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Atomization Theorems in Mathematical Physics and General Relativity

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Abstract

Formulated Atomization Theorems extend the theory of Atomic AString Functions evolving since the 1970s allowing representation of polynomials, complex analytic functions, and solutions of linear and nonlinear differential equations via Atomic Series over smooth finite Atomic Splines. Noting the preservation of analyticity for Ricci and Einstein tensors, special new theorems are formulated for General Relativity representing spacetime field via superpositions of flexible finite "solitonic atoms" resembling quanta. The novel Atomic Spacetime model correlates with A. Einstein's 1933 paper predicting a new "atomic theory". The theorems can be applied to many theories of mathematical physics, elasticity, hydrodynamics, soliton, and field theories for unified representation of fields via series over finite Atomic AString Functions which may offer a unified theory under research where fields are connected with a common mathematical ancestor.

Keywords

Atomic Function, AString, Splines, Series, Spacetime, General Relativity

1. Introduction—A Brief History of Atomic AString Functions

Theory of Atomic Functions (AF) [1]-[12] has been evolving since 1967-1971 when V. L. Rvachev[1] and V. A. Rvachev [6] discovered and researched a pulse function $up(x)$ for which derivative pulses would conveniently be similar to the original pulse shifted and stretched by the factor of 2:

$$up'(x) = 2up(2x+1) - 2up(2x-1). \tag{1.1}$$

[1]Vladimir Logvinovich Rvachev (1926-2005), https://en.wikipedia.org/wiki/Vladimir_Rvachev, Academician of National Academy of Sciences of Ukraine, author of 600 papers, 18 books, mentor of 80 PhDs, 20 Doctors and Professors including the author.

These and similar atomic functions possess unique properties of infinite differentiability, smoothness, nonlinearity, nonanalyticity, finiteness, and compact support like splines. The most significant is that other functions like polynomials, sinusoids, exponents, and other analytic functions can be represented via a converging series of shifts and stretches of AFs. So, like from "mathematical atoms" [6]-[12], smooth functions and solutions of differential equations of mathematical physics [1]-[44] can be composed via AF superpositions, and due to that those "atoms" have been called Atomic Functions in the 1970s. They are quite similar to widely-used splines but unlike classical polynomial splines [23] [24] are infinite differentiable having derivates (1.1) expressed via themselves and are sometimes called Atomic Splines [1] [6]-[12].

As per a survey [10], while some elements, analogs, or Fourier transformations of AFs sometimes named differently (Fabius function [34], hat function, compactly supported smooth function) have been known since the 1930s, the rigorous theory development supported by many books, dissertations, lecture courses and hundreds of papers observed in [1]-[12] [21] [22] [23] has started in the 1970s. The foundation of AF theory has been developed by V. L. Rvachev and V. A. Rvachev [6] [7] [8] [25] and enriched by many followers, notably by schools of V. F. Kravchenko [9] [10] [11] [12], B. Gotovac, H. Gotovac [26] [33], and the author [1] [2] [3] [4] [5] [21] [22] [23] [43] [44]. In 2017, the author noted [2] [3] that AF $up(x)$ (1.1) is a composite object consisting of two kink functions called AStrings [1] [2] [3] [4] [5] making them more generic:

$$up(x) = AString(2x+1) - AString(2x-1) = AString'(x). \tag{1.2}$$

Moreover, AString is not only a "composing branch" but also an integral of $up(x)$. AString derived from the theory of Atomic Functions is related [2] [4] [44] to the Fabius function [34] also known since the 1970s.

Mutual relationships (1.1), (1.2) imply that theorems and many theories [1]-[12] involving AFs can be reformulated via AStrings, and often they simplify the mathematical representations and introduce novel physical models [1] [2] [3] [4] [5] [23] [43] [44]. Composing AF pulse (1.1) via kink-antikink pair (1.2) of nonlinear AStrings resembles "solitonic atoms" (or bions) from the theory of soliton dislocations [5] [29] [30]. This led to the introduction of Atomic Solitons [3] [5] where AString (1.2) becomes a solitonic kink while $up(x)$ is a "solitonic atom" made of AStrings. The ability of AFs to compose polynomials and analytic functions leads to novel interpretations of spacetime and fields as superpositions of Atomic Solitons [1] [2] [3] [4] [5].

AString possesses another important property of composing/partitioning a line and curves from a superposition of AStrings resembling the ideas of atomic spacetime quantization first published in 2018 [3] as an "intuition theory". This paper provides the mathematical formalism of that theory in the form of *Atomization Theorems* stating how polynomials, analytic functions, solutions of differential equations, and finally General Relativity (GR) equations can be represented via series over Atomic and AString Functions called *Atomic Series* [1]. They lead to the concept of AString spacetime quantum/metriant [1] [2] [3] [4] [5] and "atomization of spacetime" where "atoms" are assumed to be not conventional physical atoms but "mathematical atoms" rooted in the theory of finite Atomic Functions. Interestingly, support for the novel theory comes from A. Einstein's 1933 paper [17] where he envisaged an "… atomic theory with mathematically simplest concepts and the link between them" to solve some "stumbling blocks" of continuous field theories to describe quantized fields with finite "regions of space" with "discrete energies" [17] [23] indirectly pointing to finite functions like Atomic Functions [1]-[12].

The Atomization Theorems are not limited to spacetime and can be applied to many physical theories including Quantum Mechanics, electromagnetism, elasticity, heat conductivity, soliton, and field theories [16] [18]-[24] [35]-[50] dealing with a distribution of fields in spacetime. A unified representation of fields via Atomic Series over finite Atomic and AString Functions may offer a unified theory under research now [1] [2] [3] [4] [5] [43] where, like in string theory, fields become interconnected having a common mathematical ancestor.

The paper includes a brief history and description of Atomic and AString Functions, 13 theorems with proof including new theorems for General Relativity leading to the Atomic Spacetime model, and a discussion about further research directions of a unified spacetime and field theory based on Atomic AString Functions.

2. Introducing Simple AString Metriant Function

Let's consider the problem of composing a straight x and curved $\tilde{x}(x)$ spaceline via superpositions over some finite *metriant functions* [1] $m(x), x \in [-1,1]$:

$$x = \sum_{k=-\infty}^{\infty} am((x-ka)/a); \quad \tilde{x}(x) = \sum_{k=-\infty}^{\infty} c_k m((x-b_k)/a_k) \tag{2.1}$$

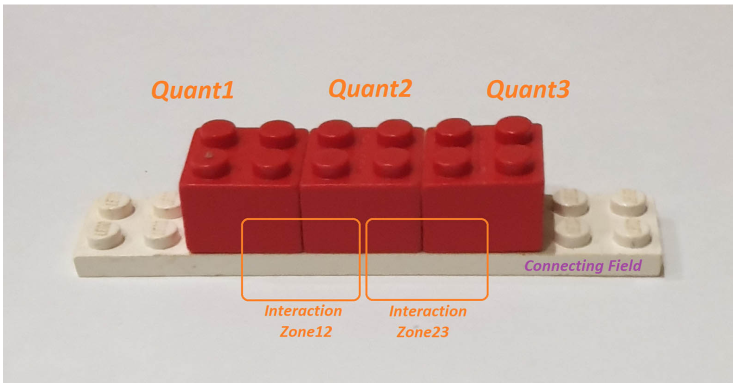
composing a spaceline from "elementary pieces" set at regular points ka resembling finite quanta of width $2a$ (**Figure 1**). We seek spaceline x to appear not only as a Lego-like translation (2.1) but also in "interaction zones" between quanta ($a = 1$) (**Figure 1(a)** and **Figure 1(b)**):

$$x \equiv \cdots + m(x-1) + m(x) + m(x+1) + \cdots;$$
$$x \equiv m\left(x - \frac{1}{2}\right) + m\left(x + \frac{1}{2}\right), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{2.2}$$

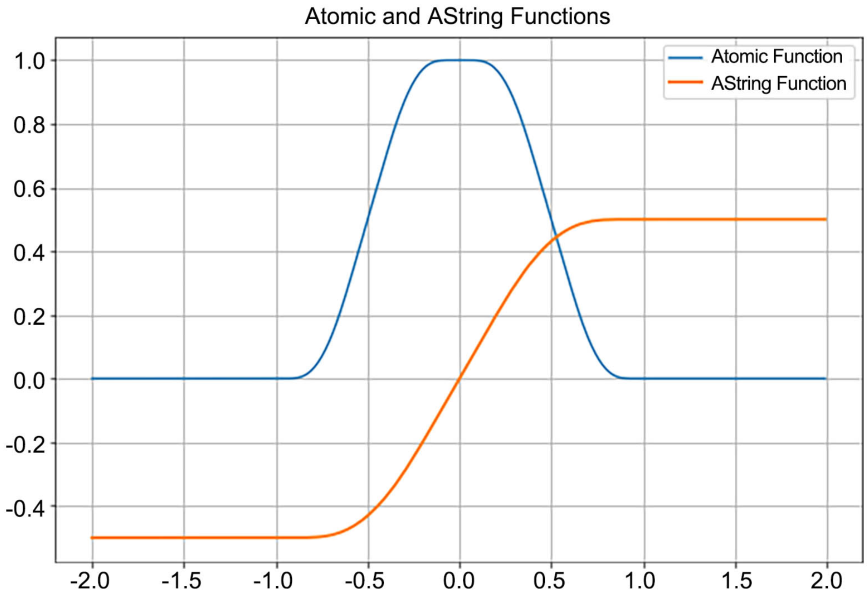
Reformulated for derivatives $p(x) = m'(x)$, the problem leads to a "partition of unity" [2]-[7] to represent a constant via a series of finite pulses:

$$1 \equiv \cdots + p(x-1) + p(x) + p(x+1) + \cdots;$$
$$1 \equiv p\left(x - \frac{1}{2}\right) + p\left(x + \frac{1}{2}\right), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]. \tag{2.3}$$

