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The Solutions and the Dynamic Behavior of the Rational Difference Equations

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Abstract

The main purpose of this paper is to study the dynamic behavior of the rational difference equation of the fourth order

$$y_{n+1} = \frac{\alpha y_{n-1} y_{n-3}}{\beta y_{n-1} + \gamma y_{n-3}}, \quad n = 0, 1, \cdots,$$

where α, β and γ are positive constants and the initial conditions $y_{-3}, y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers. Also, we obtain the solution of some special cases of this equation and investigate the existence of a periodic solutions of these equations. Finally, some numerical examples will be given to explicate our results.

Keywords

Difference Equation, Local Stability, Global Attractor, Boundedness, Periodic

1. Introduction

Over the last few years, the mathematicians have shown a lot of interest on studying the behavior of the non-linear difference equations and systems. These studies have been very productive and helpful to develop the basic theory of of the qualitative behaviour of non-linear rational difference equations. This topic experienced enormous growth in many areas where many real life phenomena were modeled using difference equations studies, for examples, from probability theory, statistical problems, stochastic time series, electrical network, genetics in biology, economics, sociology, etc. [1] [2] [3] [4]. It is known that non-linear difference equations are capable of producing a complicated behavior regardless its order. Thus, every research that studies the global attractivity, the boundedness character and the periodicity nature of non-linear difference equations are of paramount importance in their own right. The objective of this paper is to investigate some qualitative behavior of the solutions of the nonlinear difference equation:

$$y_{n+1}=\frac{\alpha y_{n-1}y_{n-3}}{\beta y_{n-1}+\gamma y_{n-3}},\ n=0,1,\cdots, \tag{1}$$

where α,β and γ are positive constants and the initial conditions y_{-3},y_{-2},y_{-1},y_0 are arbitrary positive real numbers. Also, we obtain the solution of some special cases of this equation.

In fact, many authors and researchers studied qualitative behaviors of the solution of rational difference equations for example:

In [5], Amleh *et al.* investigated the third-order rational difference equation

$$x_{n+1}=\frac{a+bx_{n-1}}{A+Bx_{n-2}}, \tag{2}$$

where a,b,A,B are non-negative real numbers and the initial conditions are non-negative real number.

In [6] [7] and [8], Cinar investigated the solutions of the following difference equations

$$x_{n+1}=\frac{ax_{n-1}}{1+bx_nx_{n-1}},\ x_{n+1}=\frac{x_{n-1}}{1+bx_nx_{n-1}},\ x_{n+1}=\frac{x_{n-1}}{ax_nx_{n-1}}, \tag{3}$$

where x_{-1} and x_0 are positive real numbers.

Elabbasy *et al.* [9] investigated the asymptotic behavior of the solutions of a new class of the rational difference equations

$$\omega_{n+1}=a+\frac{b_0\omega_{n-1}}{c_1\omega_n+c_2\omega_{n-1}}+\frac{b_1\omega_{n-3}}{c_3\omega_{n-2}+c_4\omega_{n-3}}, \tag{4}$$

where a,b_0,b_1,c_1,c_2,c_3 and $c_4\in[0,\infty)$ and the initial conditions $\omega_{-3},\cdots,\omega_{-1}$ and ω_0 are arbitrary positive real numbers.

El-Owaidy *et al.* [10] have investigated the global behavior of the difference equation

$$x_{n+1}=\frac{\alpha x_{n-1}}{\beta+\gamma x_{n-2}^{p}}, \tag{5}$$

where the parameters α,β,γ, and p are non-negative real numbers and the initial conditions x_{-2},x_{-1}, and x_0 are non-negative real numbers.

El-sayed [11] has investigate the global convergence result, boundedness, and periodicity of solutions of the recursive sequence

$$x_{n+1}=ax_n+\frac{bx_{n-1}+cx_{n-2}}{dx_{n-1}+ex_{n-2}}, \tag{6}$$

where the parameters a,b,c,d and e are positive real numbers and the initial conditions x_{-2},x_{-1} and x_0 are positive real numbers.

El-Moneam [12] studied the global stability of the positive solutions of the following nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\kappa} + \frac{bx_{n-k}}{dx_{n-k} + ex_{n-l}}, \tag{7}$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and κ are positive integers. The initial conditions $x_{-\kappa}, \cdots, x_{-l}, \cdots, x_{-k}, \cdots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < \kappa$.

Abo-Zeid [13] solved and studied the global behavior of the well defined solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-3}}{Ax_{n-2} + Bx_{n-3}}, \tag{8}$$

where $A, B > 0$ and the initial values $x_{-i}, i \in \{1, 2, 3\}$ are real numbers.

Gul [14] investigated the solution of the following difference equation

$$z_{n+1} = (p_n)^{-1}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \tag{9}$$

where $p_n = a + bz_n + cz_{n1} z_n$ with the parameters a, b, c and the initial values z_{-1}, z_0 are nonzero quaternions such that their solutions are associated with generalized Fibonacci-type numbers.

Li and Li [15] studied investigate the global asymptotic stability of the following difference equation

$$x_{n+1} = \frac{p + qx_n}{1 + rx_{n-k}}, \tag{10}$$

where $p, q \in [0, \infty)$, $r > 0, k \geq 1$ is an integer and initial conditions $x_k, \cdots, x_1, x_0 \in (0, \infty)$.

In addition, other related results on rational difference equations can be found in Refs. [16] [17] [18] and [19] and the references cited therein.

2. Preliminaries

Now we recall some results that are given in [2], which will be helpful in our investigation of the difference Equation (1).

