] [30] [31], quantum error correction codes [32] [33] [34] [35], concatenated quantum error correction codes [36] [37] and articles related to topological quantum codes [38] [39]. Lately, quantum computing error control has focused on both coherent errors [40] [41] and cross-talk errors [42] [43]. Finally, we cannot forget the hardest error to control in quantum computing, the quantum decoherence [44]. As we have commented above, these quantum computing errors can be analyzed in the framework of random variables that has been set in [16] [18]. Thus, the general considerations stated in the conclusions section of the present work would also affect the study of the behaviour of these errors.

The outline of the article is as follows: in section 2 we study the fidelity of the quantum states  $\Psi$ ,  $\Psi_0$  and  $\tilde{\Phi}$ ; in section 3 we prove the relationship between them given by Formula (5); finally, in section 4 we analyze the conclusions that can be obtained from the main result.

## 2. Analysis of Fidelity

Given a  $(n, m)$ -quantum code  $\mathcal{C}$ , the following parameters are defined:  $d = 2^n$  is the dimension of  $\mathcal{C}$ ,  $d' = 2^m$  and  $d''$  is the number of discrete errors that  $\mathcal{C}$  corrects.

First we study how the fidelities of the quantum states  $\Psi$  and  $\tilde{\Phi}$ , which are  $n$ -qubits encoded with the quantum code  $\mathcal{C}$ , can be compared with the fidelity of the state  $\Psi_0$ , which is an unencoded  $m$ -qubit state. The working scheme in these two scenarios is illustrated in **Figure 1**. We assume that the  $\mathcal{C}$  correction

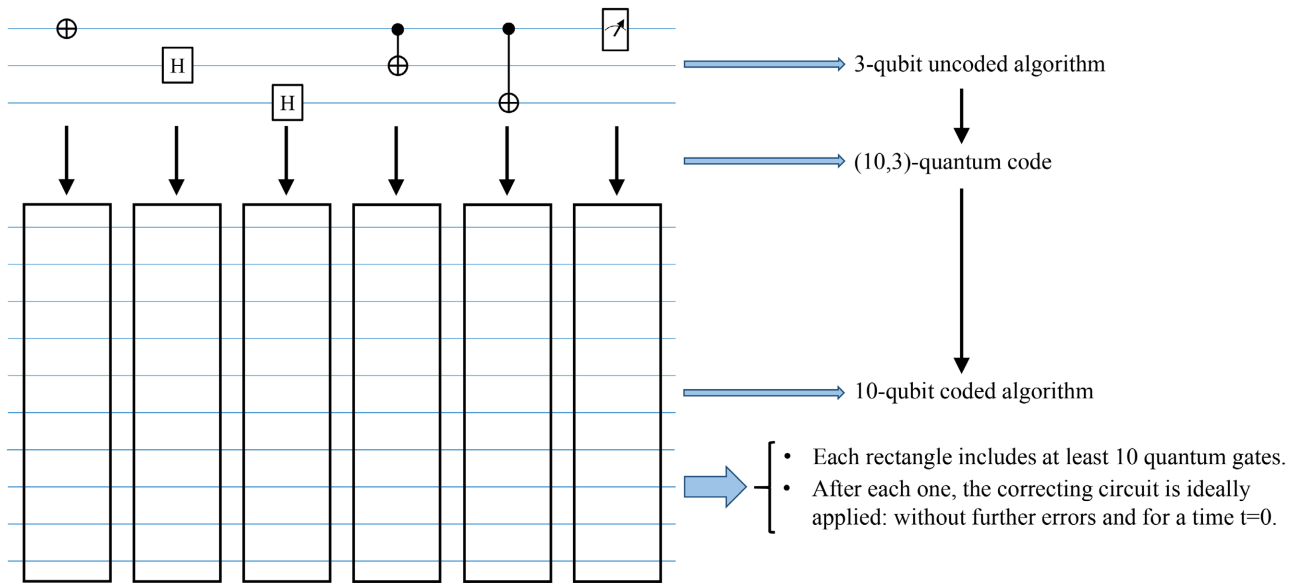


Figure 1. Uncoded/coded work scheme.

circuit, which is applied after each quantum gate in the coded algorithm, does not introduce new errors and is ideally applied in time  $t = 0$ . In this way, we study the theoretical capacity of  $\mathcal{C}$  to control isotropic errors, that is, its capacity to increase the fidelity of the final state  $\tilde{\Phi}$  with respect to  $\Psi$ . Furthermore, we can compare the fidelity of  $\tilde{\Phi}$  with the fidelity of the final state  $\Psi_0$  which is obtained in a scheme without the quantum code  $\mathcal{C}$ .

