

Computational Physics of *Mathematica* and Geometric Calculus

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Abstract

The latter half of the twentieth century yielded two tools of unprecedented power, both of which took decades to mature to their current states. The purpose of this research is to apply these to a theory of gravity and develop the consequences of the model based on these tools. This paper presents such results without mathematical details, which are presented elsewhere. The tools are: *Geometric Calculus*, developed by David Hestenes, circa 1965 and *Mathematica*, released in 1988 by Steven Wolfram. Both tools have steep learning curves, requiring several years to acquire expertise in their use. This paper explains in what sense they are optimal.

Keywords

Calabi-Yau Topology, Fermion Spin, Particle Genesis, Primordial Field, Self-Interaction Equations, Yang-Mills Gravity

1. Introduction

A theory of physics is a model of reality, typically expressed as a mathematical model. The theorist then faces the problem of choosing the most appropriate class of mathematics. For example, Einstein's general relativity, based on concepts of curved spacetime, led to 4-dimensional Differential Geometry as the model preferred by most, whereas the choice of most quantum theorists has been that of Hilbert spaces that may be infinite dimensional. These formalisms and underlying ontological concepts of time and space are incompatible with each other, leading to a century-long effort to reconcile gravity theory with quantum theory. The current state of quantum gravity theory [1] is far from reconciled.

Both relativity and quantum theory began a century ago and were fully formulated by mid-century. This effectively insured that advances in math and modelling post-1990 have had relatively little effect on the base theories. On the other hand, the current unsatisfactory state of quantum gravity theory suggests that current tools may be of value in resolving the issues. The goal of this paper is to present two relatively recent mathematical tools that are available to quantum gravity theorists and other physicists.

David Hestenes [2] developed *Geometric Calculus*, an extension of Clifford's Geometric Algebra, of 1870. Physical theories have been based on algebra and geometry as two separate academic fields; physics equations are formulated algebraically, including vector calculus operators, and geometric images are often drawn to accompany the equations and to illustrate the meaning of the algebraic terms. This approach worked well for centuries, requiring only that the physicist master algebra and geometry separately. Hestenes changed this in a manner that is not immediately obvious: his geometric algebra is formulated such that every mathematical entity has both geometric and algebraic meaning. In 3 spatial dimensions, we consider scalars, vectors, bivectors, and trivectors, each of which has a well-defined geometric nature and an algebraic representation, while algebraic operators include $+, -, *, \div, \cdot, \wedge$ and *i*. The standard operators $+, -, *, \div$ need no introduction, while the vector operations \cdot and \wedge are generally familiar to physicists. The *geometric product* is defined for two vectors, *a* and *b*:

$$ab = a \cdot b + a \wedge b \tag{1}$$

A dot product of two vectors yields a scalar, $a \cdot b$, and a wedge product yields a bivector, $a \wedge b$. Observe that geometric algebra allows one to add "unlike entities" ("apples and oranges") in the multi-vector equation. Finally, the *duality* operator, *i*, transforms one geometric algebra entity into its dual, for example

$$a \wedge b = -i(a \times b) \tag{2}$$

The bivector $a \wedge b$ is a 2-dimensional entity obtained by rotating *a* into *b* which has an area but no defined shape, and a direction of rotation; *a* into *b* or *b* into *a*: $a \wedge b = -b \wedge a$. On the right of Equation (2) the duality operator operates on the 1D cross product, a vector, transforming it into the 2D bivector dual to it. Note that dot product operating on two 1D vectors yields a 0D scalar, while the wedge operation on two 1D vectors yields a 2D bivector. This "raising and lowering" effect can be extended to any dimension, but we will herein limit ourselves to 4D scenarios.

