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Modification of the Convergence of GG-PPA for Solving Generalized Equations

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Abstract

A modified Gauss-type Proximal Point Algorithm (modified GG-PPA) is presented in this paper for solving the generalized equations like $0 \in \mathcal{T}(x)$, where $\mathcal{T}$ is a set-valued mapping acts between two different Banach spaces X and Y. By considering some necessary assumptions, we show the existence of any sequence generated by the modified GG-PPA and prove the semi-local and local convergence results by using metrically regular mapping. In addition, we give a numerical example to justify the result of semi-local convergence.

Keywords

Set-Valued Mappings, Metrically Regular Mappings, Semi-Local Convergence, Lipschitz Continuous, Fixed Point Lemma

1. Introduction

Consider X and Y are Banach spaces, $\mathcal{T}: X \rightrightarrows 2^Y$ is a set-valued mapping with a locally closed graph. We proceed with the problem of seeking a point $x \in X$, which satisfies the generalized equation

$$0 \in \mathcal{T}(x). \tag{1}$$

Robinson first introduced generalized equations as a hypothetical example for a wide range of variational issues, such as complementary issues, variational inequalities, and systems of nonlinear equations. For more information, see [1] [2]. It may specifically apply to optimality or equilibrium issues (for more details, see [3]). Select a set of Lipschitz continuous functions $g_t : X \to Y$ with $g_t(0) = 0$ that are located in the vicinity of the solution. Martinet [4] first developed the below algorithm for use in convex optimization by taking into account a sequence of scalars $\{\lambda_t\}$ that are distinct from zero. For each $t = 0, 1, 2$, the expression is

$$0 \in \lambda_t \left(x_{t+1} - x_t \right) + \mathcal{T} \left(x_{t+1} \right). \tag{2}$$

Method (2) was thoroughly examined by Rockafellar [5] within the broad context of maximal monotone inclusions. Specifically, Rockafellar [5] demonstrated that for a scalar sequence $\{\lambda_t\}$ which is away from 0, iteration (2) creates a sequence $\{x_t\}$ that is to a weakly convergent result of (1) for any beginning point x_0 belongs to X if x_{t+1} represents an approximate solution of (1) and $\mathcal{T}$ is maximum monotone.

The proximal point technique has been investigated by several researchers, and they have also discovered uses for it in particular variational inequalities. Numerous iterations of the procedure for resolving generalized equations using monotone mappings, and in particular monotone variational inequalities, have received the majority of the attention in the literature on this topic. Spingarn [6] has investigated the first weaker version of monotonicity. In many situations, the mapping $\mathcal{T}$ frequently exhibits monotonicity.

However, there is interest in thinking about and researching such a system without monotony. First, monotonicity works with mappings of both a space and its dual, which typically results in a restriction of the procedure for mappings on a Hilbert space. As a result, we are able to work with mappings acting between two separate Banach spaces as metric regularity does not necessitate the condition of monotonicity. Second, omitting some mappings that are metrically regular since monotonicity sometimes proves to be a strong assumption. For the situation of non-monotone mappings, Aragon Artacho *et al.* [7] have given the generic version of the PPA for solving the inclusion problem (1). The findings of the PPA's convergence without monotonicity are surveyed in [8]. Let x_0 belongs to X. Consider

$$\mathbb{D}^t \left(x \right) = \left\{ l \in X : 0 \in g_t \left(l \right) + \mathcal{T} \left(x + l \right) \right\} \tag{3}$$

is the definition of the subset of X represented by $\mathbb{D}^t \left(x \right)$. Aragon Artacho *et al.* [7] established the GPPA that is listed below:

Algorithm 1 (GPPA):

Step 1: Set t equal to 0, x_0 belongs to X, λ is greater than 0.

Step 2: Stop if $0 \in \mathbb{D}^t \left(x_t \right)$; else, proceed to Step 3.

Step 3: Enter $\{\lambda_t\} \subseteq \left(0, \lambda \right)$ and if $0 \notin \mathbb{D}^t \left(x_t \right)$, select l_t so that $l_t \in \mathbb{D}^t \left(x_t \right)$.

Step 4: Compose x_{t+1} equal to $x_t + l_t$.

Step 5: Compute t by $t + 1$ before moving on to Step 2.

Under certain conditions, Aragon Artacho *et al.* [7] established that the aforementioned method creates only one sequence that converges linearly to the solution when the beginning point x_0 is near the solution. We can therefore conclude that this type of procedure is not suitable for real application in light of practical computations. To overcome this difficulty, Alom *et al.* [9] introduced the general version of the Gauss-type Proximal Point Algorithm (GG-PPA) and proved the semi-local and local convergence results by using both the metric regularity condition and the Lipschitz-like property for set-valued mapping with some necessary conditions. We see that the procedure of Alom *et al.* [9] is comparatively lengthy and complex. To overcome the limitations of Aragon Artacho *et al.* [7] and prove the semi-local and local convergence results in the easiest way, we propose the modified GG-PPA for resolving the generalized Equation (1) with some new ideas to the key theorem and prove this by using only metric regularity condition in place of Lipschitz-like property.

To show the existence and the proof of convergence of any sequence generated by the GG-PPA, Alom *et al.* [9] considered the main theorem that $(\eta+3)\kappa\lambda \le 1$, $\delta \le \min\left\{\frac{r_{\bar{x}}}{4}, \frac{\bar{r}}{2\lambda}, \frac{r_{\bar{y}}}{3\lambda}, 1, \frac{1-\lambda\kappa}{6\lambda\kappa}\right\}$ and $\|\bar{y}\| < \lambda\delta$, while in the key theorem in this paper, we consider that $(\eta+2)\kappa\lambda \le 1$, $\delta \le \min\left\{\frac{r_{\bar{x}}}{2}, \frac{\bar{r}}{\lambda}, \frac{2r_{\bar{y}}}{3\lambda}\right\}$ and $\|\bar{y}\| < \frac{1}{2}\lambda\delta$. We demonstrate that our approach is superior to the prior one for resolving the generalized Equation (1). The improved GG-PPA creates sequences whose any sequence is convergent, however, Algorithm 1 does not, which is the distinction between our suggested method 2 and Algorithm 1. So, the updated GG-PPA that we suggest here is given below:

Algorithm 2 (Modified GG-PPA):

Step 1: Set t equal to 0, x_0 belongs to X, λ is greater than 0 and $\eta \ge 1$.