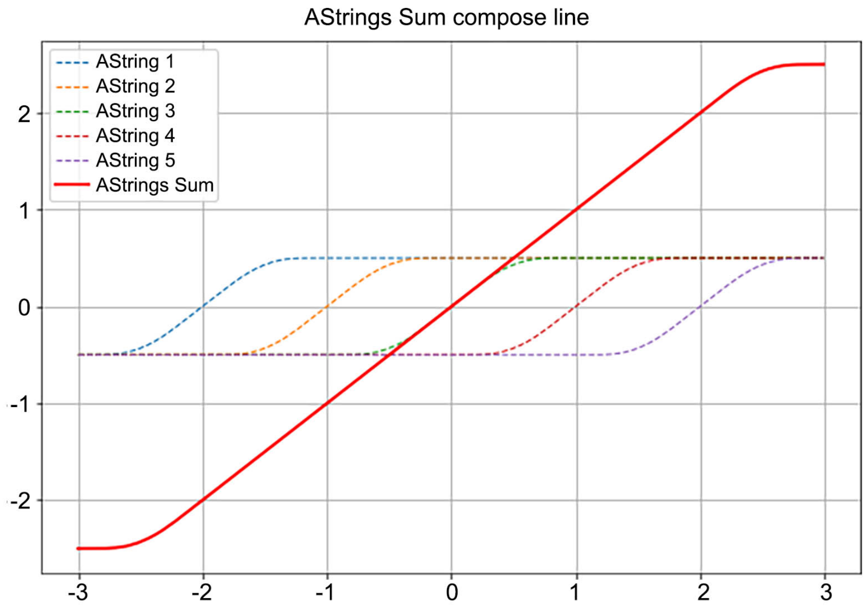
It can be achieved with polynomial splines but it leads to a "polynomial trap" problem [24] imposing artificial polynomial order on spacetime models and not being able to compose a smooth curve $\tilde{x}(x)$ of any polynomial order. Instead,


Quant1
Quant2
Quant3
Connecting Field
Interaction Zone12
Interaction Zone23

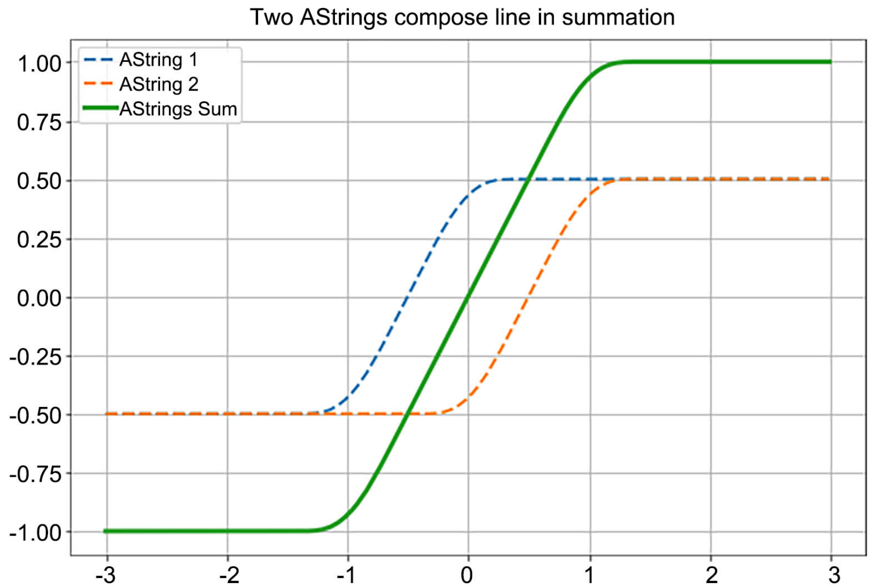
(a)


Atomic and AString Functions
Atomic Function
AString Function

(b)


AStrings Sum compose line
AString 1
AString 2
AString 3
AString 4
AString 5
AStrings Sum

(c)



(d)

Figure 1. (a) Lego model with interaction zones; (b) Desired metriant function and its derivative; (c) Expansion of space by the sum of metriant functions; (d) Emergence of line $y = x$ by summing two metriant functions in "interaction zone".

seeking a solution amongst finite functions for which derivatives are expressed via the functions themselves

$$p'(x) = f(p(x)) = cp(ax+b) + dp(ax-b) \tag{2.4}$$

yields so-called atomic function (AF) $up(x)$ [1]-[12] discovered in the 1970s.

The desired metriant function $m(x)$ would be the integral of $up(x)$ called *AString* in 2017 [2] [3]:

$$p(x) = up(x),\ m(x) = \int_0^x up(x)\mathrm{d}x = AString(x),\ x \equiv \sum_k AString(x-k). \tag{2.5}$$

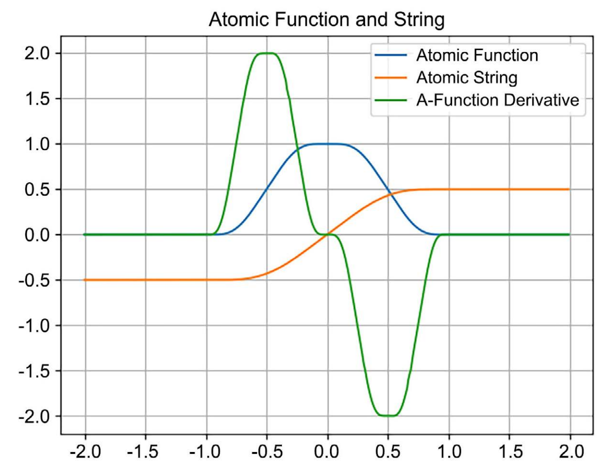
AString shaped as a kink (**Figure 1**) can compose both straight and curved lines from "elementary pieces" resembling quanta and leading to novel Atomic Spacetime models [1]-[5] described later. While AString was obtained in 2017 from spacetime and AF theories, it is a generic function that can also be used in many theories including Atomic Machine Learning [44].

3. Atomic and AString Functions

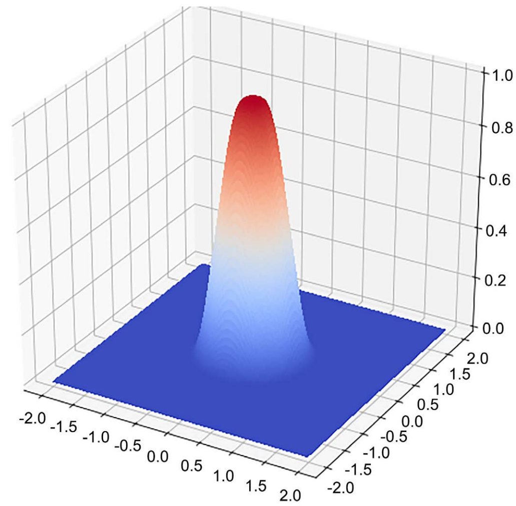
Let's describe Atomic [1]-[12] and AString [1] [2] [3] [4] [5] Functions in more detail.

3.1. Atomic Function

Atomic Function (AF) (V. L. Rvachev, V. A. Rvachev, 1971, [6]) $up(x)$ is a finite compactly supported non-analytic infinitely differentiable pulse function (**Figure 2**) with the first derivative expressible via the function itself shifted and stretched by the factor of 2:



(a)



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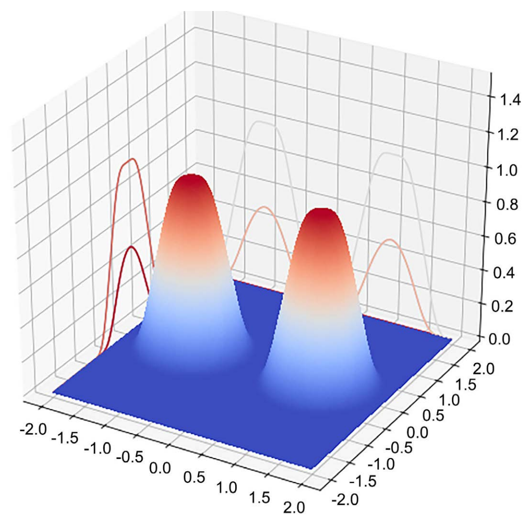


(c)

Figure 2. (a) Atomic function pulse with its derivative and integral (AString) (b) Atomic Function pulse ("solitonic atom") in 2D (c) Two Atomic Function pulses ("solitonic atoms" or "atomic solitons").

$$up'(x) = 2up(2x+1) - 2up(2x-1) \text{ for } |x| \le 1, \quad up(x) = 0 \text{ for } |x| > 1. \tag{3.1}$$

With exact Fourier series representation [1] [2] [3] [4] [5] [7]-[12]

$$up(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx} \prod_{k=1}^{\infty} \frac{\sin(t2^{-k})}{t2^{-k}} dt, \quad \int_{-1}^{1} up(x)dx = 1, \tag{3.2}$$

the values $up(x)$ can be calculated with computer scripts [2] [4] [9] [10] [11] [12] [43] [44].