We analyze the isotropic error as a decoherence error over a unit of time. This unit corresponds to the time it takes to apply a quantum gate in the coded algorithm. To compare it with the uncoded algorithm we have to bear in mind that the unit of time in this case will be at most the  $n$ -th part of the unit of time in the coded algorithm. To relate the probability distributions in both cases we use the following equality of variances:

$$V(E) = V(E_1 + E_2 + \dots + E_n),$$

where  $E$  is the decoherence error during a unit of time in the coded algorithm and  $E_1, E_2, \dots, E_n$  are independent decoherence errors corresponding to a unit of time in the uncoded algorithm. Using the following generalization of Formula (3) (proven in [18]),

$$V(E_1 + E_2 + \dots + E_n) = 2 - 2 \left( 1 - \frac{v_u}{2} \right)^n, \tag{6}$$

where  $v_u$  is the variance of each of the independent errors, we obtain the following relation of  $v_u$  with the variance  $v_c$  of the error  $E$ :

$$v_c = 2 - 2 \left( 1 - \frac{v_u}{2} \right)^n \Leftrightarrow v_u = 2 - 2 \left( \frac{2 - v_c}{2} \right)^{1/n}. \tag{7}$$

In the case of the normal probability distribution defined in [16] [18], with the density function

$$f_n(\sigma, \theta_0) = \frac{(2d-2)!!}{(2\pi)^d} \frac{1-\sigma^2}{(1+\sigma^2-2\sigma\cos(\theta_0))^d}, \tag{8}$$

where the parameter  $\sigma$  belongs to the interval  $[0,1)$ , the above variances have a very simple expression and are independent of the dimension:  $v_c = 2(1-\sigma_c)$  and  $v_u = 2(1-\sigma_u)$ . The relationship between them given in Formula (7) translates into a very simple relationship between the corresponding sigma parameters:

$$\sigma_c = \sigma_u^n \Rightarrow \sigma_u = \sigma_c^{1/n}. \tag{9}$$

From now on we are going to follow the same scheme as the one proposed in [16] to calculate the variances of states  $\Psi$  and  $\tilde{\Phi}$ . Nevertheless, this time we will use it to calculate the fidelities of these states and the fidelity of the state  $\Psi_0$ .

### 2.1. Fidelity of $\Psi$ and $\Psi_0$

The state  $\Psi$ , described in Cartesian coordinates in Formula (1) can be written in spherical coordinates as follows:

$$\begin{aligned} \Psi &= (\theta_0, \theta_1, \dots, \theta_{2d-2}) \begin{cases} 0 \leq \theta_0, \dots, \theta_{2d-3} \leq \pi \\ 0 \leq \theta_{2d-2} \leq 2\pi \end{cases}, \\ x_j &= \sin(\theta_0) \cdots \sin(\theta_{j-1}) \cos(\theta_j) \quad \text{for all } 0 \leq j \leq 2d-2, \\ x_{2d-1} &= \sin(\theta_0) \cdots \sin(\theta_{2d-2}). \end{aligned}$$

On the other hand, the translation of Formula (4) into spherical coordinates is

$$F^2(X) = E[\cos^2(\theta_0) + \sin^2(\theta_0)\cos^2(\theta_1)] = 1 - E[\sin^2(\theta_0)\sin^2(\theta_1)]. \tag{10}$$

**Theorem 1.** *The fidelity of the isotropic random variable  $\Psi$  with density function  $f(\theta_0)$  is given by*

$$F^2(\Psi) = 1 - 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-1) \bar{E}[\sin^{2d}(\theta_0)], \tag{11}$$

where  $\bar{E}[\sin^{2d}(\theta_0)] = \int_0^\pi f(\theta_0) \sin^{2d}(\theta_0) d\theta_0$ .