Step 2: Stop if $0 \in \mathbb{D}^t(x_t)$; else, proceed to Step 3.

Step 3: Enter $\{\lambda_t\} \subseteq (0,\lambda)$ and if $0 \notin \mathbb{D}^t(x_t)$, select l_t so that $l_t \in \mathbb{D}^t(x_t)$ and $\|l_t\| \le \eta d(0, \mathbb{D}^t(x_t))$.

Step 4: Compose x_{t+1} equal to $x_t + l_t$.

Step 5: Compute t by $t+1$ before moving on to Step 2.

From Algorithm 2, we note that:

1) Algorithm 2 simplifies to the traditional PPA defined by (2) if η equal to 1, $\mathbb{D}^t(x_t)$ is single valued, $g_t(u) = \lambda_t u$, and Y equal to X a Hilbert space.

2) if η equal to 1 and $\mathbb{D}^t(x_t)$ is single valued, Algorithm 2 and Algorithm 1 are equivalent.

3) Algorithm 2 is equivalent to the traditional G-PPA that Rashid *et al.* [10] introduced when $g_t(u) = \lambda_t u$, and Y equal to X is a Banach space.

There is substantial work on the analysis of the local convergence of Algorithm 1, but no additional analysis of the semi-local convergence of Algorithm 1 is available. Rashid *et al.* provided a semi-local convergence study for the traditional proximal point approach of the Gauss type in [10]. Rashid developed the G-PPA for solving the issue of variational inequality in his subsequent study [11], which also yielded local and semi-local convergence findings. Numerous contributions to the study of the analysis of semi-local convergence for the Gauss-Newton method have been made in [12]. For the purpose of solving smooth generalized equations, Alom and Rashid [13] presented the GGPPA and examined the local and semi-local convergence results. Recently, Khaton and Rashid [14] developed the extended Newton-type method and got the local and semi-local convergence results for solving the generalized Equation (1).

According to our knowledge, Algorithm 2's semi-local and local convergence analysis of the applied system has never been studied before. As a result, we draw the conclusion that the contributions made in this work appear novel. Basically, Algorithm 2 is subject to two different kinds of convergence problems. While one of them, the convergence criterion depending on information near the starting point x_0 is the focus of a technique called semi-local analysis, the other, known as local convergence analysis, is focused on the convergence ball depending on information close to a solution of (1). Our goal in the current work is to investigate the results of semi-local convergence of the modified GG-PPA as stated by Algorithm 2. Our study mainly relies on the metric regularity conditions of the mappings with set values.

The primary findings come from the convergence analysis introduced in Section 3, which is dependent on the zone of attraction surrounding the beginning point and provides certain necessary assumptions for any sequence produced by Algorithm 2 to converge to a solution. As a consequence, the result of the local convergence analysis of the modified GG-PPA is produced.

This paper is arranged in the form: In Section 3, we investigate the modified GG-PPA that was described in this part. Applying the ideas of metric regularity axiom for the mapping $\mathcal{T}$ with set values, we will establish the convergence of the sequence produced by Algorithm 2 and demonstrate its existence. In Section 4, we offer a numerical example to verify the result of the semi-local convergence analysis of the modified GG-PPA. We provide an overview of the key findings from this effort in Section 5.

2. Notations and Preliminaries

This part of the paper serves as a review of several common notations, foundational ideas, and mathematical findings that will be used often in the next part. Assume X and Y both are general Banach spaces. The expression $\mathcal{T}: X \rightrightarrows 2^Y$ is a symbol for mapping with set values from X to subset of Y. Allow r is greater than 0 and x belongs to X. $\mathbb{B}_r(x)$ indicates a closed ball with centered at x and radius r.

The terms graph of $\mathcal{T}$ is represented by $gph\mathcal{T}$ and is expressed by

$$gph\mathcal{T} = \{(a,b) \in X \times Y : b \in \mathcal{T}(a)\},$$

the domain of $\mathcal{T}$ is represented by $dom\mathcal{T}$ and is expressed by

$$dom\mathcal{T} = \{x \in X : \mathcal{T}(x) \neq \varnothing\},$$

and the inverse of $\mathcal{T}$ is represented by $\mathcal{T}^{-1}$ and is expressed by

$$\mathcal{T}^{-1}(b) = \{c \in X : b \in \mathcal{T}(c)\}.$$

By the symbol $\|\cdot\|$, we represents all the norms. Allow S and D are subsets of X respectively.

$d(x,S) = \inf\{\|x-b\| : b \in S\}$ for all x belongs to X, defines the distance from x to S, where as

$$e(D,S) = \sup\{d(x,S) : x \in D\}$$

defines the excess between sets S and D. The above definitions are collect from [10].

For a mapping with set-values, we accept the following concept of metric regularity from [15].

Definition 1. **Metrically Regular Mapping:**

Assuming $(x', y') \in gph H$, where $H: X \rightrightarrows 2^Y$ is a set-valued mapping. Let $r_{x'}$, $r_{y'}$, and κ all be greater than zero. When

$d(x, H^{-1}(y)) \le \kappa d(y, H(x))$ for each $x \in \mathbb{B}_{r_{x'}}(x')$, $y \in \mathbb{B}_{r_{y'}}(y')$,

then at (x', y') on $\mathbb{B}_{r_{x'}}(x')$ relative to $\mathbb{B}_{r_{y'}}(y')$ with constant κ, the mapping H is said to be metrically regular.

For set-valued mappings, we recall the concepts of Lipchitz-like continuity from [16]. This idea was first put by Aubin in [17].