Higher derivatives $up^{(n)}$ and integrals I_m can also be expressed via $up(x)$ [6]-[12] [25] [26]

$$up^{(n)}(x) = 2^{\frac{n(n+1)}{2}} \sum_{k=1}^{2^n} \delta_k up(2^n x + 2^n + 1 - 2k), \delta_{2k} = -\delta_k, \delta_{2k-1} = \delta_k, \delta_1 = 1;$$

$$I_m(x) = 2^{C_m^2} up(2^{-m} x - 1 + 2^{-m}), x \le 1;$$

$$I_m(x) = 2^{C_m^2} up(2^{-m+1} - 1) + \frac{(x-1)^{m-1}}{(m-1)!}, x > 1;$$

$$I_1(x) = up(2^{-1} x - 2^{-1}); \quad I_1'(x) = up(x). \tag{3.3}$$

AF satisfies *partition of unity* [1]-[12] to exactly represent the number 1 by summing up individual overlapping pulses set at regular points… −2, −1, 0, 1, 2… (**Figure 3(a)**):

$$\cdots + up(x-2) + up(x-1) + up(x) + up(x+1) + up(x+2) + \cdots \equiv 1. \tag{3.4}$$

This property is related to the following double symmetry [1]-[12]:

$$up(x) = up(-x), x \in [-1,1]; \quad up(x) + up(1-x) = 1, x \in [0,1]. \tag{3.5}$$

Generic AF pulse of width 2*a*, height *c*, and center positions *b*, *d* has the form

$$up(x,a,b,c,d=0) = d + c * up((x-b)/a), \quad \int_{-a}^{a} cup(x/a)dx = ca. \tag{3.6}$$

Multi-dimensional atomic functions [2]-[8] [24] [27] (**Figure 3** and **Figure 4**) can be constructed as either multiplications or radial atomic functions:

$$up(x,y,z) = up(x)up(y)up(z),$$

$$up(r) = up\left(\sqrt{x^2+y^2+z^2}\right), \quad \iiint cup\left(\frac{x}{a},\frac{y}{a},\frac{z}{a}\right) dxdydz = ca^3. \tag{3.7}$$

3.2. AString Function

AString function (**Figure 4**) (Eremenko, [2] [3], 2018) was proposed as both an integral (3.3) and "composing branch" of $up(x)$ (§2):

$$AString'(x) = AString(2x+1) - AString(2x-1) = up(x). \tag{3.8}$$

While AString was derived from the theory of Atomic Functions, a similar function was introduced by Fabius in 1966; Fabius function [34] is specially shifted and stretched AString [3] [44].

AString has a form of a solitary kink (**Figure 4(a)**) which can compose a straight line $y = x$ both between and as a translation of AString kinks (**Figure 4(c)**) leading to spacetime "atomization"/quantization ideas (§2, 6):

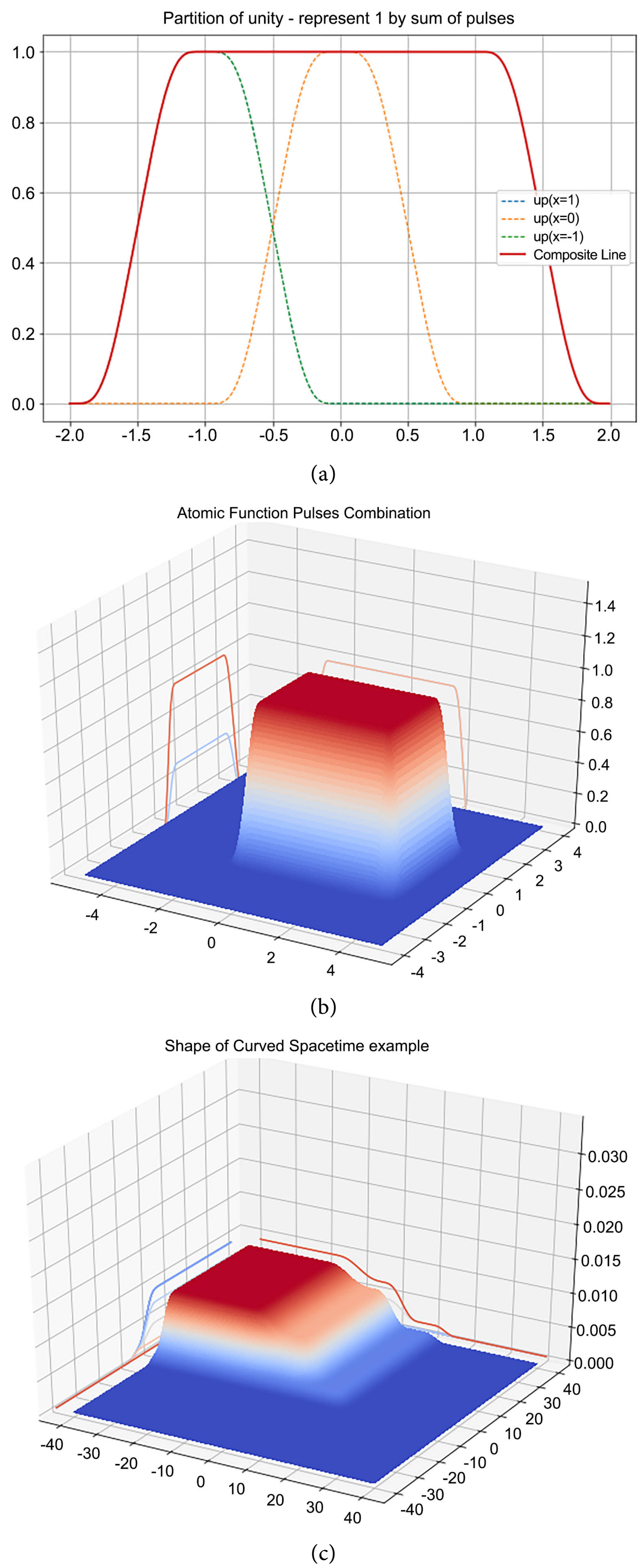
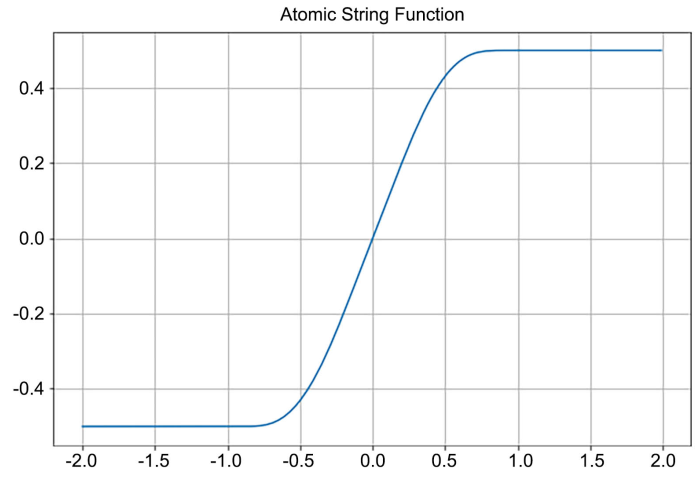
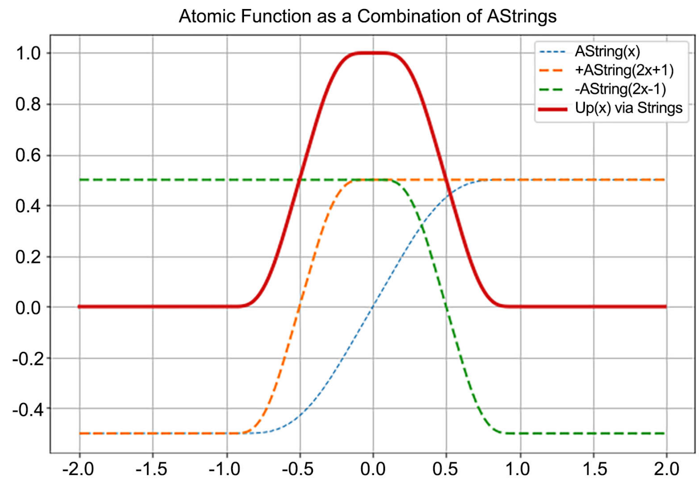


Figure 3. (a) Partition of unity with atomic functions; (b) Representation of flat surface via summation of Afs; (c) Curved surface as a superposition of "solitonic atoms".