Let I be some interval of real numbers and let

$$g : I^{l+1} \to I,$$

be a continuously differentiable function. Then for every set of initial conditions $y_{-l}, y_{-l+1}, \cdots, y_0 \in I$, the difference equation

$$y_{n+1} = g(y_n, y_{n-1}, \cdots, y_{n-l}), \ n = 0, 1, \cdots, \tag{11}$$

has a unique solution $\{y_n\}_{n=-l}^{\infty}$.

Definition 1. (Equilibrium point) A point $\bar{y} \in I$ is called an equilibrium point of the difference Equation (1) if

$$\bar{y} = g(\bar{y}, \bar{y}, \cdots, \bar{y}).$$

That is, $y_n = \bar{y}$ for $n \geq 0$ is a solution of Equation (1), or equivalently, $\bar{y}$ is a fixed point of g.

Definition 2. (Stability) Let $\bar{y} \in (0, \infty)$ be an equilibrium point of the difference Equation (1). Then, we have the following:

(i) The equilibrium point of the difference Equation (1) is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y_{-l}, y_{-l+1}, \cdots, y_0 \in I$ with

$$|y_{-l} - \bar{y}| + \cdots + |y_{-1} - \bar{y}| + |x_0 - \bar{y}| < \delta,$$

We have

$$|y_n - \bar{y}| < \varepsilon \quad \text{for all} \quad n \geq -l.$$

(ii) The equilibrium point $\bar{y}$ Equation (1) is called locally asymptotically stable if $\bar{y}$ is locally stable solution of (1) and there exists $\gamma > 0$, such that, for all $y_{-l}, y_{-l+1}, \cdots, y_0 \in I$ with

$$|y_{-l} - \bar{y}| + \cdots + |y_{-1} - \bar{y}| + |y_0 - \bar{y}| < \gamma,$$

We have

$$\lim_{n \to \infty} y_n = \bar{y}.$$

(iii) The equilibrium point $\bar{y}$ of Equation (1) is called a global attractor if for all $y_{-l}, y_{-l+1}, \cdots, y_0 \in I$, we have

$$\lim_{n \to \infty} y_n = \bar{y}.$$

(iv) The equilibrium point $\bar{y}$ of the difference Equation (1) is called a global asymptotically stable if it is locally stable, and $\bar{y}$ is also global attractor of the difference Equation (1).

(v) The equilibrium point $\bar{y}$ of the difference Equation (1) is called unstable if $\bar{y}$ is not locally stable.

Definition 3. The linearized equation of (1) about the equilibrium point $\bar{y}$ is the linear difference equation

$$z_{n+1} = \sum_{i=1}^{l} \frac{\partial H(\bar{y}, \cdots, \bar{y})}{\partial y_{n-i}} z_{n-i}. \tag{12}$$

Definition 4. (Periodicity) A sequence $\{y_n\}_{n=-l}^{\infty}$ is said to be periodic with periodic q if $y_{n+q} = y_n$ for all $n \geq -l$.

Definition 5. (Fibonacci sequence)

The sequence $\{F_i\}_{i=0}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, \cdots\}$, *i.e.*, $F_i = F_{i-1} + F_{i-2}$, $F_{-2} = 1$, $F_{-1} = 1$ is called Fibonacci sequence.

Now, assume that the characteristic equation associated with (12) is

$$p(\lambda) = p_0 \lambda^l + p_1 \lambda^{l-1} + \cdots + p_{l-1} \lambda + p_l = 0, \tag{13}$$

where

$$p_i = \frac{\partial H(\bar{y}, \cdots, \bar{y})}{\partial y_{n-i}}. \tag{14}$$

Theorem A. Assume that $p_0, p_1, \cdots, p_l \in R$, and $l \in \{0, 1, 2, \cdots\}$. Then

$$\sum_{i=1}^{l} |p_i| < 1, \tag{15}$$

is a sufficient condition for the asymptotic stability of the difference equation:

$$y_{n+l}+p_1y_{n+l-1}+\cdots+p_ly_n=0,\ \ n=0,1,\cdots. \tag{16}$$

Theorem B. Let $[p,q]$ be an interval of real numbers and assume that

$$g:[p,q]\times[p,q]\leftarrow[p,q]$$

is a continuous function satisfying the following properties:

(a) $g(x,y)$ is non-decreasing in x in $[p,q]$ for each $y\in[p,q]$, and is non-increasing in $y\in[p,q]$ for each $x\in[p,q]$.

(b) If $(m,M)\in[p,q]\times[p,q]$ is a solution of the system

$$M=g(M,m)\ \text{ and }\ m=g(m,M),$$

Then

$$M=m.$$

Then equation

$$y_{n+1}=g(y_{n-1},y_{n-3}),$$

has a unique equilibrium $\overline{y}\in[p,q]$ and every solution of this equation converge to $\overline{y}$.

3. Dynamics of Equation (1)

In this section, we obtain the equilibrium point then we study the local stability, global stability of the solutions, and the boundedness of the following difference equation

$$y_{n+1}=\frac{\alpha y_{n-1}y_{n-3}}{\beta y_{n-1}+\gamma y_{n-3}},\ \ n=0,1,\cdots,$$

where α,β and γ are positive constants and the initial conditions y_{-3},y_{-2},y_{-1},y_0 are arbitrary positive real numbers.