Definition 2. **Lipschitz-Like Continuity:**

Assume that $\gamma: Y \rightrightarrows 2^X$ is a mapping with set-values and that $(\bar{y}, \bar{x})$ belongs to $gph \gamma$. Suppose $r_{\bar{x}}$, $r_{\bar{y}}$ and k all be greater than zero. Then at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ together with constant k, the mapping γ is said to be Lipschitz-like, when the subsequent disparity exists: for every y_1, y_2 belongs to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$,

$$e\left(\gamma(y_1) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \gamma(y_2)\right) \le k \|y_1 - y_2\|.$$

We collect the following lemma from [11], which establishes the relationship between the Lipschitz-like continuity of the inverse $\mathcal{T}^{-1}$ at $(\bar{y}, \bar{x})$ and a mapping $\mathcal{T}$ of metric regularity at $(\bar{x}, \bar{y})$.

Lemma 1. Let $(\bar{x}, \bar{y}) \in gph \mathcal{T}$, where $\mathcal{T}: X \rightrightarrows 2^Y$ be a mapping with set values. Suppose $r_{\bar{x}}$ and $r_{\bar{y}}$ both are greater than zero. It follows that at $(\bar{x}, \bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ the mapping $\mathcal{T}$ is metrically regular with constant L for any $y, y' \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$, iff at $(\bar{y}, \bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ the inverse $\mathcal{T}^{-1}: Y \rightrightarrows 2^X$ is Lipschitz-like with constant L, that is,

$$e\left(\mathcal{T}^{-1}(y) \cap \mathbb{B}_{r_{\bar{x}}}(\bar{x}), \mathcal{T}^{-1}(y')\right) \le L \|y - y'\|. \quad (4)$$

We have taken the Lyusternik-Graves theorem from [18]. We conduct that a set G is a subset of X is locally closed at z belongs to G if t is greater than 0 so that the set $G \cap \mathbb{B}_t(z)$ is closed.

Lemma 2. **Lyusternik-Graves Theorem**: Suppose $(\bar{x}, \bar{y}) \in gph \mathcal{T}$, here $\mathcal{T}: X \rightrightarrows 2^Y$ is a mapping with set values and $gph \mathcal{T}$ is locally closed are taken into consideration. Let $\mathcal{T}$ have constant $\kappa > 0$ and be metrically regular at $\bar{x}$ for any $\bar{y}$. Take into consideration a function $g: X \to Y$ that has a Lipschitz constant λ and is continuous at $\bar{x}$, such that λ is less than κ^{-1}. The mapping $g + \mathcal{T}$ then has constant $\frac{\kappa}{1 - \kappa\lambda}$ and is metrically regular at $\bar{x}$ for $\bar{y} + g(\bar{x})$.

The fixed point lemma for set-valued mappings was generalized from the fixed point theorem [6] and proved by Dontchev and Hagger in [19]. To demonstrate the existence of any sequence, this lemma is absolutely essential.

Lemma 3. **Banach Fixed Point Lemma:**

Consider $\phi: X \rightrightarrows 2^X$ is a mapping with set values. Suppose η_0 belongs to X, r belongs to $(0,\infty)$ and $0<\alpha<1$ be such that

$$d\left(\eta_0,\phi(\eta_0)\right)<r(1-\alpha), \tag{5}$$

and for all $x_1,x_2\in\mathbb{B}_r(\eta_0)$,

$$d\left(x_1,\phi(x_2)\right)\le e\left(\phi(x_1)\cap\mathbb{B}_r(\eta_0),\phi(x_2)\right)\le\alpha\|x_1-x_2\| \tag{6}$$

are respectively satisfied. Consequently, there is a fixed point for ϕ in $\mathbb{B}_r(\eta_0)$, indicating that x belongs to $\mathbb{B}_r(\eta_0)$ exists and $x\in\phi(x)$. If ϕ is also single-valued, the only fixed point of ϕ is in $\mathbb{B}_r(\eta_0)$.

3. Convergence Analysis

In this part, we consider that $\mathcal{T}: X \rightrightarrows 2^Y$ is a mapping with set-values with a locally closed graph at $(\bar{x},\bar{y})$ belongs to $gph\mathcal{T}$ so that at $(\bar{x},\bar{y})$ with constant $\kappa>0$, the mapping $\mathcal{T}$ is metrically regular. Consider $g: X\to Y$ is a (single-valued) function with $g(0)$ equal to 0 and a Lipschitz constant $\lambda>0$ that is Lipschitz continuous in the area near the origin. Create a set-valued mapping $P_x: X\rightrightarrows 2^Y$ defined by

$$P_x(u)=g(u-x)+\mathcal{T}(u), \text{ for all } u\in X. \tag{7}$$

Hence, for each $z\in X$ and $y\in Y$, we have the following result:

$$z\in P_x^{-1}(y) \text{ both implies } y\in g(z-x)+\mathcal{T}(z). \tag{8}$$

As for example,

$$\bar{x}\in P_x^{-1}(y) \text{ for any } (\bar{x},\bar{y})\in gph\mathcal{T}. \tag{9}$$

Suppose $r_{\bar{x}}$ is greater than 0, $r_{\bar{y}}$ is greater than 0, and let $(\bar{x},\bar{y})$ belongs to $gph(g+\mathcal{T})$. Assuming that with constant $\dfrac{\kappa}{1-\kappa\lambda}$, the mapping $P_{\bar{x}}$ is metrically regular at $(\bar{x},\bar{y})$, we use the Lemma 2, which leads to the relation

$$e\left(P_{\bar{x}}^{-1}(y)\cap\mathbb{B}_{r_{\bar{x}}}(\bar{x}),P_{\bar{x}}^{-1}(y')\right)\le\frac{\kappa}{1-\kappa\lambda}\|y-y'\| \text{ for each } y,y'\in\mathbb{B}_{r_{\bar{y}}}(\bar{y}). \tag{10}$$

Select

$$\bar{r}=\min\left\{\frac{2r_{\bar{y}}-r_{\bar{x}}\lambda}{2},\frac{r_{\bar{x}}(1-2\kappa\lambda)}{4\kappa}\right\}. \tag{11}$$

After that,

$$\bar{r}>0 \text{ both implies } \lambda<\min\left\{\frac{2r_{\bar{y}}}{r_{\bar{x}}},\frac{1}{2\kappa}\right\}. \tag{12}$$

Due to the refinement of [11], the analysis of convergence of the modified GG-PPA depends on the following lemma.