(a)



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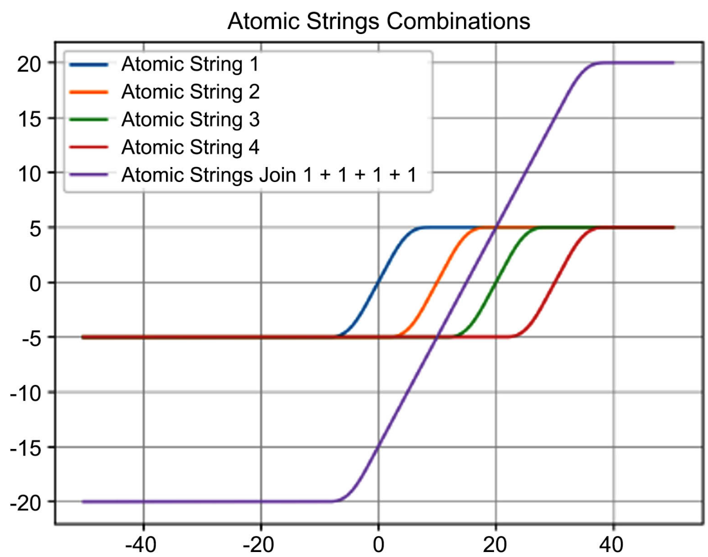


(c)

Figure 4. (a) Atomic string function (AString); (b) Atomic function as a combination of two AStrings; (c) Representation of a straight line segment by summing of AStrings.

$$x \equiv AString\left(x-\frac{1}{2}\right)+AString\left(x+\frac{1}{2}\right),\ x\in\left[-\frac{1}{2},\frac{1}{2}\right];$$

$$\begin{aligned} x \equiv & \cdots + AString(x-2)+AString(x-1)+AString(x) \\ & + AString(x+1)+AString(x+2)+\cdots \end{aligned} \tag{3.9}$$

The Elementary AString kink function can be generalized in the form

$$AString(x,a,b,c,d=0)=d+c*AString((x-b)/a). \tag{3.10}$$

Importantly, the Atomic Function pulse (3.6) can be presented as a sum of two opposite AString kinks (**Figure 4(b)**) making AStrings and AFs deeply related to each other:

$$up(x,a,b,c)=AString\left(x,\frac{a}{2},b-\frac{a}{2},c\right)+AString\left(x,\frac{a}{2},b+\frac{a}{2},-c\right). \tag{3.11}$$

3.3. Atomic Solitons

Being solutions of special kinds of nonlinear differential equations with shifted arguments (3.1), (3.8), AStrings and Atomic Functions possess some mathematical properties of lattice solitons [29] [30] [31] [32] and have been called *Atomic Solitons* [2] [3] [4] [5]. AString is a solitonic kink whose particle-like properties exhibit themselves in the composition of a line (3.9) and kink-antikink "atoms" (3.8) (**Figure 4**). Being a composite object (3.8) made of two AStrings, AF $up(x)$ is not a true soliton but rather a *solitonic atom*, like "bions" or "dislocation atoms" [2] [4] [29] [30], as described in [2] [4].

4. Atomic Series, Atomic Splines, and "Mathematical Atoms"

Atomic and AString Functions (*Atomics, or Atomic Splines*) possess unique approximation properties described later in §5, 6. Like from "mathematical atoms" [6]-[12], as founders called them, flat and curved smoothed surfaces/functions (**Figure 3**) can be composed of a superposition of Atomics via the so-called Generalized Taylor's Series [7] [8] [9] [25] [26] (or simply, *Atomic Series* [1]) with an *exact* representation of polynomials of any order

$$\frac{1}{4}\sum_{k=-\infty}^{k=+\infty} k\,up\left(x-\frac{k}{2}\right)\equiv\sum_{k=-\infty}^{k=+\infty} AString(x-k)\equiv x;$$

$$\sum_{k=-\infty}^{k=+\infty}\left(\frac{k^2}{64}-\frac{1}{36}\right)up\left(x-\frac{k}{4}\right)\equiv x^2,$$

$$\begin{aligned} x^n & \equiv \sum_{k=-\infty}^{k=+\infty} C_k\, up\left(x-k2^{-n}\right) \\ & = \sum_{k=-\infty}^{k=+\infty} C_k\left(AString\left(2\left(x-k2^{-n}\right)+1\right)-AString\left(2\left(x-k2^{-n}\right)-1\right)\right). \end{aligned} \tag{4.1}$$

Importantly, despite infinite sums in (4.1) only a limited number of neighboring finite "atoms" are required to calculate a polynomial value at a given point.

Polynomial representations (4.1) mean that Atomics can also represent/atomize any *analytic function* [29] (a function representable by converging Taylor's series via polynomials) with calculable coefficients:

$$y(x)=\sum_{m=0}^{\infty}\frac{y^{(m)}(0)}{m!}x^{m}=\sum_{m=0}^{\infty}B_{m}x^{m}=\sum_{m=0}^{\infty}\sum_{k=-\infty}^{k=+\infty}B_{m}C_{k}up\left(x-k2^{-m}\right)$$
$$=\sum_{mk=-\infty}^{\infty}c_{mk}up\left(\frac{x-b_{mk}}{a_{mk}}\right)=\sum_{l=-\infty}^{l=+\infty}AString\left(x,a_{l},b_{l},c_{l}\right). \quad (4.2)$$

Analytic functions [29] represent a wide range of polynomial, trigonometric, exponential, hyperbolic, and other functions, their sums, derivatives, integrals, reciprocals, multiplications, and superpositions. Therefore, they all can be "atomized" via superpositions of Atomic and AString Functions with a predefined degree of precision, which is the most important property.

Instead of sums (4.1), and (4.2), we will be using short notation with localized basis atomic functions $A_k(x)$ and function values y^k at node k assuming summation over repeated indices k:

$$y(x)=A_{k}(x)y^{k};\quad f(x,y,z)=A_{k}(x,y,z)f^{k}. \quad (4.3)$$

5. Atomization Theorems

Being compactly supported solitonic spline-like functions, Atomic and AString Functions (Atomics, or Atomic Splines) possess unique properties. Starting from known theorems [6]-[12] extended here for recently introduced AStrings, the formulated *Atomization Theorems* [1] [23] provide a mathematical formalism for the representation of polynomials, elementary and complex analytic functions, and solutions of linear and nonlinear differential equations including General Relativity via Atomic Series over Atomic Splines.

5.1. Polynomial Atomization Theorem

AStrings and Atomic Functions are *non-analytic* [29]—cannot be represented by polynomials via converging Taylor's series [1]-[12] [29]. But, interestingly, the opposite is true—polynomials can be exactly represented by Atomics leading to the following most important theorem proven in the 1970s [7] [8] [9] and generalized here for AStrings.

Theorem 1 (*Polynomial atomization theorem*). Polynomials of any order can be exactly represented via a series of Atomic and AString Functions.

Proof. Following [7] [8] [25] [26] and taking n-derivative of a polynomial x^n leads to a constant: $\left(x^n\right)^{(n)}=c$ which, due to the partition of unity (3.4) and (4.1) can be represented via a sum of shifted pulses $up(x)$. Integrating I_n that sum n-times and noting (3.3) that integrals of $up(x)$ are expressed via the function itself and polynomials lead to the same polynomial x^n, now expressed via the sum (4.1) of $up(x)$ with some constant calculable coefficients C_k:

$$\left(x^{n}\right)^{(n)}=c\sum_{k=-\infty}^{k=+\infty}up(x-k)\equiv c,$$
$$x^{n}=I_{n}\left(c\sum_{k=-\infty}^{k=+\infty}up(x-k)\right)=\sum_{k=-\infty}^{k=+\infty}C_{k}up\left(x-k2^{-n}\right). \quad (5.1)$$

Because $up(x)$ is a sum of two AStrings (3.8), generic polynomial $P_n(x)$

can be expressed via AFs and AStrings with known constant coefficients [7] [8]:

$$P_n(x) = x^n + a_1 x^{n-1} + \cdots + a_n \equiv \sum_{k=-\infty}^{k=+\infty} C_k up\left(\frac{x-ka}{a}\right) = \sum_{l=-\infty}^{l=+\infty} AString(x, a_l, b_l, c_l). \quad (5.2)$$

Proof obtained. This theorem can be easily understood intuitively—because Atomics derivatives and integrals are expressed via themselves, it is possible to adjust Atomic Splines coefficients so that sections of polynomials would be reproduced exactly. Here are some well-known presentations (4.1) of some polynomials [1]-[12] depicted in **Figure 5**:

$$\frac{1}{4}\sum_{k=-\infty}^{k=+\infty} kup\left(x-\frac{k}{2}\right) \equiv \sum_{k=-\infty}^{k=+\infty} AString(x-k) \equiv x,$$

$$\sum_{k=-\infty}^{k=+\infty}\left(\frac{k^2}{64}-\frac{1}{36}\right)up\left(x-\frac{k}{4}\right) \equiv x^2,$$

$$x^n \equiv \sum_{k=-\infty}^{k=+\infty} C_k up\left(x-k2^{-n}\right) = \sum_{k=-\infty}^{k=+\infty} C_k\left(AString\left(2\left(x-k2^{-n}\right)+1\right) - AString\left(2\left(x-k2^{-n}\right)-1\right)\right). \quad (5.3)$$

While these series contain infinite sums, in real calculations, due to the locality of $up(x)$, only a few neighboring pulses contribute to a value at a given point (**Figure 5**). Rather than using recurrent calculations, it is convenient to calculate coefficients via computer scripts [2] [4] [44]. This fundamental theorem signifies the difference between Atomic Splines and widely-used polynomial splines [22] [23] which, due to Strang-Fix condition [41], can only exactly reproduce polynomials up to the order of a spline (eq parabola for a quadratic spline). With a limited number of neighboring pulses (5.1) around a given point, Atomic Splines can exactly fit a polynomial of any order.