3.1. Local Stability of the Equilibrium Point

In this subsection, we study the local stability of the equilibrium point of Equation (1). Equation (1) has a unique equilibrium point and is given by

$$\overline{y}=\frac{\alpha\overline{y}\overline{y}}{\beta\overline{y}+\gamma\overline{y}},$$

$$\overline{y}=\frac{\alpha\overline{y}^2}{\beta\overline{y}+\gamma\overline{y}},$$

$$\overline{y}^2(\beta+\gamma)=\alpha\overline{y}^2.$$

If $(\beta+\gamma)\neq\alpha$, then the only equilibrium point is $\overline{y}=0$.

Theorem 3.1. Let

$$\frac{\beta+\gamma}{(\beta+\gamma)^2}<\frac{1}{\alpha}.$$

Then the equilibrium point of Equation (1) is locally asymptotically stable.

Proof.

Proof. Let $g:(0,\infty)^2 \to (0,\infty)$, be a continuous function defined by

$$g(w_1,w_2) = \frac{\alpha w_1 w_2}{\beta w_1 + \gamma w_2}.$$

Therefore, it follows that

$$\frac{\partial g(w_1,w_2)}{\partial w_1} = \frac{\alpha\gamma w_2^2}{(\beta w_1 + \gamma w_2)^2},$$

$$\frac{\partial g(w_1,w_2)}{\partial w_2} = \frac{\alpha\beta w_1^2}{(\beta w_1 + \gamma w_2)^2}.$$

We see that

$$\frac{\partial g(\overline{y},\overline{y})}{\partial w_1} = \frac{\alpha\gamma}{(\beta+\gamma)^2},$$

$$\frac{\partial g(\overline{y},\overline{y})}{\partial w_2} = \frac{\alpha\beta}{(\beta+\gamma)^2}.$$

So the linearized equation of (1) about $\overline{y} = 0$ is

$$z_{n+1} - \left(\frac{\alpha\gamma}{(\beta+\gamma)^2}\right) z_{n-1} + \left(\frac{\alpha\beta}{(\beta+\gamma)^2}\right) z_{n-3} = 0.$$

It follows by Theorem A that Equation (1) is asymptotically stable if

$$\left|\frac{\alpha\gamma}{(\beta+\gamma)^2}\right| + \left|\frac{\alpha\beta}{(\beta+\gamma)^2}\right| < 1,$$

and so,

$$\frac{\alpha(\gamma+\beta)}{(\beta+\gamma)^2} < 1,$$

Thus,

$$\frac{\gamma+\beta}{(\beta+\gamma)^2} < \frac{1}{\alpha}.$$

The proof is complete. □

3.2. Global Stability of the Equilibrium Point of Equation (1)

In this subsection, we study the global stability of the positive solutions of (1).

Theorem 3.2. The equilibrium point $\overline{y}$ of Equation (1) is global stability if $\alpha \neq \beta$.

Proof. Let p,q be a real numbers and assume that $g:[p,q]\times[p,q]\to[p,q]$ be a function define by

$$g(w_1,w_2) = \frac{\alpha w_1 w_2}{\beta w_1 + \gamma w_2}.$$

Then we can see that the function $g(w_1,w_2)$ is increasing in w_1 and w_2. Suppose that (m,M) is a solution of the system

$$M = g(M,m) \quad \text{and} \quad m = g(m,M).$$

Then from Equation (1), we can see that

$$M = \frac{\alpha M^2}{\beta M + \gamma M}, \quad m = \frac{\alpha m^2}{\beta m + \gamma m},$$

and then

$$\beta M^2 + \gamma M^2 = \alpha M^2,$$

$$\beta m^2 + \gamma m^2 = \alpha m^2.$$

Subtracting these two equations, we obtain

$$(\beta + \gamma)(M^2 - m^2) = \alpha(M^2 - m^2),$$

and if $(\beta + \gamma) \neq \alpha$, then we see that $M = m$.

According to Theorem B the equilibrium point $\overline{y}$ is a global attractor of (1). The proof is complete. □

3.3. Boundedness of Solutions of Equation (1)

In this subsection, we look at the boundedness and unboundedness solutions of Equation (1).

Theorem 3.3. Every solution of Equation (1) is bounded if $\frac{\alpha}{\gamma} < 1$.

Proof. Let $\{y_n\}_{n=-3}^{\infty}$ be a solution of (1).It follows from (1) that

$$y_{n+1} = \frac{\alpha y_{n-1} y_{n-3}}{\beta y_{n-1} + \gamma y_{n-3}} \leq \frac{\alpha y_{n-1} y_{n-3}}{\gamma y_{n-3}} = \left(\frac{\alpha}{\gamma}\right) y_{n-1}.$$

Then when $\frac{\alpha}{\gamma} < 1$, we see that

$$y_{n+1} \leq y_{n-1} \quad \text{for all} \quad n \geq 0.$$

Then the sequences $\{y_n\}_{n=-3}^{\infty}$ is decreasing and so is bounded from the above by $M = \max\{y_{-3}, y_{-2}, y_{-1}, y_0\}$. □

3.4. Numerical Examples of Equation (1)

In order to illustrate the results of the previous sections and to support our theoretical discussions, we assume some numerical examples in this section.

Example 1. Figure 1 shows that the zero solution of the difference Equation (1) is local stability with the initial conditions $y_{-3} = 14$, $y_{-2} = 12$, $y_{-1} = 13$ and $y_0 = 17$ and the parameters $\alpha = 7$, $\beta = 8$ and $\gamma = 7$.