Lemma 4. Assume that satisfying (11) and (12), the mapping $P_{\bar{x}}: X\rightrightarrows 2^Y$ is metrically regular at $(\bar{x},\bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant

$\frac{\kappa}{1-\kappa\lambda}$. Consider $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ and $\mathbb{B}_{r_{\bar{x}}}(0)$ are the subsets of the neighbourhood of origin. Then the mapping $P_x^{-1}: Y \rightrightarrows 2^X$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $\frac{\kappa}{1-2\kappa\lambda}$, *i.e.* for each $y_1, y_2 \in \mathbb{B}_{\bar{r}}(\bar{y})$,

$$e\left(P_x^{-1}(y_1)\cap\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x}), P_x^{-1}(y_2)\right) \le \frac{\kappa}{1-2\kappa\lambda}\|y_1-y_2\|.$$

For the benefit, we take into account a collection of functions $g_t: X \to Y$ with $g_t(0)=0$ that are Lipschitz continuous near the origin, identical for all *t*, with Lipschitz constants λ_t satisfying

$$\lambda = \sup_t \lambda_t < \frac{3}{19\kappa}. \tag{13}$$

Remark 1. For each $t = 0,1,2,\cdots$, we define the set-valued mapping $P_x^t: X \rightrightarrows 2^Y$ by using g_t in place of *g* in (7) that

$$P_x^t(u) = g_t(u-x) + \mathcal{T}(u) \quad \text{for all} \quad u \in X. \tag{14}$$

By the relation (13), we can write $\lambda\kappa < \frac{3}{19} < 1$. Thus by Lemma 1 and Lemma 2, we conclude that the mapping $P_{\bar{x}}^{t^{-1}}(u)$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ relative to $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ with constant $\frac{\kappa}{1-\kappa\lambda}$ which satisfies (10).

Therefore, we obtain

$$\mathbb{D}^t(x) = \left\{l \in X : 0 \in P_x^t(x+l)\right\}. \tag{15}$$

Additionally, we specify the set-valued mapping $Z_x: X \rightrightarrows 2^Y$ for each *x* belongs to *X* by

$$Z_x^t(u) = g_t(u-\bar{x}) - g_t(u-x), \tag{16}$$

and $\Phi_x^t: X \rightrightarrows 2^X$ is a set-valued mapping followed by

$$\Phi_x^t(u) = P_{\bar{x}}^{t^{-1}}\left[Z_x^t(u)\right] \quad \text{for all} \quad u \in X. \tag{17}$$

Thus, for all x', x'' belongs to *X*,

$$\begin{aligned}\left\|Z_x^t(x') - Z_x^t(x'')\right\| &= \left\|g_t(x'-\bar{x}) - g_t(x'-x) - g_t(x''-\bar{x}) + g_t(x''-x)\right\| \\ &\le \left\|g_t(x'-\bar{x}) - g_t(x''-\bar{x})\right\| + \left\|g_t(x'-x) - g_t(x''-x)\right\|.\end{aligned} \tag{18}$$

Let us consider that

$$\lim_{x\to\bar{x}} d(\bar{y}, \mathcal{T}(x)) = 0. \tag{19}$$

The key finding of this work was as follows, which outlines certain necessary conditions for the modified GG-PPA to converge with the initial point x_0.

Theorem 1. Consider that η is greater than 1, $\mathcal{T}$ is metrically regular at $(\bar{x},\bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant κ and $\{g_t\}$ is a sequence of Lipschitz continuous function with Lipschitz constants λ_t where

$g_t(0)=0$. Suppose $\bar{r}=\min\left\{\frac{2r_{\bar{y}}-r_{\bar{x}}\lambda}{2},\frac{r_{\bar{x}}(1-2\kappa\lambda)}{4\kappa}\right\}$. Allow $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ is a subset of the neighbourhood of origin and $0<\delta\le 1$ be defined as follows:

1) $(\eta+2)\kappa\lambda\le 1$.

2) $\delta\le\min\left\{\frac{r_{\bar{x}}}{2},\frac{\bar{r}}{\lambda},\frac{2r_{\bar{y}}}{3\lambda}\right\}$.

3) $\|\bar{y}\|<\frac{1}{2}\lambda\delta$.

Then, there are some $\hat{\delta}$ is greater than 0 so that every sequence $\{x_n\}$ created by Algorithm 2 beginning with x_0 belongs to $\mathbb{B}_{\hat{\delta}}(\bar{x})$ linearly converges to a result x^* of (1), *i.e.* 0 belongs to $\mathcal{T}(x^*)$ is satisfied by x^*.

Proof. According to the statement of the theorem, $\mathcal{T}$ is metrically regular at $(\bar{x},\bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant κ and $\{g_t\}$ is a sequence of Lipschitz continuous function with Lipschitz constants λ_t which are away from zero. Thus, we get $\lambda\kappa<1$ by (13). Then by applying Lyusternik-Graves theorem on (14), we have the set-valued mapping $P_{\bar{x}}^t$ is metrically regular at $(\bar{x},\bar{y})$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_{\bar{y}}}(\bar{y})$ with constant $\frac{\kappa}{1-\lambda\kappa}$. For our convenience, we choose the constant