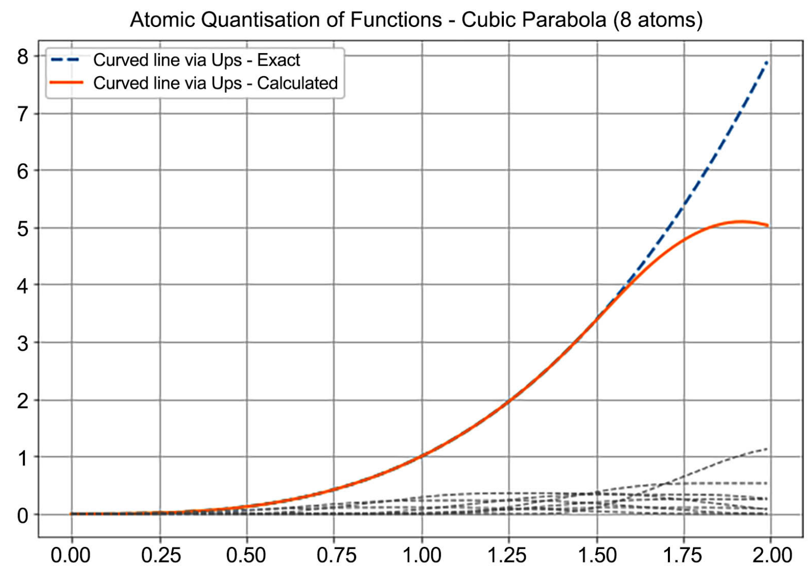


Figure 5. Representing sections of polynomials with AStrings and atomic functions—cubic parabola via 8 atomic functions.

The physical meaning of this theorem is that spacetime and fields described by polynomials can be exactly represented via shifts and stretches of Atomic AString Functions (Atomic Splines). Because AString resembles spacetime quantum/metriant (§2), spacetime field can be interpreted as some weighted superpositions of flexible overlapping quanta [1] [2] [3] [4] [5] (§2, 6, 7).

5.2. Analytic Atomization Theorem

The ability of Atomics to exactly represent polynomials leads to a more generic method developed in the 1980s [6]-[12] of "atomization" of exponential, trigonometric, hyperbolic $\exp(kx), \sin(kx), \sinh(kx)$ and other analytic functions [29] representable by converging Taylor's series and extended here for recently introduced AStrings [1] [2] [3] [4] [5].

Theorem 2 (*Analytic atomization theorem*). Analytic functions representable by converging Taylor's series via polynomials can be represented via converging Atomic Series of localized Atomic and AString Functions.

Proof. By definition, analytic functions [29] $y(x)$ are those representable by converging Taylor's series via polynomials, which in turn are representable by Atomic Splines, leading to the following series with calculable coefficients:

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} \frac{y^{(m)}(0)}{m!} x^m = \sum_{m=0}^{\infty} B_m x^m \\ &= \sum_{m=0}^{\infty} \sum_{k=-\infty}^{k=+\infty} B_m C_k up\left(x - k2^{-m}\right) \\ &= \sum_{mk=-\infty}^{k=+\infty} c_{mk} up\left(\frac{x - b_{mk}}{a_{mk}}\right) \\ &= \sum_{l=-\infty}^{l=+\infty} AString\left(x, a_l, b_l, c_l\right). \end{aligned} \tag{5.4}$$

Proof obtained. This theorem tells that not only polynomials but also a wide variety of analytic functions representable by polynomials like $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ or $\exp(x) = \sum_n \frac{x^n}{n!}$ would also be representable via Atomics. Naturally, kink-like AString functions better to represent growing functions while pulse-looking Atomic Functions better suit localized functions but ultimately AStrings and AFs are interconnected via (3.8).

The meaning of this theorem is that spacetime and physical fields described by exponential, trigonometric, hyperbolic, and other analytic functions can be interpreted as superpositions of flexible overlapping "mathematical atoms", as founders called them [6]-[12].

5.3. Atomization Theorem for Complex Analytic Functions

Theorem 2 can be generalized to various combinations of analytic functions [1] [6]-[12] [29].

Theorem 3 (*Complex analytic atomization theorem*). Complex functions $y(x)$ that are sums $y = y_1 + y_2$, products $y = y_1 y_2$, reciprocals $y = 1/y_1$ ($y_1 \neq 0$), inverse $y(y_1) = x$, derivatives $y = y_1'$, integrals $I(y)$, and superposition

$y = y_1(y_2)$ of analytic functions $y_1(x), y_2(x)$ can be represented by Atomic Series over finite Atomic and AString Functions (Atomic Splines).

Proof. Those combinations of analytic function $y_1(x), y_2(x)$ are also analytic [29] (where they are not infinite), hence representable by Taylor's series via polynomials and then via Atomic Splines by Atomic Series (5.4), (5.3):

$$y(x) = y_1(y_2(x));$$
$$y(x) \equiv \sum_{l=-\infty}^{l=+\infty} up(x, a_l, b_l, c_l) = \sum_{k=-\infty}^{k=+\infty} AString(x, a_k, b_k, c_k). \quad (5.5)$$

Sums, products, derivatives, integrals, and superpositions of converging power Taylors' series (5.4) would also be polynomial power series. Due to Lagrange inversion theorem [46], reciprocals and inversions of invertible analytic functions would also be analytic ones representable by polynomial Taylor's series, hence via Atomic Splines via (5.4), (5.3).

Proof obtained. This theorem is easy to understand intuitively; if two functions are represented by power series via polynomials, the complex superpositions of them would also be polynomial representable via Atomics.

This important theorem covers a wide range of composite functions like $x\sinh(x), \mathrm{sech}(x), \mathrm{sech}^2(x), \tanh(x)$, gaussian $\exp(-x^2)$, $\arctan(\exp(x))$, $\sqrt{1+x}$, $x/\sqrt{1+x^2}$ appearing in the linear and nonlinear soliton theories, quantum mechanics, relativity, and spacetime physics, for example, to represent Schwarzschild metrics in General Relativity [13] [14] [15] (**Figure 6**).

5.4. Atomization Theorem for Differential Equations

In general, elementary polynomic, trigonometric, exponential, and hyperbolic functions are the solutions of some linear differential Equations (LDE), for example, $y' - y = 0, y = \exp(x); y'' + y = 0, y = \sin(x)$. It implies that Atomization Theorems can be extended from functions to differential equations [1]-[12].

Theorem 4 (*LDE atomization theorem*). Solutions of linear differential Equations (LDE) with constant coefficients

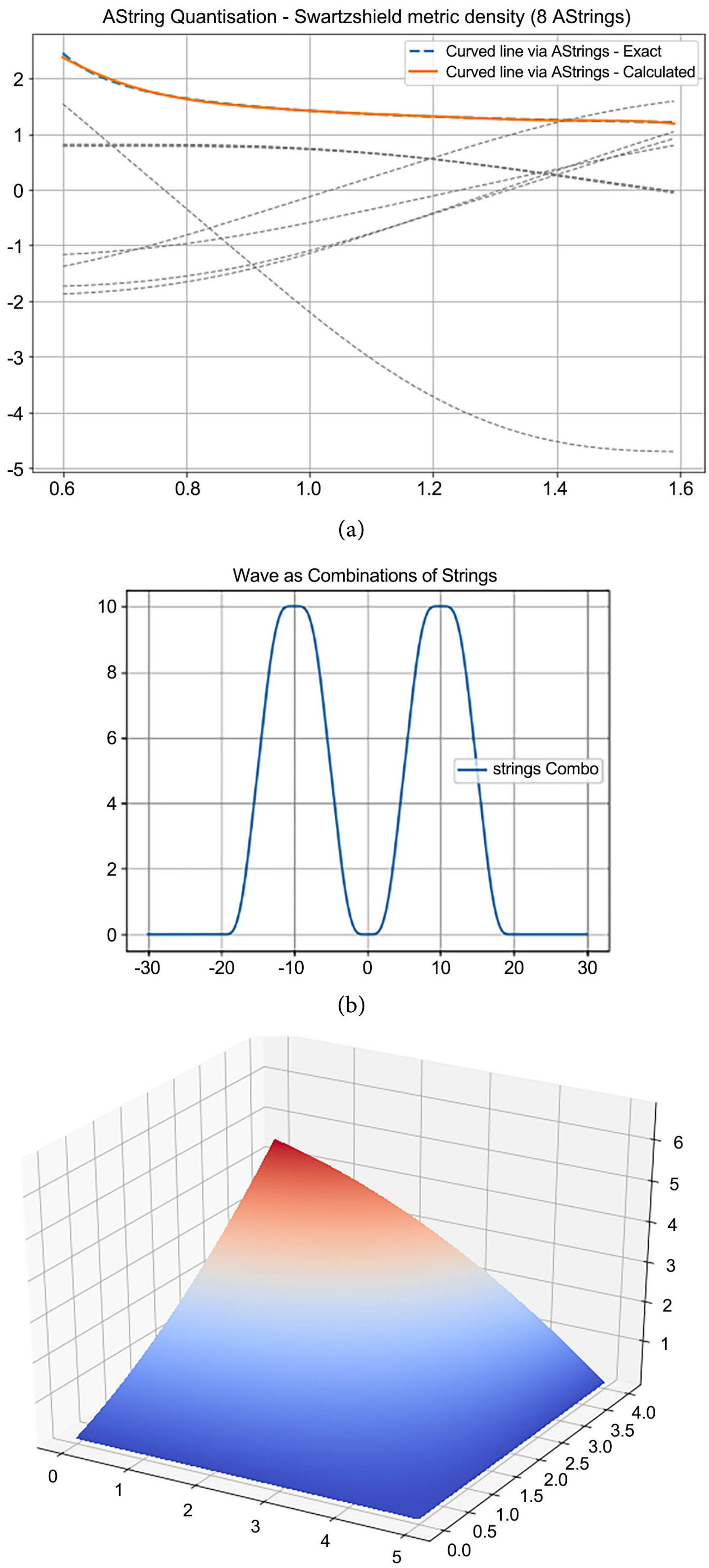
$$L(y) = y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = 0 \quad (5.6)$$

can be represented via Atomic Series over Atomic and AString Functions.

$$L(y) = 0; \; y(x) \equiv \sum_{l=-\infty}^{l=+\infty} up(x, a_l, b_l, c_l) = \sum_{k=-\infty}^{k=+\infty} AString(x, a_k, b_k, c_k). \quad (5.7)$$

Proof. Using Taylor's series, let's seek a solution of (5.6) as a polynomial $y(x) = \sum_{m=0}^{\infty} B_m x^m$. Injecting it into (5.6) would yield the sum of polynomials, the coefficients of which can be recurrently chosen to satisfy (5.6) exactly. But because those polynomials are exactly representable via Atomic Splines (Theorem 1), the solution of LDE (5.6) can also be representable via Atomics.