Example 2. In Figure 2, we choose the parameters $\alpha = 0.15$, $\beta = 0.1$ and $\gamma = 0.1$ with the initial conditions $y_{-3} = 0.57$, $y_{-2} = 1.1$, $y_{-1} = 0.3$ and $y_0 = 0.5$. Then, the plot if the behavior of the zero solution of the difference Equation (1) is global stability where $\alpha \neq \beta$.

Example 3. In Figure 3, consider the difference Equation (1) with initial conditions $y_{-3} = 0.57$, $y_{-2} = 1.1$, $y_{-1} = 0.3$, $y_0 = 0.5$. Moreover, choosing the parameters $\alpha = 0.015$, $\beta = 0.1$ and $\gamma = 0.1$. Then, the plot of the behavior of the solution of Equation (1) is bounded as shown in **Figure 3**.

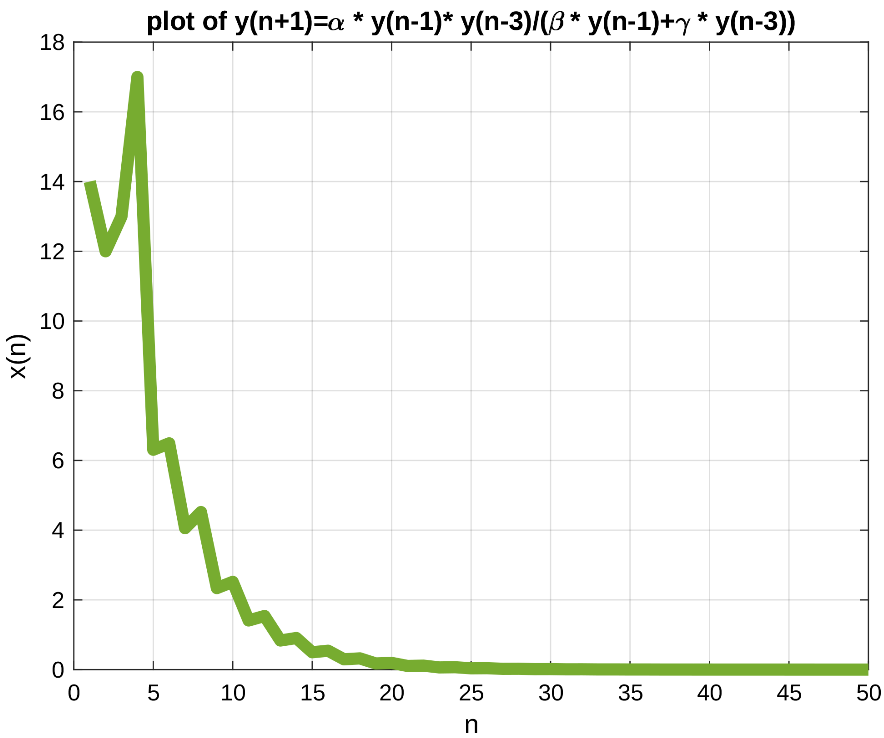

Figure 1. The stable solution corresponding to difference Equation (1).

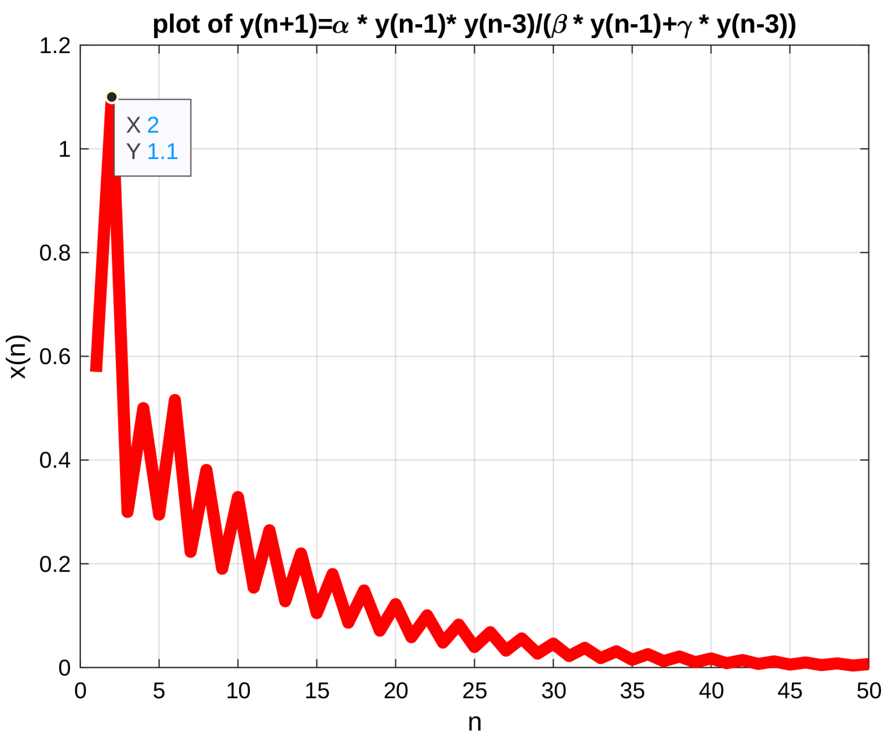

Figure 2. Plot the behavior of the zero solution of (1) is global stable.