$$h=\frac{\eta\kappa\lambda}{1-2\kappa\lambda}. \tag{20}$$

Then, by applying the assumption $(\eta+2)\kappa\lambda\le 1$, it becomes

$$h\le\frac{\frac{\eta}{\eta+2}}{1-\frac{2}{\eta+2}}=1. \tag{21}$$

We can take $0<\hat{\delta}\le\delta$ under the conditions of assumption $\|\bar{y}\|<\frac{1}{2}\lambda\delta$ and (19) such that for each $x_0\in\mathbb{B}_{\hat{\delta}}(\bar{x})$,

$$d(0,\mathcal{T}(x_0))<\frac{1}{2}\lambda\delta. \tag{22}$$

Using logical deduction, we'll proceed to demonstrate that Algorithm 2 creates one or more sequences and that each sequence $\{x_t\}$ produced by Algorithm 2 fulfills the following claims:

$$\|x_t-\bar{x}\|\le 2\delta, \tag{23}$$

and for every $t=0,1,2,\cdots$,

$$\|x_{t+1}-x_t\|\le h^{t+1}\delta. \tag{24}$$

To prove the inequalities (23) and (24), we choose another constant r_x by

$$r_x=\frac{8\kappa}{5(1-\lambda\kappa)}\left(\|\bar{y}\|+\lambda\|x-\bar{x}\|\right)\ \text{ for every }\ x\in X. \tag{25}$$

By applying assumptions $(\eta+2)\kappa\lambda\le 1$ and $\|\bar{y}\|<\frac{1}{2}\lambda\delta$ with $\eta>1$, we have for every $x \in \mathbb{B}_{2\delta}(\bar{x})$,

$$r_x \le \frac{4\kappa\lambda}{1-\lambda\kappa}\delta \le \frac{4}{\eta+1}\delta \le 2\delta. \tag{26}$$

The fact that (23) holds true for $t = 0$ is simple. We must demonstrate that x_1 exists, *i.e.* $\mathbb{D}^0(x_0)$ not equal to empty, in order to demonstrate that (24) is valid for $t = 0$. Lemma 3 is applied to the mapping $\Phi^0_{x_0}$ with η equal to $\bar{x}$, we can demonstrate that $\mathbb{D}^0(x_0) \ne \varnothing$. Let's verify that Lemma 3's assertions (5) and (6) are true for $r = r_{x_0}$ and $\alpha = \frac{3}{8}$.

From (17), we can write $\Phi^0_{x_0}(\bar{x}) = P^{0^{-1}}_{\bar{x}}\left[Z^0_{x_0}(\bar{x})\right]$. We, therefore, obtain

$$d\left(\bar{x}, \Phi^0_{x_0}(\bar{x})\right) = d\left(\bar{x}, P^{0^{-1}}_{\bar{x}}\left[Z^0_{x_0}(\bar{x})\right]\right). \tag{27}$$

Now, by the definition of metric regularity, we can write that

$$d\left(\bar{x}, P^{0^{-1}}_{\bar{x}}\left[Z^0_{x_0}(\bar{x})\right]\right) \le \frac{\kappa}{1-\lambda\kappa} d\left(Z^0_{x_0}(\bar{x}), P^0_{\bar{x}}(\bar{x})\right) = \frac{\kappa}{1-\lambda\kappa}\left\|\bar{y} - Z^0_{x_0}(\bar{x})\right\| \tag{28}$$

as $\bar{y} \in P^0_{\bar{x}}(\bar{x})$ by the definition of the set-valued mapping $P^t_x : X \rightrightarrows 2^Y$. Thus from (27) and (28), we obtain

$$d\left(\bar{x}, \Phi^0_{x_0}(\bar{x})\right) \le \frac{\kappa}{1-\lambda\kappa}\left\|\bar{y} - Z^0_{x_0}(\bar{x})\right\|. \tag{29}$$

From the definition of the set-valued mapping $Z^t_x : X \rightrightarrows 2^Y$ in (16), we get

$$\begin{aligned}
\left\|Z^0_{x_0}(x) - \bar{y}\right\| &= \left\|g_0(x-\bar{x}) - g_0(x-x_0) - \bar{y}\right\| \\
&\le \left\|g_0(x-\bar{x}) - g_0(x-x_0)\right\| + \left\|\bar{y}\right\| \\
&\le \lambda_0\left\|x_0 - \bar{x}\right\| + \left\|\bar{y}\right\| \\
&= \lambda\left\|x_0 - \bar{x}\right\| + \left\|\bar{y}\right\|
\end{aligned} \tag{30}$$

by using the definition of Lipschitz continuous mapping and the choice of λ. As $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x})$, which is a subset of $\mathbb{B}_{\delta}(\bar{x})$, then by the assumption $\frac{3}{2}\lambda\delta \le r_{\bar{y}}$ in 2) and by the assumption $\left\|\bar{y}\right\| < \frac{1}{2}\lambda\delta$ in 3), we write from (30) that

$$\left\|Z^0_{x_0}(x) - \bar{y}\right\| \le \frac{3}{2}\lambda\delta \le r_{\bar{y}}. \tag{31}$$

This shows that for every $x \in \mathbb{B}_{\delta}(\bar{x})$, $Z^0_{x_0}(x) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. More specifically,

$$\begin{aligned}
\left\|Z^0_{x_0}(\bar{x}) - \bar{y}\right\| &= \left\|-g_0(\bar{x}-x_0) - \bar{y}\right\| \\
&\le \left\|g_0(0) - g_0(\bar{x}-x_0)\right\| + \left\|\bar{y}\right\| \\
&\le \lambda_0\left\|x_0 - \bar{x}\right\| + \left\|\bar{y}\right\| \\
&= \lambda\left\|x_0 - \bar{x}\right\| + \left\|\bar{y}\right\| \\
&\le \frac{3}{2}\lambda\delta \le r_{\bar{y}}.
\end{aligned} \tag{32}$$

This implies that $Z^0_{x_0}(\bar{x}) \in \mathbb{B}_{r_{\bar{y}}}(\bar{y})$. By using (28) in (29), we obtain

$$d\left(\bar{x},\Phi_{x_0}^0\left(\bar{x}\right)\right)\le\frac{\kappa}{1-\lambda\kappa}\left(\left\|\bar{y}\right\|+\lambda\left\|x_0-\bar{x}\right\|\right). \tag{33}$$