2. Fundamental Principles

To demonstrate the power inherent in this math we consider the primordial field of the universe, ψ , which we assume to exist at the moment of creation [3]. The primordial nature implies that *nothing else* existed at this moment, hence, if the field is to interact, as it must, to evolve into our current Universe, the field must interact with itself; nothing else exists to interact with. This yields the *Self-Interaction Principle* represented by the Self-Interaction equation:

$$\nabla \psi = \psi \psi \tag{3}$$

where ψ represents the primordial field and ∇ represents the change opera-

tor. If the field depends upon parameter ξ , the change operator becomes $\nabla \rightarrow \partial_{\xi}$, which leads to two formal solutions for Equation (3): a scalar solution $\psi(\xi) = -\xi^{-1}$ and a vector solution $\psi(\xi) = \xi^{-1}$. Associate the scalar field with time ξ = time and the vector field as a function of position \mathbf{r} . Defining primordial field $\psi = \mathbf{G}(\mathbf{r}, t) + i\mathbf{C}(\mathbf{r}, t)$ with corresponding operator $\nabla = \nabla + \partial_t$, Equation (3) becomes

$$(\nabla + \partial_t)(G + iC) = (G + iC)(G + iC)$$
(4)

A Hestenes' Geometric Calculus expansion of this equation immediately leads to the following:

Self-Interaction equations

$$\nabla \cdot G = G \cdot G - C \cdot C$$

$$i \nabla \cdot C = i 2G \cdot C$$

$$\partial_t G - \nabla \times C = G \times C + C \times G$$

$$i \nabla \times G + i \partial C = 0$$
(5)

Since ψ is the primordial field, its components G and C are fields and the terms $G \cdot G$ and $C \cdot C$ represent the field energy density of these fields, while $G \times C$ represents Poynting-like momentum. We define ρ as the density of the equivalent mass. Substituting this into Equation (5) we obtain:

Heaviside equations

$$\nabla \cdot \boldsymbol{G} = -\rho$$

$$\nabla \cdot \boldsymbol{C} = 0$$

$$\nabla \times \boldsymbol{C} = -\rho \boldsymbol{v} + \partial_t \boldsymbol{G}$$

$$\nabla \times \boldsymbol{G} = -\partial_t \boldsymbol{C}$$
(6)

identical to Heaviside's 1893 formulation [4] of gravity G and gravitomagnetic field C. Note that the concept of *field strength* is absent in the derivation, other than the implicit assumption of strong fields existing at the big bang. Heaviside's equations derived by linearizing Einstein's equations are erroneously labeled the *weak field approximation* to Einstein's equations and have led physicists to regard Einstein's geometric equations as the "true" physics. But Heaviside's formulation is equivalent to Einstein at *all* field strengths, holds at all scales, including the particle scale, and clearly shows the dependence on mass density ρ . Equations (6) are based on gravitational fields G(r,t) and C(r,t)), however the field equation $\nabla \cdot C = 0$ implies that we can make use of vector identity $\nabla \cdot \nabla \times A = 0$ to replace C with a potential vector $\nabla \times A$ and derive the gauge field Eqns used by Yang-Mills [5].

3. Inverse Operations in Solving Equations

The inverse differential ∇^{-1} is used frequently in solving electromagnetic field equations, and is defined in terms of Green's functions:

$$\nabla^{-1} \Longrightarrow \int f\left(r^{-1}\right) \tag{7}$$

A simpler solution to gravitomagnetic field equations is [6] the recently in-

troduced inverse curl:

(

$$(\boldsymbol{\nabla} \times)^{-1} = (\boldsymbol{r} \times) \text{ and } (\boldsymbol{\nabla} \times) = (\boldsymbol{r} \times)^{-1}$$
 (8)

Allowing relatively simple calculation of the induced field circulation:

$$\boldsymbol{\nabla} \times \boldsymbol{C} = -\boldsymbol{P} / \int \mathrm{d}^3 \boldsymbol{x} \tag{9}$$

$$(\mathbf{r} \times)(\nabla \times)\mathbf{C} = -\left(\mathbf{r} \times \frac{\mathbf{P}}{r^3}\right) \Longrightarrow \mathbf{C} = -\mathbf{r} \times \mathbf{p}$$
 (10)

where P is momentum and p is momentum density of the local field. Geometric approaches, such as Einstein's general relativity are unable to define local field density and instead use concepts such as "*quasi local mass*" which I treat in [7]. The concept of local field density is well defined in the Heaviside approach to the gravitational field. As seen in Equation (10) the *C*-field contains angular momentum density proportional to frequency, *i.e.*, *C* is the characteristic frequency of the local angular momentum density: $\int d^3x C = -r \times P \sim L$. The minus sign represents left-handed circulation of the field induced by P.