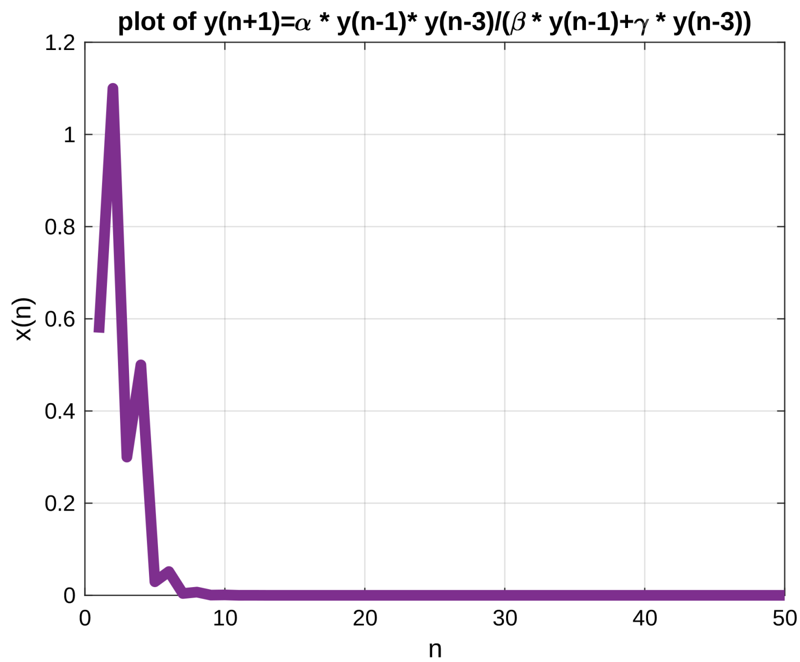

Figure 3. Sketch the behavior of the solution of (1) is bounded.

4. Special Cases of Equation (1)

In this section we investigate the following special case:

$$y_{n+1} = \frac{\alpha y_{n-1} y_{n-3}}{\beta y_{n-1} + \gamma y_{n-3}}, \quad n = 0, 1, \cdots,$$

where α, β and γ are constants and the initial conditions $y_{-3}, y_{-2}, y_{-1}, y_0$ are arbitrary nonzero real numbers.

4.1. First Case

In this subsection, we solve the special case of Equation (1) when $\alpha = \beta = \gamma = 1$.

Theorem 4.1. The solution of the following difference equation

$$y_{n+1} = \frac{y_{n-1} y_{n-3}}{y_{n-1} + y_{n-3}}, \tag{17}$$

is given by the following formulas for $n = 0, 1, 2, \cdots$.

$$y_{2n-3} = \frac{bd}{f_{n-1} b + f_n d},$$

$$y_{2n-2} = \frac{ac}{f_{n-1} a + f_n c}.$$

where the initial conditions $y_{-3} = d, y_{-2} = c, y_{-1} = b, y_0 = a$ are arbitrary positive real numbers with $y_{-2} \neq y_0$, $y_{-3} \neq y_{-1}$.

Proof. By using mathematical induction we can proves as follow. For $n = 0$ the result holds. Assume that the result holds for $n - 1$, as follows

$$y_{2n-4} = \frac{ac}{f_{n-2} a + f_{n-1} c},$$

$$y_{2n-5} = \frac{bd}{f_{n-2}b + f_{n-1}d},$$

$$y_{2n-6} = \frac{ac}{f_{n-3}a + f_{n-2}c},$$

$$y_{2n-7} = \frac{bd}{f_{n-3}b + f_{n-2}d}.$$

where f_i is Fibonacci sequence and $\{f_i\}_{i=-1}^{\infty} = \{1,0,1,1,2,3,5,8,13,21,\cdots\}$.

Then, from Equation (17), it follows that

$$y_{2n-2} = \frac{y_{2n-4}x_{2n-6}}{y_{2n-4} + y_{2n-6}} = \frac{\left(\frac{ac}{f_{n-2}a + f_{n-1}c}\right)\left(\frac{ac}{f_{n-3}a + f_{n-2}c}\right)}{\frac{ac}{f_{n-2}a + f_{n-1}c} + \frac{ac}{f_{n-3}a + f_{n-2}c}}$$

$$= \frac{(ac)^2}{(ac)(f_{n-3}a + f_{n-2}c) + (ac)(f_{n-2}a + f_{n-1}c)}$$

$$= \frac{ac}{f_{n-3}a + f_{n-2}c + f_{n-2}a + f_{n-1}c}.$$

Thus,

$$y_{2n-2} = \frac{ac}{f_{n-1}a + f_n c}.$$

Similarly, one can prove the other relations. Thus the proof is completed. □

Example 5. Figure 4 shows the behaviour of the solution of Equation (17) when the initial conditions $y_{-3} = 0.8$, $y_{-2} = 5$, $y_{-1} = 3$ and $y_0 = 2$.