By using (33) in (25) with $r=r_{x_0}$ and $\alpha=\frac{3}{8}$, we get

$$d\left(\bar{x},\Phi_{x_0}^0\left(\bar{x}\right)\right)\le\left(1-\frac{3}{8}\right)r_{x_0}=\left(1-\alpha\right)r.$$

It demonstrates that Lemma 3's assertion (5) is true. Now, we demonstrate that Lemma 3's assertion (6) is true. Suppose x',x'' belongs to $\mathbb{B}_{r_{x_0}}\left(\bar{x}\right)$. Therefore we get $x',x''\in\mathbb{B}_{r_{x_0}}\left(\bar{x}\right)\subseteq\mathbb{B}_{2\delta}\left(\bar{x}\right)$ by using (26) and by the first assumption $2\delta\le r_{\bar{x}}$ in 2), we can write $x',x''\in\mathbb{B}_{r_{x_0}}\left(\bar{x}\right)\subseteq\mathbb{B}_{2\delta}\left(\bar{x}\right)\subseteq\mathbb{B}_{r_{\bar{x}}}\left(\bar{x}\right)$. Now, from (31), $Z_{x_0}^0\left(x'\right),Z_{x_0}^0\left(x''\right)$ belongs to $\mathbb{B}_{r_{\bar{y}}}\left(\bar{y}\right)$. Therefore by using (17) and the definition of metric regularity, we can write

$$\begin{aligned}d\left(x',\Phi_{x_0}^0\left(x''\right)\right)&=d\left(x',P_{\bar{x}}^{0^{-1}}\left[Z_{x_0}^0\left(x''\right)\right]\right)\\&\le\frac{\kappa}{1-\lambda\kappa}d\left(Z_{x_0}^0\left(x''\right),P_{\bar{x}}^0\left(x'\right)\right)\\&=\frac{\kappa}{1-\lambda\kappa}d\left(Z_{x_0}^0\left(x''\right),Z_{x_0}^0\left(x'\right)\right)\\&=\frac{\kappa}{1-\lambda\kappa}\left\|Z_{x_0}^0\left(x'\right)-Z_{x_0}^0\left(x''\right)\right\|.\end{aligned} \tag{34}$$

Now, by using (18) in (34) and the choice of λ in (13), we observe that

$$\begin{aligned}&d\left(x',\Phi_{x_0}^0\left(x''\right)\right)\\&\le\frac{\kappa}{1-\lambda\kappa}\left(\left\|g_0\left(x'-\bar{x}\right)-g_0\left(x''-\bar{x}\right)\right\|+\left\|g_0\left(x'-x_0\right)-g_0\left(x''-x_0\right)\right\|\right)\\&\le\frac{2\lambda_0\kappa}{1-\lambda\kappa}\left\|x'-x''\right\|\le\frac{2\lambda\kappa}{1-\lambda\kappa}\left\|x'-x''\right\|.\end{aligned} \tag{35}$$

Since $\lambda\kappa<\frac{3}{19}$ by (13), so put $\lambda\kappa=\frac{3}{19}$ in (35), we get

$$d\left(x',\Phi_{x_0}^0\left(x''\right)\right)\le\frac{3}{8}\left\|x'-x''\right\|=\alpha\left\|x'-x''\right\|.$$

It follows that condition (6) of Lemma 3 is also valid. Since conditions (5) and (6) of Lemma 3 are true, we may infer that there is a fixed point $\hat{x}_1$ belongs to $\mathbb{B}_{r_{x_0}}\left(\bar{x}\right)$ so that $\hat{x}_1$ belongs to $\Phi_{x_0}^0\left(\hat{x}_1\right)$, that translates to $Z_{x_0}^0\left(\hat{x}_1\right)$ belongs to $P_{\bar{x}}^0\left(\hat{x}_1\right)$, *i.e.* 0 belongs to $g_0\left(\hat{x}_1-x_0\right)+\mathcal{T}\left(\hat{x}_1\right)$, and so $\mathbb{D}^0\left(x_0\right)$ not equal to empty.

We now demonstrate that (24) is true for $t=0$. Keep in mind that $\bar{r}$ is greater than 0 by (12). Therefore, (12) is true for (13). As the set-valued mapping $P_{\bar{x}}^{t^{-1}}\left(\cdot\right)$ is Lipschitz-like at $\left(\bar{y},\bar{x}\right)$ on $\mathbb{B}_{r_{\bar{y}}}\left(\bar{y}\right)$ relative to $\mathbb{B}_{r_{\bar{x}}}\left(\bar{x}\right)$, the Lemma 4 states that with constant $\frac{\kappa}{1-2\lambda\kappa}$, the mapping $P_{x}^{t^{-1}}\left(\cdot\right)$ is Lipschitz-like at $\left(\bar{y},\bar{x}\right)$ on $\mathbb{B}_{\bar{r}}\left(\bar{y}\right)$ relative to $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}\left(\bar{x}\right)$ for every x belongs to $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}\left(\bar{x}\right)$.