*Proof.* We have to calculate the expected value of an expression that depends only on the angles  $\theta_0$  and  $\theta_1$ . Furthermore, the isotropic density function depends only on the angle  $\theta_0$ . Therefore, using Formula (10):

$$\begin{aligned} F^2(\Psi) &= 1 - |S^{2d-3}| \bar{E}[\sin^{2d}(\theta_0)] \int_0^\pi \sin^{2d-1}(\theta_1) d\theta_1 \\ &= 1 - \frac{(2\pi)^{d-1}}{(2d-4)!!} 2 \frac{(2d-2)!!}{(2d-1)!!} \bar{E}[\sin^{2d}(\theta_0)] \\ &= 1 - 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-1) \bar{E}[\sin^{2d}(\theta_0)]. \end{aligned}$$

Note that we have used equalities from the Appendix.  $\square$

**Corollary 1.** *The fidelity of the isotropic random variable  $\Psi$  with normal distribution  $f_n(\sigma_c, \theta_0)$  is equal to*

$$F^2(\Psi) = \frac{1 + (d-1)\sigma_c^2}{d}. \tag{12}$$

*Proof.* Using the definition of the normal distribution given in Formula (8) and some results from the Appendix:

$$\begin{aligned} F^2(\Psi) &= 1 - 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-1) \bar{E}[\sin^{2d}(\theta_0)] \\ &= 1 - 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-1) \frac{(2d-2)!!}{(2\pi)^d} (1-\sigma_c^2) \frac{(2d-1)!!}{(2d)!!} \pi \\ &= 1 - \frac{d-1}{d} (1-\sigma_c^2) = \frac{1+(d-1)\sigma_c^2}{d}. \end{aligned}$$

□

Theorem 1 and Corollary 1 also apply to state  $\Psi_0$ , just substituting the parameter  $d$  by  $d'$ .

**Corollary 2.** *The fidelity of the isotropic random variable  $\Psi_0$  with density function  $f(\theta_0)$  is equal to*

$$F^2(\Psi_0) = 1 - 4 \frac{(2\pi)^{d'-1}}{(2d'-1)!!} (d'-1) \bar{E}[\sin^{2d'}(\theta_0)], \tag{13}$$

where  $\bar{E}[\sin^{2d'}(\theta_0)] = \int_0^\pi f(\theta_0) \sin^{2d'}(\theta_0) d\theta_0$ . Furthermore, if the probability distribution of  $\Psi_0$  is normal with density function  $f_n(\sigma_u, \theta_0)$ , the fidelity is equal to:

$$F^2(\Psi_0) = \frac{1 + (d'-1)\sigma_u^2}{d'}. \tag{14}$$

To compare the fidelities of  $\Psi_0$  and  $\tilde{\Phi}$  we need to obtain their values as a function of their variances  $v_u$  and  $v_c$ , respectively. The relationship between these variances, obtained in Formula (7), allows us to relate the fidelities of these states.

**Theorem 2.** *The fidelity of the isotropic random variable  $\Psi_0$  with density function  $f(\theta_0)$  satisfies*

$$F^2(\Psi_0) \geq 1 - \frac{2d'-2}{2d'-1} \left( v_u - \left( \frac{v_u}{2} \right)^2 \right). \tag{15}$$

*Proof.* First we prove, in a similar way as in Theorem 1, the following:

$$\begin{aligned} F^2(\Psi_0) &= 1 - |S^{2d'-3}| \bar{E}[\sin^{2d'}(\theta_0)] \int_0^\pi \sin^{2d'-1}(\theta_1) d\theta_1 \\ &= 1 - |S^{2d'-3}| \bar{E}[\sin^{2d'}(\theta_0)] \int_0^\pi \sin^{2d'-3}(\theta_1) d\theta_1 \frac{\int_0^\pi \sin^{2d'-1}(\theta_1) d\theta_1}{\int_0^\pi \sin^{2d'-3}(\theta_1) d\theta_1} \\ &= 1 - \int_{S^{2d'-1}} f(\theta_0) \sin^2(\theta_0) \frac{\int_0^\pi \sin^{2d'-1}(\theta_1) d\theta_1}{\int_0^\pi \sin^{2d'-3}(\theta_1) d\theta_1} \end{aligned}$$