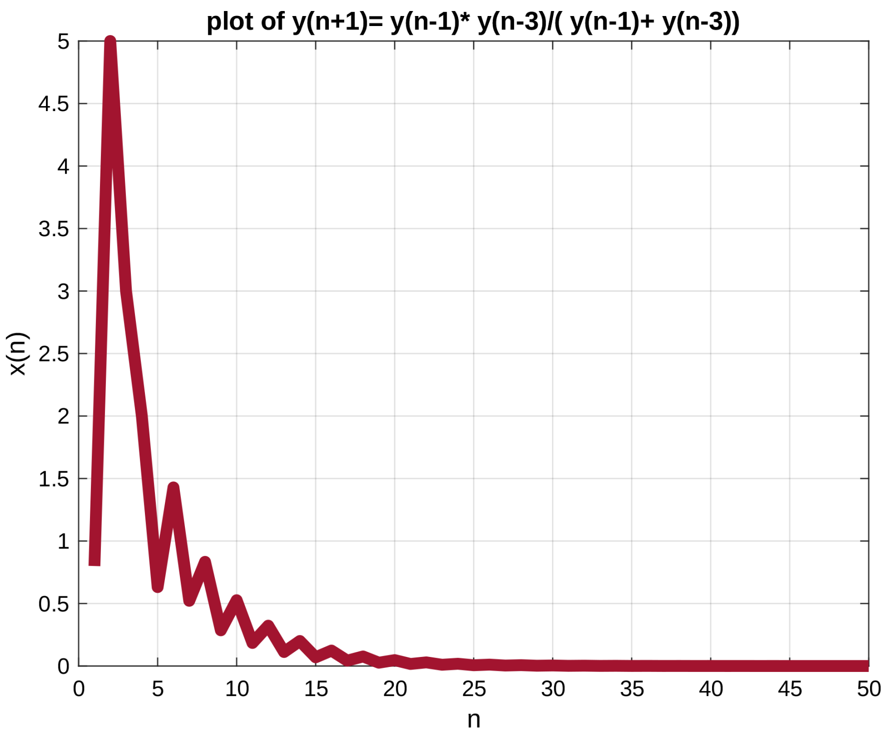

Figure 4. Sketch the behavior of the solution of (17).

4.2. Second Case

In this subsection, we deal with the specific case of the Equation (1) when $\alpha=\beta=1$ and $\gamma=-1$.

Theorem 4.2. For $n=0,1,2,\cdots$, the solution of difference equation

$$y_{n+1}=\frac{y_{n-1}y_{n-3}}{y_{n-1}-y_{n-3}}, \tag{18}$$

has the following formulas:

$$y_{2n-3}=\frac{(-1)^n bd}{f_{n-1}b-f_n d},$$

$$y_{2n-2}=\frac{(-1)^n ac}{f_{n-1}a-f_n c}.$$

where the initial conditions $y_{-3}=d, y_{-2}=c, y_{-1}=b, y_0=a$ are arbitrary non-zero real numbers with $y_{-2}\neq y_0$, $y_{-3}\neq y_{-1}$.

Proof. The results hold for $n=0$. Assume that the result holds for $n-1$.

$$y_{2n-4}=\frac{(-1)^{n-1} ac}{f_{n-1}a-f_n c},$$

$$y_{2n-5}=\frac{(-1)^{n-1} bd}{f_{n-1}b-f_n d},$$

$$y_{2n-6}=\frac{(-1)^{n-2} ac}{f_{n-1}a-f_n c},$$

$$y_{2n-7}=\frac{(-1)^{n-2} bd}{f_{n-1}b-f_n d}.$$

where f_i is Fibonacci sequence and $\{f_i\}_{i=-1}^{\infty}=\{1,0,1,1,2,3,5,8,13,21,\cdots\}$.

Now, it follow from Equation (18), that

$$y_{2n-3}=\frac{y_{2n-5}y_{2n-7}}{y_{2n-5}-y_{2n-7}}=\frac{\left(\frac{(-1)^{n-1} bd}{f_{n-1}b-f_n d}\right)\left(\frac{(-1)^{n-2} bd}{f_{n-1}b-f_n d}\right)}{\frac{(-1)^{n-1} bd}{f_{n-1}b-f_n d}-\frac{(-1)^{n-2} bd}{f_{n-1}b-f_n d}}$$

$$=\frac{(-1)^{n-1}(bd)(-1)^{n-2}(bd)}{(-1)^{n-1}(bd)(f_{n-3}b-f_{n-2}d)+(-1)^{n-2}(bd)(f_{n-2}b-f_{n-1}d)}$$

$$=\frac{(-1)^{n-1}(-1)^{n-2}(bd)^2}{(-1)^{n-1}(-1)^{n-2}(bd)\left[(f_{n-3}b-f_{n-2}d)-(f_{n-2}b-f_{n-1}d)\right]}.$$

Thus,

$$y_{2n-3}=\frac{(-1)^n bd}{f_{n-1}b+f_n d}.$$

Hence, we can easily proof the other relations. The proof has been done. □

Example 6. Figure 5 illustrates the solution of Equation (18) when the initial

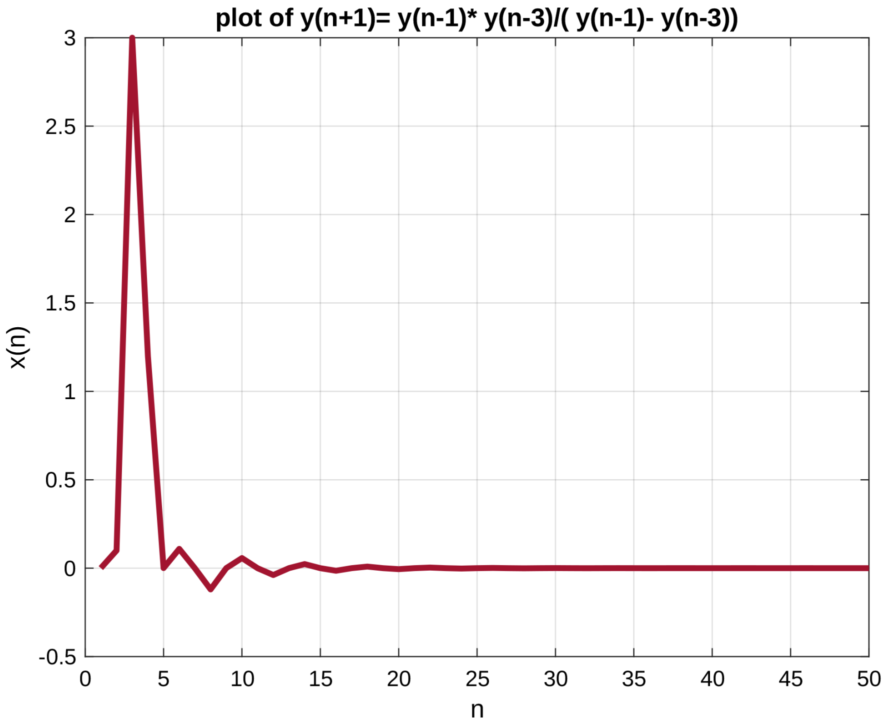

Figure 5. Sketch the behavior of the solution of (18).