Proof obtained. The idea of differential equations' atomization is quite intuitive. Because Atomics are localized and their derivatives are expressed via themselves, injecting them into LDE (5.6) would yield another Atomic Series over localized Atomic Splines $c_i up((x - b_i)/a_i)$, the coefficients of which can be chosen to satisfy LDE at a given point. A similar idea is used in Fourier analysis



(c)

Figure 6. Representing sections of polynomials and analytic functions with AStrings and Atomic Functions. (a) Schwarzschild metric function; (b) Wave-like formation; (c) 2D surface composed of a few Atomic Splines.

operating with trigonometric functions, derivatives of which are also expressed via themselves. But here we use finite Atomic Splines which, by the way, can represent trigonometric functions in the Fourier series (Theorem 2).

5.5. Atomization of Equations with Variable Coefficients

The last theorem can be extended to a more generic case of variable analytic coefficients [1]-[12].

Theorem 5 (*Variable LDE atomization theorem*). Solutions of linear differential equation

$$L\left(y,a_k\left(x\right)\right)=y^{(n)}\left(x\right)+a_1\left(x\right)y^{(n-1)}\left(x\right)+\cdots+a_{n-1}\left(x\right)y'\left(x\right)+a_n\left(x\right)y\left(x\right)=0 \quad (5.8)$$

with variable coefficients $a_k\left(x\right)$ representable by analytic functions can be represented via Atomic Series over Atomic and AString Functions.

Proof. Using Taylor's series, let's seek a solution in a form of a generic polynomial $y\left(x\right)=\sum_{m=0}^{\infty}B_m x^m$, the derivatives of which would also be polynomials, and assume a similar representation for analytic functions $a_k\left(x\right)=\sum_{m=0}^{\infty}A_{km}x^m$. Due to Theorem 3, the products $a_k\left(x\right)y^{(k)}\left(x\right)$ would also be polynomials making LDE (5.8) a sum of some polynomials, the coefficients of which can be recurrently chosen to match zero on the right side. But because the polynomials are exactly atomizable via Atomic Splines (Theorem 1), the solutions of LDE (5.8) can also be representable via Atomics:

$$\begin{aligned}&L\left(y,a_k\left(x\right)\right)=0;\\&y\left(x\right)\equiv\sum_{l=-\infty}^{l=+\infty}up\left(x,a_l,b_l,c_l\right)=\sum_{k=-\infty}^{k=+\infty}AString\left(x,a_k,b_k,c_k\right).\end{aligned} \quad (5.9)$$

Proof obtained. This theorem covers a wide variety of differential equations of mathematical physics, for example, Newtonian mechanics dealing with second derivatives.

5.6. Atomization of Nonlinear Differential Equations and "Preservation of Analyticity"

Atomization (5.4), (5.5) of composite functions like $\arctan\left(\exp\left(x\right)\right)$, $\operatorname{sech}\left(x\right)$ satisfying *nonlinear* sine-Gordon and Schrodinger differential equations well-known in soliton theories [2] [4] [29] [30] [31] [32] imply that the Atomization procedure traditionally applicable for linear differential equations [6]-[12] can be extended to some nonlinear differential Equations (NDE) important in hydrodynamics, plasticity, relativity, soliton, and other nonlinear theories. It leads to the following theorem.

Theorem 6 (*NDE atomization theorem*). Solutions of nonlinear differential Equations (NDE) with linear differential operator $L\left(y\right)$ and nonlinear analytic function $f\left(y\right)$

$$\begin{aligned}&L\left(y,a_k\left(x\right)\right)\\&=y^{(n)}\left(x\right)+a_1\left(x\right)y^{(n-1)}\left(x\right)+\cdots+a_{n-1}\left(x\right)y'\left(x\right)+a_n\left(x\right)y\left(x\right)\\&=f\left(y\right)\end{aligned} \quad (5.10)$$

can be represented via Atomic Series over Atomic and AString Functions.

Proof. Presumably analytic, function $f(y)$ can be represented via converging Taylor's series $f(y)=\sum_{m=0}^{\infty} C_m y^m$. Seeking solution via polynomials $y(x)=\sum_{m=0}^{\infty} B_m x^m$ and noting that due to Theorem 5, the powers, derivatives, multiplications, and superpositions of polynomials would also be polynomials, leading to polynomials for both sides of (5.10), the coefficients of which can be recurrently chosen to satisfy (5.10) exactly. But because those polynomials are exactly representable via Atomic Splines (Theorem 1), the solutions of NDE (5.10) are also representable via the superposition of Atomics:

$$
\begin{aligned}
& L(y)=f(y); \\
& y(x) \equiv \sum_{l=-\infty}^{l=+\infty} up(x, a_l, b_l, c_l)=\sum_{k=-\infty}^{k=+\infty} AString(x, a_k, b_k, c_k).
\end{aligned} \tag{5.11}
$$

Proof obtained. The idea can be illustrated with the sine-Gordon equation [4] [32]

$$
y''(x)=\sin(y)=y-\frac{y^3}{3!}+\frac{y^5}{5!}+\cdots
$$

leading to analytic function $y(x)=\arctan(\exp(x))$ used in differential geometry and many soliton theories [2] [4] [30] [31] [32]. Another "atomizable" example is an envelope soliton function $\operatorname{sech}(x)$ satisfying stationery nonlinear Schrodinger equation [4] [30] [31] $y-y''(x)=2y^3$ of the form (5.10).

It looks like the *preservation of analyticity* (when a differential operator applied to an analytic function would yield another analytic function) and the fact of Theorem 3 that compositions $y=y_1(y_2)$ of analytic functions are also analytic are the two crucial properties for the wide applicability of Atomization Theorems even for complex nonlinear differential equations including General Relativity (§6). The theorems 1 - 6 can be generalized in the following theorem.

Theorem 7 (*Complex NDE atomization theorem*). Solutions of nonlinear differential equations with functional-differential operator F preserving analyticity of function $y(x)$ can be represented via Atomic Series over Atomic and AString Functions:

$$
\begin{aligned}
& F\left(y(x), \frac{\partial^m y}{\partial x^n}\right)=0; \\
& y(x) \equiv \sum_{l=-\infty}^{l=+\infty} up(x, a_l, b_l, c_l)=\sum_{k=-\infty}^{k=+\infty} AString(x, a_k, b_k, c_k).
\end{aligned} \tag{5.12}
$$

Proof. Injecting polynomial solution $y(x)=\sum_{m=0}^{\infty} B_m x^m$ into operator F which supposedly preserves analyticity would convert a solution into another polynomial, the coefficients of which can be selected to match zero on the right side of (5.12). Because an analytic function [29] can be presented via polynomials by converging Taylor's series, the function $y(x)$ would also be representable via Atomic Series (5.12). Examples of different operators F preserving analyticity are presented in Theorems 1 - 6.

Proof obtained. The idea can be illustrated by the stationary nonlinear Korteveg-de Vries equation $F(y)=y'''(x)-cy'(x)-6yy'(x)=0$ for soliton wave function $y(x)=\operatorname{sech}^2(x)$ [4] [27] [30] [31] representable by Atomics (5.5) due to Theorem 3.

5.7. Generic Atomization for Nonlinear Nonanalytic Equations

The abovementioned Atomization Theorems are applicable for the representation of a wide range of analytic functions, their superpositions as well as linear and nonlinear differential equations. However, there are *nonanalytic* functions like $\exp(-1/x)$, bump functions $\exp\left(-1/\left(1-x^2\right)\right)$ and other functions analytic only in some areas as well as generic nonlinear differential equations for which Taylor's power series may not be converging [29]:

$$F_1\left(y_i\left(x_j,t\right),\frac{\partial^m y_i}{\partial x_j^n}\right)=0;\ F_2\left(y_i\left(x_j,t\right)\right)=0. \tag{5.13}$$

Here, the atomization procedure based on Atomic Series (5.12) may not deliver converging representations for field functions $y_i\left(x_j,t\right)$. However, the quite universal idea of atomization of field equations still holds if we recall (3.14), (3.16), and (4.1) that spacetime coordinates themselves x,y,z,t are representable via AStrings:

$$\begin{aligned} x_j &= \sum_{k=-\infty}^{k=+\infty} aAString\left(\frac{x_j-ka}{a}\right)=\sum AString\left(x_j,a,ka,a\right); \\ t &= \sum AString\left(t,\Delta,k\Delta,\Delta\right). \end{aligned} \tag{5.14}$$

In this case, generic nonlinear Equations (5.13) become complex functions of sums of AStrings, the interpretation of which can only be given within a physical theory which those equations describe:

$$\begin{aligned} &F_1\left(\sum AString\left(x_j\right),\sum AString\left(t\right),\frac{\partial^m y_i}{\partial\sum AString\left(x_j\right)}\right); \\ &F_2\left(y_i\left(\sum AString\left(x_j\right),\sum AString\left(t\right)\right)\right)=0. \end{aligned} \tag{5.15}$$

In this generic form, the atomization procedure becomes quite universal and applicable to all equations of mathematical physics dealing with x,y,z,t, similar to finite elements and finite differences methods [21] [22] which can be universally used for discretization of almost any equation. "Atomization" is a kind of "advanced discretization" with the preservation of smoothness between finite elements.