Then, using the formulas in the Appendix, we obtain:

$$F^2(\Psi_0) = 1 - E[\sin^2(\theta_0)] \frac{2d' - 2}{2d' - 1}.$$

Using Jensen's inequality we obtain a lower bound for  $E[\sin^2(\theta_0)]$ :

$$\begin{aligned} (E[1 - \cos(\theta_0)])^2 &\leq E[(1 - \cos(\theta_0))^2] \\ &= E[1 + \cos^2(\theta_0) - 2\cos(\theta_0)] \\ &= E[2 - 2\cos(\theta_0) - \sin^2(\theta_0)] \\ &= v_u - E[\sin^2(\theta_0)]. \end{aligned}$$

And then:

$$E[\sin^2(\theta_0)] \leq v_u - (E[1 - \cos(\theta_0)])^2 = v_u - \left(\frac{v_u}{2}\right)^2.$$

Substituting the previous lower bound of  $E[\sin^2(\theta_0)]$  into the formula of  $F^2(\Psi_0)$ , the proof is concluded:

$$F^2(\Psi_0) \geq 1 - \frac{2d' - 2}{2d' - 1} \left( v_u - \left(\frac{v_u}{2}\right)^2 \right).$$

### 2.2. Fidelity of $\tilde{\Phi}$

The formula for the fidelity of the state  $\tilde{\Phi}$  is very similar to that of the state  $\Psi$ , Formula (11). The proof, though, is more complex because the quantum code  $\mathcal{C}$  is involved.

**Theorem 3.** *The fidelity of the isotropic random variable  $\tilde{\Phi}$  with density function  $f(\theta_0)$  is equal to*

$$F^2(\tilde{\Phi}) = 1 - 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-d'') \bar{E}[\sin^{2d}(\theta_0)], \tag{16}$$

where  $\bar{E}[\sin^{2d}(\theta_0)] = \int_0^\pi f(\theta_0) \sin^{2d}(\theta_0) d\theta_0$ .

*Proof.* Taking into account Theorem 3 and Corollary 1 of [16], the fidelity of  $\tilde{\Phi}$  is

$$F^2(\tilde{\Phi}) = E\left[ P_0 |\langle \Phi | \Pi_0 \Psi \rangle|^2 \right] + (d'' - 1) E\left[ P_1 |\langle E_1 \Phi | \Pi_1 \Psi \rangle|^2 \right],$$

where  $P_0$  and  $P_1$  are the probabilities of measuring the syndromes 0 and 1 respectively,  $\Pi_0$  and  $\Pi_1$  are the (normalized) projectors corresponding to the discrete errors  $E_0 = I$  and  $E_1$  associated with the aforementioned syndromes and  $E_1 \Phi = E_1 |0\rangle = |2d'\rangle$ .

The first expected value in the above expression is equal to  $F^2(\Psi)$  by the Formula (10) and so, using Theorem 1, we obtain:

$$\begin{aligned} E\left[ P_0 |\langle \Phi | \Pi_0 \Psi \rangle|^2 \right] &= E\left[ 1 - \sin^2(\theta_0) \sin^2(\theta_1) \right] \\ &= 1 - 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-1) \bar{E}[\sin^{2d}(\theta_0)]. \end{aligned}$$

The second expected value in the formula is



$$E \left[ P_1 \left| \langle E_1 \Phi | \Pi_1 \Psi \rangle \right|^2 \right] = E \left[ \sin^2(\theta_0) \cdots \sin^2(\theta_{2d'-1}) \left( 1 - \sin^2(\theta_{2d'}) \sin^2(\theta_{2d'+1}) \right) \right].$$

Then, using the Appendix, we obtain:

$$\begin{aligned} & E \left[ \sin^2(\theta_0) \cdots \sin^2(\theta_{2d'-1}) \right] \\ &= \bar{E} \left[ \sin^{2d}(\theta_0) \right] \int_0^\pi \sin^{2d-1}(\theta_0) d\theta_1 \cdots \int_0^\pi \sin^{2d-2d'+1}(\theta_{2d'-1}) d\theta_{2d'-1} |S_{2d-2d'-1}| \\ &= \bar{E} \left[ \sin^{2d}(\theta_0) \right] 2 \frac{(2d-2)!!}{(2d-1)!!} \pi \frac{(2d-3)!!}{(2d-2)!!} \cdots 2 \frac{(2d-2d')!!}{(2d-2d'+1)!!} \frac{(2\pi)^{d-d'}}{(2d-2d'-2)!!} \\ &= \bar{E} \left[ \sin^{2d}(\theta_0) \right] 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-d'). \end{aligned}$$