conditions $y_{-3}=0.0007$, $y_{-2}=0.1$, $y_{-1}=3$, $y_0=1.2$.

4.3. Third Case

In this subsection, we study the following special case of Equation (1) when $\alpha=1$, $\beta=-1$ and $\gamma=1$.

Theorem 4.3. Every solution of the following difference equation

$$y_{n+1}=\frac{y_{n-1}y_{n-3}}{-y_{n-1}+y_{n-3}}, \tag{19}$$

is periodic with period 12. Moreover, the solution of (10) takes the following form

$$y_{12n-3}=d,\quad y_{12n-2}=c,\quad y_{12n-1}=b,$$

$$y_{12n}=a,\quad y_{12n+1}=\frac{bd}{-b+d},\quad y_{12n+2}=\frac{ac}{-a+c}$$

$$y_{12n+3}=-d,\quad y_{12n+4}=-c,\quad y_{12n+5}=-b,$$

$$y_{12n+6}=-a,\quad y_{12n+7}=\frac{bd}{b-d},\quad y_{12n+8}=\frac{ac}{a-c}.$$

where the initial conditions $y_{-3}=d, y_{-2}=c, y_{-1}=b, y_0=a$ are arbitrary non-zero real numbers, and $y_{-2}\neq y_0$, $y_{-3}\neq y_{-1}$.

Proof. For $n=0$ the result holds. Suppose that the result holds for $n-1$.

$$y_{12n-15}=d,\quad y_{12n-14}=c,\quad y_{12n-13}=b,$$

$$y_{12n-12}=a,\quad y_{12n-11}=\frac{bd}{-b+d},\quad y_{12n-10}=\frac{ac}{-a+c}$$

$$y_{12n-9} = -d, \quad y_{12n-8} = -c, \quad y_{12n-7} = -b,$$

$$y_{12n-6} = -a, \quad y_{12n-5} = \frac{bd}{b-d}, \quad y_{12n-4} = \frac{ac}{a-c}.$$

We see from Equation (19) that

$$y_{12n-4} = \frac{y_{12n-6}y_{12n-8}}{-y_{12n-6} + y_{12n-8}} = \frac{(-a)(-c)}{-(-a)+(-c)}.$$

Thus,

$$y_{12n-4} = \frac{ac}{a-c}.$$

Similarly,

$$y_{12n+3} = \frac{y_{12n+1}y_{12n-1}}{-y_{12n+1} + y_{12n-1}} = \frac{\left(\frac{bd}{-b+d}\right)b}{-\left(\frac{bd}{-b+d}\right)+b} = \frac{b^2 d}{-bd - b^2 + bd}.$$

Thus,

$$y_{12n+3} = -d.$$

Other relations can be found in similar way. Hence, the proof is completed. □

Example 7. Figure 6 shows that the solution of Equation (19) has a periodic solution with period 12 when the initial conditions $y_{-3} = 0.01$, $y_{-2} = 0.3$, $y_{-1} = 0.7$, $y_0 = 5$.

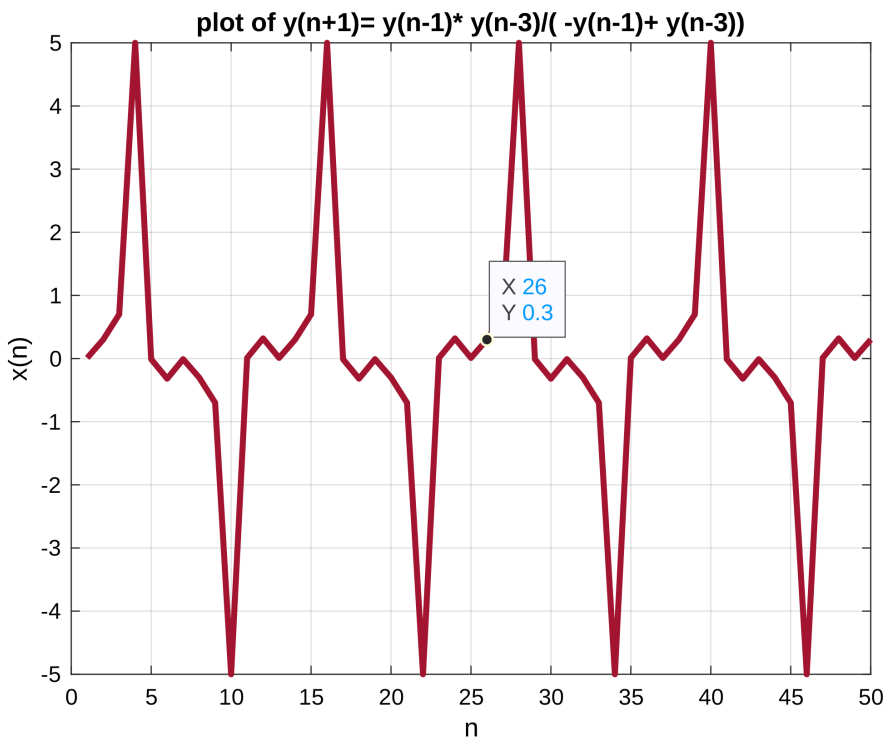

Figure 6. Sketch the periodicity of the solution of (19).