Based on the above it is quite easy to calculate the *C*-field at an arbitrary observation point, O, as shown in Figure 1. Observe that we have drawn the physical objects as 3D objects. A local momentum vector P induces a local field circulation, $\nabla \times C = -p$, where the local momentum is shown as a red arrow and the circulation as a bivector centered on the momentum.

Gravitomagnetic loops in an ultra-dense turbulent gravity field can "reconnect" to produce a coaxial balanced field dynamics as seen in Figure 1(a). The bivector $\nabla \times C$ is circulation of the *C*-field. The magnitude of the square of the field, $C \cdot C$ is local field energy density (of the spin), and is easily computed at any point in time, as the traveling energy density induced by momentum density



Figure 1. (a) Two coaxial anti-parallel momenta move apart from a reconnection event at the origin. The momenta are represented by red arrows, the local field mass density by the sphere on the red arrow, and the local field circulation by the disc surrounding the sphere. The *xy*-plane defines the region where induced fields cancel each other. (b) Two parallel momenta, with accompanying induced *C*-field circulations, interact with respect to observation point O.

p. This energy can be plotted at the observation point over time, and due to cylindrical coaxial symmetry can be visualized as the strength of field energy at point O due to P from P(t=0) to $P(\infty)$ as seen in Figure 2.

In [8] I construct the physics of h_{ij} for a dynamic spatially homogeneous anisotropic Bianchi vacuum model that solves Einstein's equations in terms of the physically real primordial field, otherwise devoid of matter. Kasner derived the solution to $R^{\mu\nu} = 0$ in 1921. Narlikar and Karmarkar's later formulation is:

$$ds^{2} = c^{2}dt^{2} - \sum_{j=1}^{D-1} (1+nt)^{2p_{j}} dx_{j}^{2} .$$
(11)

While Equation (11) is subject to constraints on p_j , the meaning of parameter *n* has been obscure. I interpret *n* to be primordial field C(t) induced by momentum p_j , assumed to exist because of a reconnection event.

4. Aspects of Mathematica for Hestenes' Geometric Calculus

In 1988 Steven Wolfram [9] released *Mathematica* 1.0 which has evolved to the current version 13.1. Of particular interest to us is the fact that *Mathematica* supports 2D graphics and 3D graphics at the fundamental level, with common geometries "built-in" in parametrizable fashion. These objects can be readily mapped into Geometric Calculus formulations as shown in section 3.

Mathematica models allow a degree of certainty regarding physical equations that is absent when the equations are solved by hand. In such cases one gains certainty by reviewing the logic of the system until convinced that no logical errors exist. If the *Mathematica* model produces stable behaviors where one expects stable behaviors, one rather quickly exercises modes of behavior that hand solutions do *not* exercise.



Figure 2. The *C*-field energy density as a function of time as measured at the observation point in **Figure 1(a)** with respect to a reconnection event in an anisometric open universe described by the Kasner metric. The time axis is mapped onto the reconnection axis corresponding to the *z*-axis.

In addition, *Mathematica* provides the ability to manipulate parameters "live" based on a large variety of "controls" and to observe the resultant behaviors from easily selectable 3D perspectives. These aspects allow exploration of behavioral details hidden from the analytic solver of equations.