In a similar way,

$$E \left[ \sin^2(\theta_0) \cdots \sin^2(\theta_{2d'+1}) \right] = \bar{E} \left[ \sin^{2d}(\theta_0) \right] 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-d'-1).$$

With the last two results, the following expression is obtained:

$$E \left[ P_1 \left| \langle E_1 \Phi | \Pi_1 \Psi \rangle \right|^2 \right] = \bar{E} \left[ \sin^{2d}(\theta_0) \right] 4 \frac{(2\pi)^{d-1}}{(2d-1)!!}.$$

Ans finally we get the result:

$$\begin{aligned} F^2(\tilde{\Phi}) &= E \left[ P_0 \left| \langle \Phi | \Pi_0 \Psi \rangle \right|^2 \right] + (d''-1) E \left[ P_1 \left| \langle E_1 \Phi | \Pi_1 \Psi \rangle \right|^2 \right] \\ &= 1 - \bar{E} \left[ \sin^{2d}(\theta_0) \right] 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-1-(d''-1)) \\ &= 1 - \bar{E} \left[ \sin^{2d}(\theta_0) \right] 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-d''). \end{aligned}$$

□

If the probability distribution of  $\Psi$  is normal the fidelity of  $\tilde{\Phi}$  has a much simpler expression.

**Corollary 3.** *If  $\Psi$  has a normal probability distribution with parameter  $\sigma_c$  the fidelity of  $\tilde{\Phi}$  satisfies*

$$F^2(\tilde{\Phi}) = \frac{1+(d'-1)\sigma_c^2}{d'}. \tag{17}$$

*Proof.* To prove the result, it is enough to substitute the value of the integral  $\bar{E} \left[ \sin^{2d}(\theta_0) \right]$  from the Appendix in Theorem 3 and noticing that  $d = d'd''$ . □

To compare the fidelities of  $\Psi_0$  and  $\tilde{\Phi}$  we need to obtain  $F^2(\tilde{\Phi})$  as a function of the variances  $v_c$  of the state  $\Psi$ .

**Theorem 4.** *If the state  $\Psi$  has an isotropic distribution with density function  $f(\theta_0)$  such that*

$$\int_0^\pi (1 - \cos(\theta_0)) \cos(\theta_0) f(\theta_0) \geq 0, \tag{18}$$

then the fidelity of  $\tilde{\Phi}$  satisfies

$$F^2(\tilde{\Phi}) \leq 1 - \frac{d-d''}{2d'-1} v_c. \tag{19}$$

*Proof.* First we prove, in the same lines of the proofs of Theorems 1 and 2, the following:

$$\begin{aligned} F^2(\tilde{\Phi}) &= 1 - 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} (d-d'') \bar{E}[\sin^{2d}(\theta_0)] \\ &= 1 - 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} \frac{d-d''}{|S^{2d-2}|} \bar{E}[\sin^{2d}(\theta_0)] |S^{2d-2}| \\ &= 1 - 4 \frac{(2\pi)^{d-1}}{(2d-1)!!} \frac{(2d-3)!!}{2(2\pi)^{d-1}} (d-d'') E[\sin^2(\theta_0)] \\ &= 1 - 2 \frac{d-d''}{2d-1} E[\sin^2(\theta_0)]. \end{aligned}$$

Now, using Formula (18), we obtain the following lower bound:

$$\begin{aligned} E[\sin^2(\theta_0)] &= E[(1-\cos(\theta_0))(1+\cos(\theta_0))] \\ &\geq E[1-\cos(\theta_0)] = \frac{v_c}{2}. \end{aligned}$$

The proof is concluded by introducing the previous lower bound in the expression previously obtained for  $F^2(\tilde{\Phi})$ .

### 3. Relationship between the Fidelity of the States $\Psi_0$ , $\tilde{\Phi}$ and $\Psi$

The results obtained in the previous section allow us to state the following theorem.