4.4. Fourth Case

In this subsection, we investigate the following special case of Equation (1) when $\alpha = 1$, $\beta = -1$ and $\gamma = -1$ and $y_{-2} \neq y_0$, $y_{-3} \neq y_{-1}$.

Theorem 4.4. For $n = 0, 1, \cdots$, the solution of the following difference equation

$$y_{n+1} = \frac{y_{n-1} y_{n-3}}{-y_{n-1} - y_{n-3}}, \tag{20}$$

is periodic with period 6. Moreover, the solution of (20) takes the following form

$$y_{6n-3} = d, \quad y_{6n-2} = c,$$
$$y_{6n-1} = b, \quad y_{6n} = a,$$
$$y_{6n+1} = \frac{bd}{-b-d}, \quad y_{6n+1} = \frac{ac}{-a-c}.$$

where the initial conditions $y_{-3}, y_{-2}, y_{-1}, y_0$ are arbitrary nonzero real numbers, and $y_{-2} \neq y_0$, $y_{-3} \neq y_{-1}$.

Proof. For $n = 0$ the result holds. Now, suppose that $n > 0$ and that our assumption holds for $n-1$. That is

$$y_{6n-9} = d, \quad y_{6n-8} = c,$$
$$y_{6n-7} = b, \quad y_{6n-6} = a,$$
$$y_{6n-5} = \frac{bd}{-b-d}, \quad y_{6n-4} = \frac{ac}{-a-c}.$$

From Equation (20) we have

$$y_{6n-3} = \frac{y_{6n-5} y_{6n-7}}{-y_{6n-5} - y_{6n-7}} = \frac{\left(\frac{bd}{-b-d}\right) b}{-\left(\frac{bd}{-b-d}\right) - b} = \frac{b^2 d}{-bd + b^2 + bd} = d.$$

Also,

$$y_{6n-1} = \frac{x_{6n-3} x_{6n-5}}{-x_{6n-3} - x_{6n-5}} = \frac{d\left(\frac{bd}{-b-d}\right)}{-d - \frac{bd}{-b-d}} = \frac{bd^2}{bd + b^2 - bd} = b.$$

Hence, the rest of the relations can be found in similar way. The proof has been completed. □

Example 8. We show, in Figure 7, the behavior of the solution of equation (20) when the initial conditions $y_{-3} = 0.01$, $y_{-2} = 0.3$, $y_{-1} = 0.7$, $y_0 = 5$. Moreover, Figure 7 shows that the solution of Equation (20) has a periodic solution with period 6.

5. Conclusion

In this article we present the qualitative behavior of a rational difference equation. Firs, we prove the existence of the equilibrium point. Then it investigated the local stability, global stability and studied the boundededness of the difference

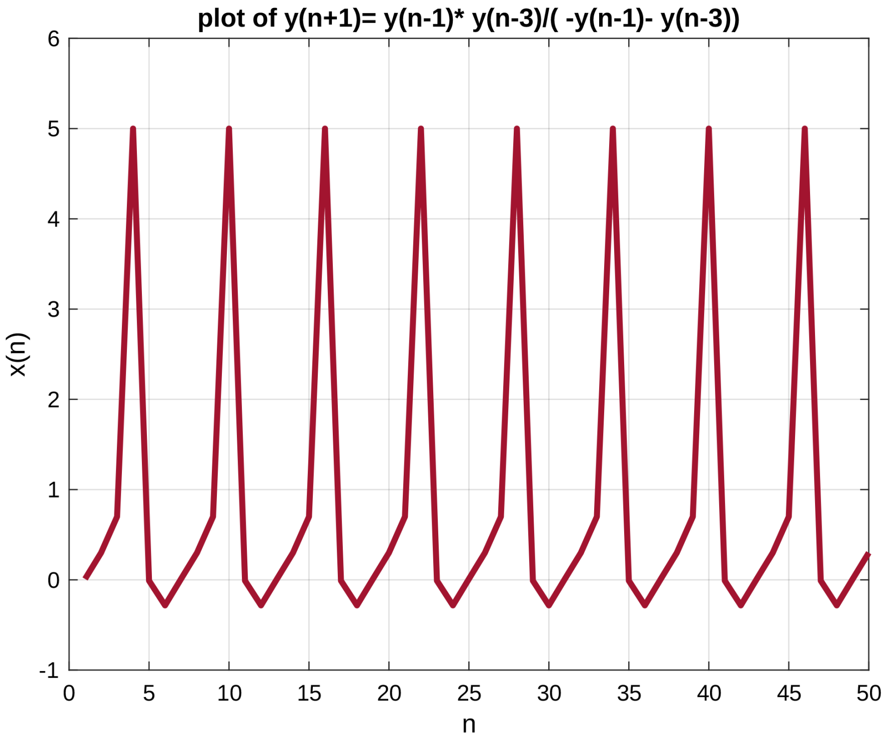

Figure 7. Sketch the periodicity of the solution of (20).

Equation (1). In Section 4, we obtained the form of the solution of four special cases of the difference Equation (1) and investigated the existence of a periodic solution of these equations, and we gave interesting numerical examples of each of the case with different initial values.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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