5.8. Atomic Representation of Waves

The abovementioned theorems imply that any smooth analytic function including wave functions can be represented via the converging superposition of Atomics. Some presentations known in the theory of Atomic Functions [8] [9] [10] [11] [12] [23] uphold this idea extended here for AStrings.

Theorem 8 (*Waves atomization theorem*). Any smooth function with a *finite spectrum* can be represented via Atomic Series over finite Atomic and AString Functions.

Proof. A smooth function $y(x)$ with finite spectrum can be represented via limited Fourier series $\sum_i d_i \sin(e_i x - f_i)$, and because $\sin(x)$ is also representable via Taylor's series hence via Atomic Series (Theorem 2), the function $y(x)$ can be also presented via Atomic Series over Atomic Splines:

$$\begin{aligned} y(x) &\equiv \sum_i d_i \sin(e_i x - f_i) = \sum_{l=-\infty}^{l=+\infty} up(x, a_l, b_l, c_l) \\ &= \sum_{k=-\infty}^{k=+\infty} AString(x, a_k, b_k, c_k). \end{aligned} \tag{5.16}$$

Proof obtained. The important physical application of this theorem is to represent electro-magnetic, sound, and water waves via some superpositions of Atomics giving rise to the quite wide applicability of Atomic Functions in radio electronics and signal processing [8] [9] [10] [11] [12].

5.9. Atomization Based on Complex Atomic Functions

Let's note another well-known [7]-[12] [23] [25] [26] [27] method of representing solutions of LDE (5.6), (5.8) $L(y)=0$ via *more generic* Atomic Functions more complex than $up(x)$. For example, if we build a new finite atomic function $\varphi(x)$ satisfying a more generic equation with shifted arguments, we can represent $y(x)$ via shift and stretches of AF pulses $\varphi(x)$ [7]-[12] [25] [26] [27]:

$$\begin{aligned} &L(\varphi) = c_1 \varphi(ax+b) + c_2 \varphi(ax-b); \; L(y) = 0; \\ &y(x) = \sum_k c_k \varphi(ax - b_k). \end{aligned} \tag{5.17}$$

Injecting y into (5.17) yields a series of localized pulses φ where it is possible to select coefficients c_k, b_k in such a way that Equation (5.17) would be satisfied. For 50 years of history, many useful atomic functions have been built and used in new kinds of finite element collocation methods for radio-electronics, signal processing, and others [7]-[12] [25] [26].

5.10. Atomization Theorems in Many Dimensions

Atomization Theorems can be extended to multiple dimensions [2] [7]-[12] [28] with the following theorem extended for AStrings [1] [2] [3] [4] [5].

Theorem 9 (3*D atomization theorem*). Representable by converging Taylor's power series, multidimensional analytic functions with their sums, multiplications, reciprocals, derivatives, integrals, and superpositions can be represented via Atomic Series over localized multidimensional Atomic and AString Functions.

Proof. Multidimensional n-order polynomial in m-dimensions $P_{mn} = P_n(x_1, \cdots, x_m)$, which are some multiplications of 1D polynomials exactly representable by Atomics (Theorem 1), are also exactly representable by multiplications of Atomic Functions (multidimensional atomic functions (3.9) $\boldsymbol{UP}(a_k, b_k, c_k)$ which in turn are AStrings combinations (3.8).

$$\begin{aligned} P_{mn} &= P_n(x_1, \cdots, x_m) = \prod_{i=1}^{m} P_n(x_i) = \sum_{k=-\infty}^{k=+\infty} \prod_{i=1}^{m} up_i(x_i, a_{ik}, b_{ik}, c_{ik}) \\ &= \sum_{k=-\infty}^{k=+\infty} \prod_{i=1}^{m} AString_i(x_i, a_{il}, b_{il}, c_{il}) = \sum_{k=-\infty}^{k=+\infty} \boldsymbol{UP}(a_k, b_k, c_k) \\ &= \sum_{l=-\infty}^{l=+\infty} \boldsymbol{AString}(a_l, b_l, c_l). \end{aligned} \tag{5.18}$$

Therefore, multidimensional analytic functions [29], being representable via converging multi-dimensional Taylor's power series, are also representable by multidimensional Atomics. Superposition of analytic functions preserves analyticity (Theorem 3), and integrals and derivatives of polynomials are also polynomials, so multidimensional analytical functions with their derivatives, integrals, and superpositions are also representable via Atomics.

Proof obtained. This theorem can be illustrated by the atomization of a function $\arctan\left(\exp\left(x^2 y^5 z\right)\right)$. Being exactly representable via (5.3) by Atomics in the relevant dimension

$$x^2 = \sum_k \left(\frac{k^2}{64} - \frac{1}{36}\right) up\left(x - \frac{k}{4}\right),$$

$$y^5 \equiv \sum_l C_l up\left(y - l2^{-5}\right),\ z = \sum_m \frac{1}{4} up\left(z - \frac{m}{2}\right),$$

$r = x^2 y^5 z$ would be the sum of multiplications which in 3D forms can be denoted via 3D atomic function assuming summation over all 3 dimensions:

$$\begin{aligned} x^2 y^5 z &= \sum_{klm} C_{klm} up\left(x, a_k, b_k, c_k\right) up\left(y, a_l, b_l, c_l\right) up\left(z, a_m, b_m, c_m\right) \\ &= \sum_n \mathbf{up}\left(x, y, z, a_n, b_n, c_n\right). \end{aligned}$$

Next, an exponent representable via Taylor's series as $\exp(r) = \sum_n \frac{r^n}{n!}$ would also be the sum of sums of AFs along with arctan applied to the exponent. But because two AStrings compose $up(x)$ (3.8), the final product can be expressed via the superposition of AStrings.

Despite the seemingly complex procedure, the atomization method tells that like in the Lego game the smooth analytic manifolds can be composed of "small pieces" with preservation of smoothness between them (Figure 1 and Figure 5).

Similar to the 1D case with Theorems 4 and 5, the atomization idea can be extended to multi-dimensional differential equations

$$\begin{aligned} & L\left(y_1, \cdots, y_m\right)\left(x_1, \cdots, x_m\right) = a_{ijmn} \frac{\partial^m y_i}{\partial x_j^n} = 0; \\ & y_i\left(x_j\right) \equiv \sum_{ijjkl} up_i\left(x_j, a_{ijl}, b_{ijl}, c_{ijl}\right) = \sum_{jjkl} AString_i\left(x_j, a_{ijk}, b_{ijk}, c_{ijk}\right) \end{aligned} \tag{5.19}$$

typically containing linear differential operators like Laplacian and Poisson operators widely used in mathematical physics:

$$\nabla_i = \frac{\partial}{\partial x_i};\ \Delta = \sum \frac{\partial^2}{\partial x_i^2};\ \Delta + k;\ \Delta\Delta. \tag{5.20}$$

Due to locality, derivatives expressed via themselves, and the ability to compose polynomials and analytic functions, the solutions of these equations can be expressed by Atomic Series (5.19) via a combination of "mathematical atoms" $up(x_i)$ made of AString kinks (3.8), as shown in Figures 1-5.

6. Atomization Theorems in General Relativity

Theorems 1 - 9 lead to the following new theorems targeting Einstein's General Relativity (GR) theory [13] [14] [15] and Atomic Spacetime theory based on

Atomic AString Functions developing since 2017 [1] [2] [3] [4] [5] [23] [43].

6.1. Atomization Theorems for Metric, Curvature, and Ricci Tensors

Considering together multidimensional Atomic Series (5.9), (5.10) and Atomization Theorems 1 - 6 leads to the following theorems important for General Relativity.

Theorem 10 (*Tensor's atomization theorem*). First $\partial_i = \frac{\partial}{\partial x_i}$ and second derivatives $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$ as well as the metric tensor g_{ij} defining interval on a curved surface $\mathrm{d}s^2 = g_{ij}\left(x_n\right)\mathrm{d}x_i\mathrm{d}x_j$ preserve analyticity and being applied to analytic functions $y_k\left(x_l\right)$ leading to analytic functions representable/ atomizable by Atomic Series over Atomic and AString Functions.