**Theorem 5.** *If the state  $\Psi$  has an isotropic distribution, the following relationship between the fidelities of  $\tilde{\Phi}$  and  $\Psi$  holds:*

$$F^2(\tilde{\Phi}) \geq F^2(\Psi). \tag{20}$$

*Proof.* Theorems 1 and 3 allow us to prove the result directly, taking into account that  $d-1 \geq d-d''$ .  $\square$

To compare the fidelities of states  $\Psi_0$  and  $\tilde{\Phi}$  we use Theorems 2 and 4. However, we shall need to establish a previous result.

**Lemma 1.** *Given  $n \in \mathbb{N}$ ,  $n \geq 2$ , and  $x \in \mathbb{R}$ ,  $0 \leq x \leq 4$ , the following holds true:*

$$g(n, x) = 2 - 2 \left(1 - \frac{x}{2}\right)^n - \left(x - \left(\frac{x}{2}\right)^2\right) \geq 0.$$

*Proof.* The change of variable  $y = \left(1 - \frac{x}{2}\right)$  allows us to better analyze the function:

$$g(n, y) = 1 + y^2 - 2y^n \quad \text{and} \quad x \in [0, 4] \Leftrightarrow y \in [-1, 1].$$

Property 1,  $y^2 \geq |y^n|$  for all  $y \in [-1, 1]$  allows us to conclude that  $g(n, y) \geq 0$

for all  $y \in [-1, 1]$  and this in turn implies

$$g(n, x) \geq 0 \text{ for all } x \in [0, 4].$$

□

The previous lemma allows us to obtain the main result of this article.

**Theorem 6.** *If states  $\Psi_0$  and  $\Psi$  have isotropic distributions with variances  $v_u$  and  $v_c$  respectively and the density function of  $\Psi$  satisfies Formula (18), the following relationship between the fidelities of  $\Psi_0$  and  $\tilde{\Phi}$  holds true:*

$$F^2(\Psi_0) \geq F^2(\tilde{\Phi}). \tag{21}$$

*Proof.* Theorems 2 and 4 allow us to prove the result, just by checking that the following inequality holds:

$$\frac{d - d''}{2d' - 1} v_c \geq \frac{2d' - 2}{2d' - 1} \left( v_u - \left( \frac{v_u}{2} \right)^2 \right).$$

Taking into account that  $d = d'd''$ , the above inequality is equivalent to the following:

$$v_c \geq \frac{2}{d''} \left( v_u - \left( \frac{v_u}{2} \right)^2 \right).$$

The fact that  $d'' \geq 2$  is enough to prove the first of the following two inequalities:

$$v_c \geq v_u - \left( \frac{v_u}{2} \right)^2 \geq \frac{2}{d''} \left( v_u - \left( \frac{v_u}{2} \right)^2 \right).$$

Substituting the value of  $v_c$  given in Formula (7) and using the function  $g(n, x)$  of Lemma 1 yield

$$v_c \geq v_u - \left( \frac{v_u}{2} \right)^2 \Leftrightarrow g(n, v_u) \geq 0.$$

Finally, Lemma 1 allows us to conclude the proof, using the fact that the variance  $v_u \in [0, 4]$ . □