The system readily supports algebraic computations of the type discussed above. In other words, the *inverse curl operation* allows us to work with the field C, it's energy density $C \cdot C$, and its circulation $\nabla \times C$. Note that the helical structure of the circulating C-field is encompassed by term $e^{i\theta} \sim e^{iCt}$ where θ is a dimensionless parameter and C has dimensions of frequency, $\sim \frac{1}{t}$. This circulating wave behavior is known as U(1)-symmetry. The solution to Maxwell's field wave equations has U(1) symmetry, $e^{i\theta} \sim \cos(\theta) + i\sin(\theta)$. In other words, the propagating field has helical structure.

The physical regimes of interest are ultra-high-density gravitational fields, exemplified by big bang and atom-atom collisions at CERN. Both such regimes are extremely turbulent such that collisions of helices, including self-intersection occurs, potentially forming tori. In such cases the symmetry is essentially $U(1) \times U(1)$ as illustrated in Figure 3.

The speed at any point on the helix is constant as is shown in **Figure 4(a)**. Our $U(1) \times U(1)$ conceptual model shows every circle disconnected from every other circle; *not* a helix. To reflect the physical ontology of the torus, we desire helical flow lines. The tangent, and hence flow velocity, has the same definition, and since the radius is constant around the U(1) circle, the velocity is constant. The parametric helix is $r = |r| \{ \cos(t), \sin(t), t \}$ hence, the *Mathematica* model for **Figure 4(a)** is:

$$x[t_]:=Cos[t]; y[t_]:=Sin[t]; z[t_]:=t;$$
(12)

 $velocities = Table [\{\{x'[\theta], y'[\theta], z'[\theta]\}\}, \{\theta, 0, 4\pi, \pi/180\}] / N$ (13)

$$ListPlot[Table[{velocities[[n]][[1]].velocities[[n]][[1]], n}, \{n, 361\}]]//N \quad (14)$$

Figure 4(b) shows the value of the velocity squared. In comparison, velocity of any point of a helix has constant magnitude. We elaborate on simple helical flow because it is easy to grasp and yet differs from toroidal flow, despite that we constructed a torus from a helix, by curving the helix until its ends join; this joining changes U(1) helical symmetry to $U(1) \times U(1)$ symmetry of the torus.



Figure 3. (a) U(1) (circles) centered on red U(1) circle yield; (b) torus with $U(1) \times U(1)$ symmetry.



Figure 4. Unlike the constant velocity of helical flow, the [squared] velocity of toroidal flow is smoothly distributed between minimum and maximum velocities. Velocities here range from ~6.5 to ~11 as the parametric path is followed from zero to 360 degrees. This differs from the velocity of the helix because the size of the torus has changed, nevertheless, this distribution of velocities represents any size torus.

We show the difference in **Figure 4(b)** by plotting the velocity of the "helical" flow around the torus.

If the donut retains a circular cross section, we might initially guess that the flow velocity would have constant magnitude like the helix. We investigate why this is not the case.

Creation of the model, representative of the theory, is an immediate realization of a behavioral mechanism, almost every point of which can be examined, typically visually. One creates the model, then brings it to "life" in Mathematica Dynamic Modules that can be manipulated in unlimited fashion via analog and digital controls.

For example, the primordial field model was *not* created based on tangents on a manifold, but the tangents exist, and when one reasons about flow on a manifold it is useful to consider these tangents. In fact, one can partition the velocities at key points on the manifold based on intuitive understanding of the flow requirements imposed by conservation of momentum. The mass flow of the toroidal field energy in $U(1) \times U(1)$ circulates *around* the hole and *through* the hole in the torus. If the torus is horizontally flat in the *xy*-plane and centered on the *z*-axis of the frame, the torus has in effect two equators, an outer equator and an inner equator. One can intuitively derive flow behavior in the horizontal plane and in a vertical plane that contains the *z*-axis. Surprisingly, even using color-coded visualization techniques, the dynamic flow on such a curved manifold is too complex to be confident that one's equations correctly describe such flow. In **Figure 5** a flow path is depicted in black with points in the flow shown by red and green arrowheads, with red arrows when the flow is above the equators and green arrows when the flow is below (with outer equator shown in white).