Particularly, by the choice of $\hat{\delta}$ and the assumption $\delta \le \frac{r_{\bar{x}}}{2}$ in 2) we can say that the mapping $P_{x_0}^{0\ -1}(\cdot)$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $\frac{\kappa}{1-2\lambda\kappa}$ as $x_0 \in \mathbb{B}_{\hat{\delta}}(\bar{x}) \subseteq \mathbb{B}_{\delta}(\bar{x}) \subseteq \mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$. Additionally, by the assumption $\|\bar{y}\| < \frac{1}{2}\lambda\delta$ and the assumption $\delta \le \frac{\bar{r}}{\lambda}$ in 2), we obtain that

$$\|\bar{y}\| < \frac{1}{2}\lambda\delta \le \bar{r}. \tag{36}$$

It shows that 0 belongs to $\mathbb{B}_{\bar{r}}(\bar{y})$. Consequently, according to Lemma 1, with constant $\frac{\kappa}{1-2\lambda\kappa}$, the mapping $P_{x_0}^{0}(\cdot)$ is metrically regular at $(\bar{x},\bar{y})$ on $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x}) \times \mathbb{B}_{\bar{r}}(\bar{y})$. As a result, by using Lemma 1, we obtain

$$d\left(x_0, P_{x_0}^{0\ -1}(0)\right) \le \frac{\kappa}{1-2\lambda\kappa} d\left(0, P_{x_0}^{0\ -1}(x_0)\right). \tag{37}$$

Equation (22) implies that this is the case

$$\begin{aligned} d\left(x_0, P_{x_0}^{0\ -1}(0)\right) &\le \frac{\kappa}{1-2\lambda\kappa} d\left(0, P_{x_0}^{0}(x_0)\right) \\ &= \frac{\lambda\kappa}{1-2\lambda\kappa} d\left(0, \mathcal{T}(x_0)\right) \\ &= \frac{\lambda\kappa}{1-2\lambda\kappa}\delta. \end{aligned} \tag{38}$$

This leads us to the conclusion that

$$d\left(0, \mathbb{D}^0(x_0)\right) = d\left(x_0, P_{x_0}^{0\ -1}(0)\right) \le \frac{\lambda\kappa}{1-2\lambda\kappa}\delta, \tag{39}$$

which we get from (15) and use in (38). We discover that

$$\|x_1 - x_0\| = \|l_0\| \le \eta d\left(0, \mathbb{D}^0(x_0)\right) \le \frac{\eta\lambda\kappa}{1-2\lambda\kappa}\delta = h\delta \tag{40}$$

from Algorithm 2 and by using (20) and (39). It follows from this that (24) is valid for $t = 0$. Assume that points $x_1, x_2, \cdots, x_n$ have been found and that (23), (24) are valid for $t = 0,1,2,\cdots,n-1$. We shall demonstrate that there is a location x_{n+1} where (23), (24) are satisfied for $t = n$ as well. The assumptions (23) and (24) are true for every $t \le n-1$, hence we obtain the relation below:

$$\|x_n - \bar{x}\| \le \sum_{i=0}^{n-1}\|l_i\| + \|x_0 - \bar{x}\| \le \delta\sum_{i=0}^{n-1} h^{i+1} + \delta \le \frac{\delta h}{1-h} + \delta \le 2\delta. \tag{41}$$

It follows that (23) is valid for $t = n$. Now, using a nearly identical argument to the one we used to prove that $t = 0$ applies, we can discover that the mapping $P_{x_n}^{n\ -1}(\cdot)$ is Lipschitz-like at $(\bar{y},\bar{x})$ on $\mathbb{B}_{\bar{r}}(\bar{y})$ relative to $\mathbb{B}_{\frac{r_{\bar{x}}}{2}}(\bar{x})$ with constant $\frac{\kappa}{1-2\lambda\kappa}$. Then, by using Algorithm 2 once more, we obtain

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \|l_n\| \le \eta d\left(0,\mathbb{D}^n(x_n)\right) \le \eta d\left(x_n, P_{x_n}^{n^{-1}}(0)\right) \\ &\le \frac{\eta\kappa}{1-2\lambda\kappa} d\left(0,P_{x_n}^n(x_n)\right) \le \frac{\eta\kappa}{1-2\lambda\kappa} d\left(0,\mathcal{T}(x_n)\right) \\ &\le \frac{\eta\kappa}{1-2\lambda\kappa} d\left(0,-g_{n-1}(x_n-x_{n-1})\right) \\ &\le \frac{\eta\kappa}{1-2\lambda\kappa} \|g_{n-1}(0)-g_{n-1}(x_n-x_{n-1})\| \\ &\le \frac{\eta\kappa\lambda_{n-1}}{1-2\lambda\kappa}\|x_n-x_{n-1}\| \le \frac{\eta\kappa\lambda}{1-2\lambda\kappa}\|x_n-x_{n-1}\| \\ &\le \frac{\eta\kappa\lambda}{1-2\lambda\kappa} h^n\delta \le h^{n+1}\delta.\end{aligned} \tag{42}$$

It demonstrates that (24) is true for $t=n$. As a result, Theorem 1's proof is finished.

Consider that

$\lim_{x\to\bar{x}} d\left(0,\mathcal{T}(x)\right)$ equal to 0.

In the specific scenario where $\bar{x}$ is the outcome of (1), *i.e.* $\bar{y}=0$, the following corollary of Theorem 1 is created and it provides the analysis of local convergence of the sequence produced by the modified GG-PPA specified by Algorithm 2.

Corollary 1. Consider that η is greater than 1, $\mathcal{T}$ is metrically regular at $(\bar{x},0)$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_0}(0)$ with constant κ and $\{g_t\}$ is a sequence of Lipschitz continuous function with Lipschitz constants λ_t where $g_t(0)=0$. Assume that $\bar{x}$ verifies $0\in\mathcal{T}(\bar{x})$. Consider that $\tilde{r}>0$ is such that $\mathbb{B}_{2\tilde{r}}(\bar{x})$ is a subset of the neighbourhood of origin. Then, every sequence $\{x_t\}$ produced by Algorithm 2 with an starting point in $\mathbb{B}_{\hat{\delta}}(\bar{x})$ converges to a result x^* of (1), meaning x^* satisfies that $0\in\mathcal{T}(x^*)$. This holds for some $\hat{\delta}$ is greater than 0.

Proof. According to the statement of the corollary, $\mathcal{T}$ is metrically regular at $(\bar{x},0)$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_0}(0)$ with constant κ and $\{g_t\}$ is a sequence of Lipschitz continuous function with Lipschitz constants λ_t which are away from zero. Thus, we get $\lambda\kappa<1$ by (13).