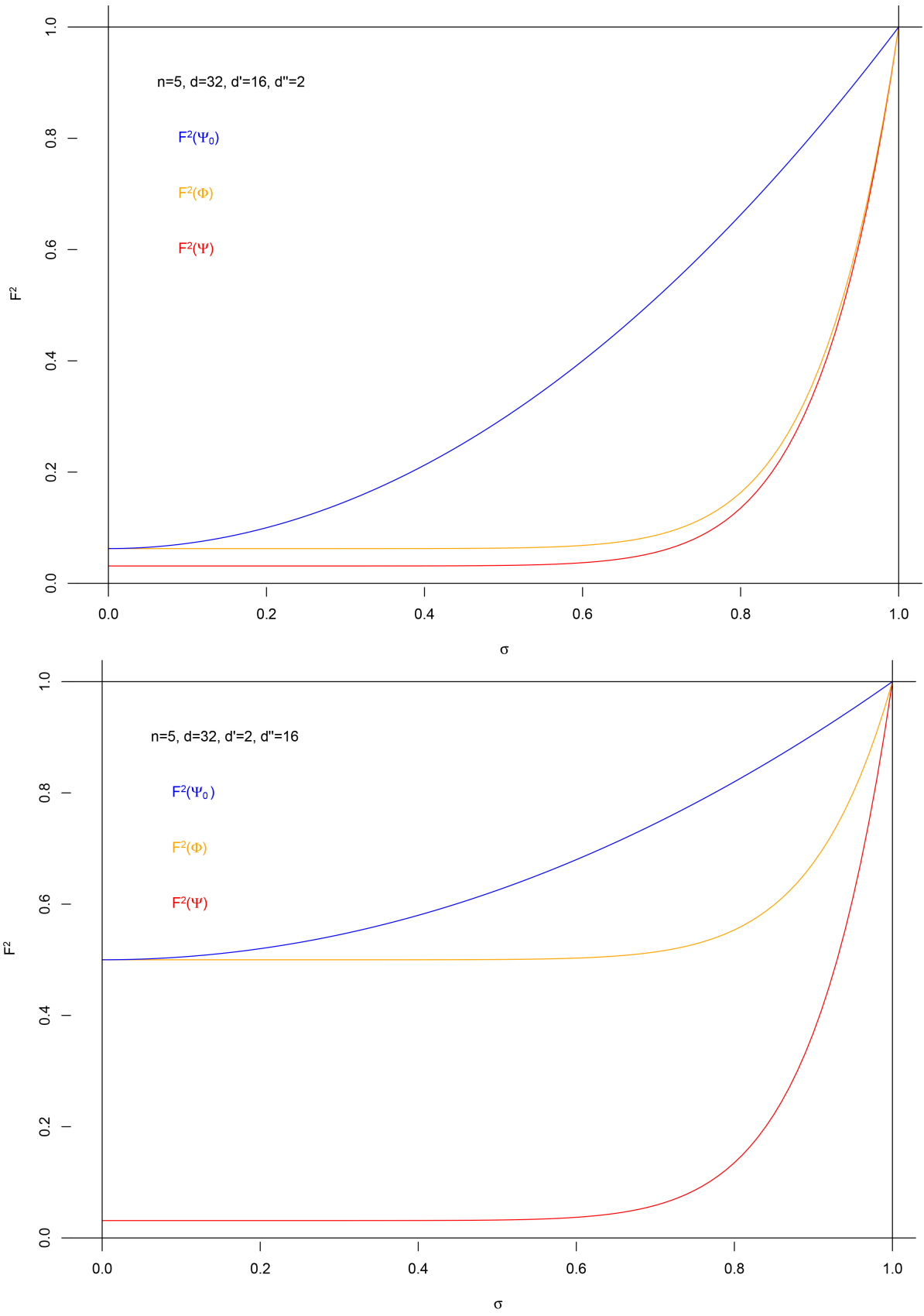
If the isotropic distributions of  $\Psi$  and  $\Psi_0$  are normal the condition given in Formula (18) for Theorems 4 and 6 is not necessary. Indeed, Corollaries 1, 2 and 3 clearly imply that

$$F(\Psi_0) \geq F(\tilde{\Phi}) \geq F(\Psi). \tag{22}$$

On the other hand, the condition given by Formula (18) for Theorems 4 and 6 is a sufficient condition. However, it is not necessary since it has been obtained by underestimating the fidelity of  $\Psi_0$  and overestimating that of  $\tilde{\Phi}$ . The condition holds for very general isotropic distributions, such as density functions  $f(\theta_0)$  that satisfy

$$f(\theta_0) = 0 \text{ for all } \theta_0 \in \left( \frac{\pi}{2}, \pi \right].$$

**Figure 2** shows the curves of  $F^2(\Psi_0)$ ,  $F^2(\tilde{\Phi})$  and  $F^2(\Psi)$  for normal



**Figure 2.** Representation of fidelities as a function of  $\sigma$ .

isotropic distributions and  $n = 5$  ( $d = 32$ ), in the extreme cases  $d' = 16$  ( $d'' = 2$ ) and  $d' = 2$  ( $d'' = 16$ ).

The conclusion of the study carried out in the present work, in view of the results summarized in Formula (22), is that the best option to obtain the highest fidelity against isotropic errors is not to use quantum codes. On the other hand, the improvement of the fidelity of  $\tilde{\Phi}$  versus that of  $\Psi$  seems to be closely related to the dimension of the subspaces to which these states belong:  $d'$  for  $\tilde{\Phi}$  versus  $d$  for  $\Psi$  (see Theorems 1 and 3 and Corollaries 1 and 3).