The dynamic visualization of the field behavior intuitively confirms the "correctness" of the model/theory, but the flow is shown for the local velocity consisting of both horizontal and vertical components of velocity. Separating these



Figure 5. Highlighting specific points on a given $U(1) \times U(1)$ path for a toroidal flow.

components make a complicated dynamic even more complicated. Once, however, one is convinced of the theory's correctness, then one can perform experiments on the model, that is, make *measurements*.

5. Measurements on a Dynamic Model

Rather than complicating the visual dynamic flow further, by dividing it into two components as it flows around the torus, I decided to simply print out the values of the horizontal and vertical components of velocity at every point. Since I typically employ 360 points for each U(1) path, and can, via *Mathematica* controls, determine the speed of simulation, it is quite simple to walk my way around the path, slowing down at each of the critical points (the red and green arrow heads in **Figure 5**) examining the velocity components, equatorial vertical velocities $v_{iz} = v_{oz}$ with $v_o > v_i$ and corresponding horizontal components $v_{o\theta} > v_{i\theta}$ with

$$\mathbf{v}_o = \mathbf{v}_{oz} + \mathbf{v}_{0\theta}$$
 and $\mathbf{v}_i = \mathbf{v}_{iz} + \mathbf{v}_{i\theta}$. (15)

We thereby build a table as seen in **Table 1**. The radii are defined in **Figure 6**, with r_i the inner radius, r_o the outer radius, and *R* the radius to the core of the torus.

At any point on the manifold the velocity $v = v_{\theta} + v_{z}$. If we square both sides, term $v_{\theta} \cdot v_{z} = 0$ since v_{θ} and v_{z} are orthogonal, hence

v

$$=\sqrt{v_{\theta}^2 + v_Z^2} \tag{16}$$

For example, at 30° the velocity is $\sqrt{9^2 + 9^2} = 12.7279220$ while at 90° $v = \sqrt{3^2 + 9^2} = 9.48$. We see from the table that the measurements confirm the intuitively derived relations based on the conservation of momentum reasoning. In short, the dynamic visualization of the field behavior intuitively confirms the correctness of the model/theory, while the measurement access to arbitrary parameters can serve as proof of the flow model worked out by conservation equations and the $U(1) \times U(1)$ -symmetry. When these measurements on the model agree in detail with intuitively and/or analytically derived behavior, the feeling is as if one has "struck gold". One can only sincerely thank David Hestenes and Steven Wolfram for their contributions to this task.



Figure 6. The radii r_{p} r_{a} and *R* used in measurements in **Table 1**.

deg	0	30	60	90	120	150	180	210
$\boldsymbol{v}_{ heta}$	10.8	9	10.8	3	10.8	9	10.8	3
v_{Z}	0	9	0	9	0	9	0	9
v	10.8	12.7279	10.8	9.48	10.8	12.72	10.8	9.48
radi	R	r _o	-R	- <i>r</i> _i	R	r _o	-R	r _i

Table 1. Measurement of velocity components.

6. Summary and Conclusions

The current paradigms of relativistic and quantum math precede the advent of modern computers, whereas Hestenes' development of a calculus of geometric algebra, based on $ab = a \cdot b + a \wedge b$ has the property that every entity has both algebraic and geometric meaning. Thus, theoretical equations can be represented both algebraically and geometrically, a claim that no other mathematical formalism can match. As computational power began to reach *Everyman*, post-1980, Wolfram designed an equation solver of monstrous proportions and it has evolved into a first-class tool for solving, calculating, and displaying 3D objects associated with geometric calculus. *Mathematica* solves the algebra and displays the geometry as it was designed to do.