Then by applying Lyusternik-Graves theorem on (14), we have the set-valued mapping $P_{\bar{x}}^t$ is metrically regular at $(\bar{x},0)$ on $\mathbb{B}_{r_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_0}(0)$ with constant $\frac{\kappa}{1-\lambda\kappa}$.

There are positive constants $\frac{\kappa}{1-\lambda\kappa}$, r_0 and $\hat{r}(\bar{x})$ so that with constant $\frac{\kappa}{1-\lambda\kappa}$, the set-valued mapping $P_{\bar{x}}^t(\cdot)$ is metrically regular at $(\bar{x},0)$ on $\mathbb{B}_{\hat{r}_{\bar{x}}}(\bar{x})$ relative to $\mathbb{B}_{r_0}(0)$. Then for each $0<r\le\hat{r}_{\bar{x}}$ with for every $x\in\mathbb{B}_{r_{\bar{x}}}(\bar{x})$, $y\in\mathbb{B}_{r_0}(0)$,

$$d\left(x,P_{\bar{x}}^{t^{-1}}(y)\right) \le \frac{\kappa}{1-\lambda\kappa} d\left(y,P_{\bar{x}}^t(x)\right).$$

Suppose $\sup_t \lambda_t = \lambda \in (0,1)$ is set up so that $\lambda\kappa$ is less than or equal to $\frac{1}{\eta+2}$. To ensure that $\frac{r_{\bar{x}}}{2}$ is less than or equal to $\tilde{r}$ and $r_0 - \frac{\lambda r_{\bar{x}}}{2}$ is greater than 0, use $r_{\bar{x}}$ belongs to $(0, \hat{r}_{\bar{x}})$. Therefore,

$$\bar{r} = \min\left\{\frac{2r_0 - r_{\bar{x}}\lambda}{2}, \frac{r_{\bar{x}}(1-2\kappa\lambda)}{4\kappa}\right\} > 0,$$

and

$$\min\left\{\frac{r_{\bar{x}}}{2}, \frac{\bar{r}}{\lambda}, \frac{2r_0}{3\lambda}\right\} > 0.$$

As a result, we can select $0 < \delta \le 1$ such that

$$\delta \le \min\left\{\frac{r_{\bar{x}}}{2}, \frac{\bar{r}}{\lambda}, \frac{2r_0}{3\lambda}\right\}.$$

Checking the validity of inequalities 1) through 3) from Theorem 1 is now routine. As a result, Theorem 1 can be used to support the corollary 1 and complete the proof.

4. Numerical Experiment

Within this part, we'll give a numerical illustration to support the modified GG-PPA's semi-local convergence finding.

Example 1. In the case $X = Y = \mathbb{R}$, $\eta = 2$, $x_0 = -0.2$, $\kappa = 0.4$ and $\lambda = 0.3$, consider a mapping $\mathcal{T}$ with set values on $\mathbb{R}$ by $\mathcal{T}(x) = \{-8x+1, 2x^3-3\}$. Think about g_n a sequence of continuous Lipschitz functions, where $g_n(0) = 0$ and $g_n(x) = \frac{1}{2}x$. Finally, Algorithm 2 produces a sequence that converges to the value x^* equal to 0.125.

Solution: The claim that at $(-0.2, 0.6) \in gph\mathcal{T}$, a metrically regular mapping $\mathcal{T}$ and a continuous Lipschitz function g_n in the vicinity of the solution with a Lipschitz constant $\lambda = \sup_t \lambda_t = 0.3$ is clear. Think about $\mathcal{T}(x) = -8x+1$. Therefore, based on (3), we obtain that

$$\mathbb{D}^t(x_t) = \{l_t \in \mathbb{R} : 0 \in g_t(l_t) + \mathcal{T}(x_t + l_t)\} = \left\{l_t \in \mathbb{R} : l_t = \frac{2-16x_t}{15}\right\}.$$

However, if $\mathbb{D}^t(x_t)$ not equal to empty is true, we see that

$$0 \in g_t(x_{t+1} - x_t) + \mathcal{T}(x_{t+1}),$$

$$\Rightarrow x_{t+1} = \frac{2-x_t}{15}.$$

As a result, from (42) we deduce that

$$\|l_t\| \le \frac{\eta\kappa\lambda}{1-2\lambda\kappa}\|l_{t-1}\|.$$

We can see that $\frac{\eta\kappa\lambda}{1-2\kappa\lambda} = \frac{0.24}{0.76} = 0.316(\text{approx.}) < 1$ for the specified values

Table 1. Finding a solution of generalized equation.

x	$\mathcal{T}(x)$
−0.2000	0.6000
0.1467	−0.1733
0.1236	0.0116
0.1251	−0.0008
0.1250	0.0001
0.1250	−0.0000
0.1250	0.0000

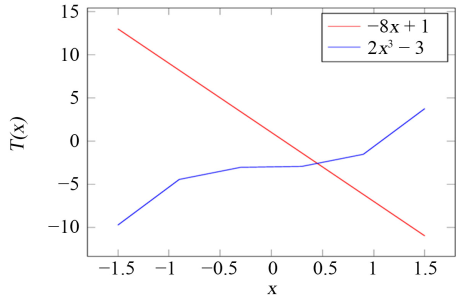

Figure 1. The graph of $\mathcal{T}(x)$.

of κ, η, λ. It follows that the sequence produced by Algorithm 2 converges linearly in this case. Table 1, produced by the Matlab program, shows that the generalized equation's solution is 0.125 for n equal to 6.

Figure 1 is the graphical representation of $\mathcal{T}(x)$.

5. Concluding Remarks

For the purpose of resolving the generalized equation $0 \in \mathcal{T}(x)$, we have established the results of local and semi-local convergence for the modified GG-PPA by taking into account the assumptions η is greater than 1, $\mathcal{T}$ is metrically regular and $g_t : X \to Y$ is a sequence of Lipschitz continuous functions with $g_t(0) = 0$ in the neighborhood of origin. A numerical experiment has also been provided to verify the semi-local convergence result of the modified GG-PPA. This result strengthens and broadens the one found in [7] [9].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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