## 4. Conclusions

In this article we have analyzed the ability of quantum codes to increase fidelity of quantum states affected by isotropic decoherence errors. The results obtained, despite being those expected for this type of quantum errors, are not good enough from the point of view of error control in quantum computing. The ability of quantum codes to reduce errors does not make up for the multiplication of the number of gates that they require. This fact implies that the best option against isotropic errors is not to use quantum codes. This result is similar to that obtained in [16] (quantum codes do not reduce the variance of isotropic errors) and in [19] (the 5-qubit quantum code does not reduce the variance of qubit independent errors). The latter is more worrying since it negatively affects the standard model of error in quantum computing. For this reason, to study the behavior of fidelity in this case seems a priority.

Taken altogether, these results strongly suggest that continuous errors must be taken into account, since it is not possible to ensure that the golden rule of error control “correct all small errors exactly” is fulfilled. Therefore, the study of the stochastic model of quantum errors, focused on discrete errors, must be extended to continuous errors.

For future research, we believe that the continuous quantum computing error model should be further developed. The results on the ability of quantum codes to increase the fidelity or to reduce the variance of quantum errors should be extended to other types of error. It is also important to develop models of the behavior of quantum errors in highly entangled quantum systems. We need to achieve a better understanding of the error behavior in this type of systems since they are capital in quantum computing. Finally, all these approaches should lead to a reformulation of fault-tolerant quantum computing for continuous errors.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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## Appendix

The values of the integrals that have been used throughout the article are included in this Appendix.

$$\int_0^\pi \sin^k(\theta) d\theta = \begin{cases} 2 \frac{(k-1)!!}{k!!} & k = 1, 3, 5, \dots \\ \pi \frac{(k-1)!!}{k!!} & k = 2, 4, 6, \dots \end{cases}$$

$$\int_0^\pi \frac{\sin^{2d-2}(\theta_0)}{(1 + \sigma^2 - 2\sigma \cos(\theta_0))^d} d\theta_0 = \frac{(2d-3)!!}{(2d-2)!!} \frac{\pi}{1-\sigma^2} \quad d = 1, 2, 3, \dots$$

$$\int_0^\pi \frac{\cos(\theta_0) \sin^{2d-2}(\theta_0)}{(1 + \sigma^2 - 2\sigma \cos(\theta_0))^d} d\theta_0 = \frac{(2d-3)!!}{(2d-2)!!} \frac{\sigma}{1-\sigma^2} \pi \quad d = 1, 2, 3, \dots$$

$$\int_0^\pi \frac{\sin^{2d}(\theta_0)}{(1 + \sigma^2 - 2\sigma \cos(\theta_0))^d} d\theta_0 = \frac{(2d-1)!!}{(2d)!!} \pi \quad d = 0, 1, 2, \dots$$

Starting from the first integral, the surface of a unit sphere of arbitrary even ( $2d$ ) or odd ( $2d-1$ ) dimension can be calculated.

$$\begin{aligned} |\mathcal{S}^{2d}| &= \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \sin^{2d-1}(\theta_0) \dots \sin^1(\theta_{2d-2}) d\theta_0 \dots d\theta_{2d-2} d\theta_{2d-1} \\ &= 2 \frac{(2d-2)!!}{(2d-1)!!} \frac{(2d-3)!!}{(2d-2)!!} \pi 2 \frac{(2d-4)!!}{(2d-3)!!} \dots \frac{(2-1)!!}{2!!} \pi 2 \frac{(1-1)!!}{1!!} 2\pi \\ &= \frac{2(2\pi)^d}{(2d-1)!!} \end{aligned}$$

$$\begin{aligned} |\mathcal{S}^{2d-1}| &= \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \sin^{2d-2}(\theta_0) \dots \sin^1(\theta_{2d-3}) d\theta_0 \dots d\theta_{2d-3} d\theta_{2d-2} \\ &= \frac{(2d-3)!!}{(2d-2)!!} \pi 2 \frac{(2d-4)!!}{(2d-3)!!} \dots \frac{(2-1)!!}{2!!} \pi 2 \frac{(1-1)!!}{1!!} 2\pi \\ &= \frac{(2\pi)^d}{(2d-2)!!} \end{aligned}$$