A theory is a model. One can create a theory centered in any of many mathematical disciplines and try to analyze one's theory and make predictions. It is typically the case that the theory arose in one's mind, and was translated into equations in one's head, and these equations are used to predict behavior. One can make reasonable progress based on intuitive models in one's mind, but when this is complemented with a stable model implementing the theory, the predicative power is significantly enhanced. Based on the self-interaction of the primordial field I have derived Heaviside's equations which are iteratively equivalent to Einstein's general relativity of "curved spacetime" and have shown that a century of relativistic paradox and perplexity are resolved by this approach. I have reinterpreted dynamic metrics in this context as hinted at in section 3.

Other treatments of physics based on *Mathematica* exist [10]. An excellent treatment based on *Geometric Algebra* is [11].

A recent paper [12] states that "Modern cosmological models are formulated in the framework of field theory [however] fields do not evolve per se—a solu*tion to field equations specifies the field content in the entire spacetime.*" This is true for quantum field theory, in which specific fields occupy all of space, with excitations in the field giving rise to specific particles, but contrasts with our primordial field theory wherein the field *does* evolve through self-interaction.

Eckstein and Horodecki further state that, while modern physics is founded on two mainstays; mathematical modeling and empirical verification, a contradiction exists, as any experiment performed in a physical system is—by necessity—invasive and thus establishes inevitable limits to the accuracy of any mathematical model. As hinted at in this paper, the tools of Geometric Calculus and *Mathematica* have recently been applied to quantum gravity. A cosmological analysis based on an extension of the Kasner metric has produced a dynamic universe, empty of matter, and a topological variant has been produced with the goal of deriving fermions from the primordial field. Evolution of the field into fundamental particles is essentially unmeasurable, so the experiments are downstream, involving particle interactions of the type produced at CERN-LHC. In this case, the limits on accuracy of the mathematical model are of little consequence; the model should exhibit a *mass-gap*, *half-integral spin*, and *discrete charge*. This has been accomplished and is in process of submission.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- Armas, J. (2021) Conversations on Quantum Gravity. Cambridge University Press, Cambridge. <u>https://doi.org/10.1017/9781316717639</u>
- Hestenes, D. and Sobczyk, G. (1984) Clifford Algebra to Geometric Calculus. Reidel Pub., Boston. <u>https://doi.org/10.1007/978-94-009-6292-7</u>
- Klingman, E.E. (2020) The Primordial Principle of Self-Interaction. Journal of Modern Physics, 12, 65-81. <u>https://doi.org/10.4236/jmp.2021.122007</u>
- [4] Heaviside, O. (1893) A Gravitational and Electromagnetic Analogy. *The Electrician*, 31, 81-82
- [5] Klingman, E.E. (2022) Particle Creation from Yang-Mills Gravity. *Journal of Modern Physics*, 13, 1128-1145. <u>https://doi.org/10.4236/jmp.2022.137065</u>
- Klingman, E.E. (2020) Exact Inverse Operator on Field Equations. *Journal of Applied Mathematics and Physics*, 8, 2213-2222. https://doi.org/10.4236/jamp.2020.810166
- Klingman, E.E. (2022) A Re-Interpretation of Quasi-Local Mass. *Journal of Applied Mathematics and Physics*, 13, 347-367. <u>https://doi.org/10.4236/jmp.2022.134025</u>
- [8] Klingman, E.E. (2019) A Primordial Space-time Metric. *Prespacetime Journal*, 10, 671-680. https://prespacetime.com/index.php/pst/article/view/1602
- [9] <u>https://www.wolfram.com/</u>
- Sarafian, H. (2022) Designing Physics Problems with Mathematica. American Journal of Computational Mathematics, 12, 297-305. https://doi.org/10.4236/ajcm.2022.123019

- [11] Arthur, J.W. (2011) Understanding Geometric Algebra for Electromagnetic Theory. John Wiley & Sons, Hoboken, NJ. https://doi.org/10.1002/9781118078549
- [12] Eckstein, M. and Horodecki, P. (2022) The Experiment Paradox in Physics. Foundations of Science, 27, 1-15. <u>https://doi.org/10.1007/s10699-020-09711-y</u>