Proof. Being linear differential operators, both first and second derivative operators preserve analyticity because derivatives of multidimensional polynomials $B_{lm}x_l^m$ would also be polynomials exactly representable via multidimensional Atomic Functions and AStrings (5.9) using Atomic Series (5.2). For curved spacetime shapes/geometries described by some analytic functions $\tilde{x}_i = \tilde{x}_i\left(x_j\right)$; $\mathrm{d}\tilde{x}_i = \frac{\partial \tilde{x}_i}{\partial x_j}\mathrm{d}x_j$, the derivatives and their multiplications would also be analytic, hence representable by Atomic Splines (Theorems 2, 3, 9). This theorem can be proved in another way by noting that all derivatives and integrals of Atomics are expressed via themselves (3.3), (3.8), and if space geometry analytic functions $\tilde{x}_i\left(x_j\right)$ are the sum of Atomics, then all derivatives and metric tensors would also be some Atomics combinations:

$$g_{ij}\left(x_n\right) = \sum_{ijnk} up\left(x_n, a_{ijnk}, b_{ijnk}, c_{ijnk}\right) = \sum_{ijnl} AString\left(x_n, a_{ijnl}, b_{ijnl}, c_{ijnl}\right). \quad (6.1)$$

Proof obtained. This theorem means that for analytic spacetime geometries/ configurations, their deformations, curvatures, metrics, and geodesics would also be some Atomics superpositions, with a range of analytical surfaces and spacetime metrics known in GR [13] [14] [15] described later. This theorem can be intuitively understood in the sense that if a spacetime geometry is described by polynomials then deformations and curvatures (which are derivatives and multiplications) of the spacetime field would also be polynomials and hence representable by Atomic Splines.

Furthermore, due to the properties of analytic function superpositions to preserve analyticity (Theorem 3), the last theorem can be extended to nonlinear Ricci tensors important in GR [13] [14] [15].

Theorem 11 (*Ricci tensor atomization theorem*). Nonlinear Ricci tensor R_{jk} and Christoffel operators Γ_{ij}^k preserve analyticity and applied to analytic functions would yield analytic functions representable by Atomic Series via Atomic and AString Functions.

Proof. Christoffel operators [13] [14] [15], which include multiplications of functions to their spatial derivatives, transform analytic metric tensor functions (6.1) representable by polynomials into more complex polynomials representable by Atomics via Atomic Series (5.3). Similarly, Ricci tensors are also a combination of derivatives and multiplications of Christoffel symbols [13] [14] [15] which preserve analyticity, hence representable via Atomic Splines:

$$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right); \ R_{jk} = \partial_i \Gamma_{jk}^{i} - \partial_j \Gamma_{ik}^{i} \Gamma_{ip}^{i} \Gamma_{jk}^{p} - \Gamma_{jp}^{i} \Gamma_{ik}^{p} \quad (6.2)$$

$$R_{ij}\left(x_n\right) = \sum\nolimits_{ijnk} up\left(x_n, a_{ijnk}, b_{ijnk}, c_{ijnk}\right) = \sum\nolimits_{ijnl} AString\left(x_n, a_{ijnl}, b_{ijnl}, c_{ijnl}\right). \quad (6.3)$$

Proof obtained. This theorem can be understood in the sense that polynomials are "hard to destroy" by common differential operators because their multiplications, derivatives, integrals, and superpositions would also be polynomials representable by Atomics. It also means that not only spacetime metrics but also curvature tensors can be "atomized" using Atomic Splines describing finite mathematical objects resembling flexible spacetime quanta §2, [1] [2] [3] [4] [5].

6.2. Atomization Theorem for General Relativity

The sequence of Theorems 1 - 10 finally converges into the following new theorem for Einstein's General Relativity [13] [14] [15].

Theorem 12 (*Atomic Spacetime Theorem*). For analytic manifolds, Einstein's curvature tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ preserves analyticity and yields spacetime shapes, deformations, curvatures, and matter/energy tensors $T_{\mu\nu}$ representable via multi-dimensional Atomic AString Functions superpositions. Solutions of General Relativity equations can be represented via converging Atomic Series over finite Atomic and AString Functions:

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \\ &= \sum\nolimits_{\mu\nu i} \boldsymbol{UP}\left(x_i, a_{\mu\nu i}, b_{\mu\nu i}, c_{\mu\nu i}\right) \\ &= \sum\nolimits_{\mu\nu i} \boldsymbol{AString}\left(x_i, a_{\mu\nu i}, b_{\mu\nu i}, c_{\mu\nu i}\right). \end{aligned} \quad (6.4)$$

Proof. For analytic manifolds—spacetime geometries described by analytic functions $\tilde{x}_i = \tilde{x}_i\left(x_j\right)$ representable by converging Taylor's power series—the metric tensors $g_{\mu\nu}$ composed of derivatives and their multiplications would also be some analytic functions (Theorem 10). Being injected into Christoffel operators and then Ricci tensors $R_{\mu\nu}$ (6.2), they would yield another set of analytic functions (Theorem 11) representable by Taylor's polynomial series because the derivatives, multiplications, and superposition of analytic functions would also be analytic (Theorem 3). The curvature scalar $R = g^{\mu\nu} R_{\mu\nu}$ in (6.4) preserves analyticity because of the cross-multiplication of polynomials and their derivatives would also be polynomials. Injected into (6.4), those tensors produce Einsteinian tensor $G_{\mu\nu}$ and energy-momentum tensor $T_{\mu\nu}$ ($8\pi G/c^4$ is a constant) presumably representable by polynomials via multi-dimensional Taylor's series. Because a polynomial of any order is exactly representable via Atomic

Splines (Theorem 1), the spacetime curvature, metric, and energy/momentum tensors would be the superpositions of multi-dimensional Atomic ***UP*** and ***AString*** functions, derivatives of which are expressed via themselves. Due to fundamental relation (3.8)

$$up(x) = AString'(x) = AString(2x+1) - AString(2x-1),$$

the Atomic Function $up(x)$ is a sum of two AStrings which may be associated with a finite quantum/metriant being able, within one model, to compose straight $x = \sum_k AString(x, a, ka, a)$ and curved $\tilde{x} = \sum_k AString(x, a_k, b_k, c_k)$ lines from elementary AString kinks resembling flexible quanta (§2, 6, 7).

Proof obtained. In a nutshell, this theorem tells that the spacetime field is representable ("atomizable") via AStrings and Atomic Functions, the derivatives of which are expressed via themselves meaning the spacetime shape, deformations, curvatures, and energy/momentum tensors can also be represented as some superposition of Atomics. Now, this idea first hypothesized in 2017 [3] is based on a set of theorems. It offers the Atomic Spacetime model [1] [2] [3] [4] [5] quite resonating with A. Einstein's 1933 lecture [17] where he predicted a "*perfectly thinkable*" "*atomic theory*" dealing with "*simplest concepts and links between them*" to solve some "*stumbling blocks*" of continuous field theories to describe quantized fields.

Let's note that Atomization is not a simple discretization of space—separation of a volume into adjacent finite elements [22] [24] [38]. Here, the "finite elements" (AStrings) are smoothly overlapping (§2, **Figure 1**) and capable to describe both expansions of space (3.9) and localized "atoms" $up(x)$ (3.8).

6.3. Deriving Atomics from General Relativity Equations

Previous Atomization Theorems were based on the historical assumption that we know the mathematical properties of finite Atomic Functions [1]-[12] and try to introduce them to spacetime physics as it was done in [2] [3] [4] [5]. The intriguing question is whether it is possible to do the opposite—to derive Atomic Functions from GR, so in theory, A. Einstein could have done it himself, especially in 1933 when in his paper [17] he envisaged an "*atomic theory*" with "*… region of three-dimensional space at whose boundary electrical density vanishes everywhere*" resembling finite functions like Atomics. The following theorem shows how it can be achieved by applying backward the Atomization Theorems 1 - 12.

Theorem 13 (*Atomic Spacetime quantum theorem*). It is possible to find an infinitely differentiable finite pulse spline function that can represent analytic solutions of GR and polynomials of any order via superpositions, and such a function should have a form of Atomic Function $up(x)$ with derivative

$$up'(x) = 2up(2x+1) - 2up(2x-1) \tag{6.5}$$

and integral

$$AString'(x) = up(x) = AString(2x+1) - AString(2x-1). \tag{6.6}$$

Being localized solitary functions capable to compose flat and curved spacetime fields in overlapping superposition, those spline functions may be interpreted as flexible quanta of spacetime.

Proof. Preservation of analyticity in Einstein's curvature tensor, Ricci tensor, and Christoffel operators (Theorems 11, 12) implies that analytic metric tensor functions $g_{ij}(x_n)$ being injected in those operators would produce other analytic functions representable by polynomials—because the multiplication of derivatives of polynomials to other polynomials would also be polynomials. Analyticity of metric tensor $g_{ij}(x_n)$ representable by Taylor's series via polynomials (for which derivatives and integrals would also be polynomials) implies that spacetime geometry $\tilde{x}_i(x_n)$ and geodesics should also be analytic functions representable by polynomials (Theorem 10). This theorem would be proved if we find some basis spline pulse-like function $p(x)\in[-1,1]$ which in translation would exactly compose polynomials of any order (Theorem 1) $x^n=\sum_k c_k p\left(\frac{x-b_k}{a_k}\right)$. Firstly, following §2, we have to eliminate the polynomial spline candidates (like B-splines or cubic Hermitian splines [22] [23]) as they are unable to exactly compose a polynomial of *any* order. The desired spline function should be a polynomial of infinite order, or simply belong to class $C(\infty)$ of absolutely smooth functions. Secondly, we have to eliminate trigonometric and other exponential-based spline functions like Gaussians or Sigmoids because by summing only a few pulses they are unable to exactly reproduce even the simplest polynomials (a line, or a constant). To satisfy Theorem 1, the choice has narrowed to infinitely differentiable spline functions $p(x)$ capable to compose any polynomial $x^n=\sum_k c_k p\left(\frac{x-b_k}{a_k}\right)$ and also satisfy the "partition of unity" $c=\sum_k cp(x-k)$. Desired infinite differentiability implies that the spline function's derivative should be expressed via the function itself $p'(x)=F(p(x))$, or in simplest linear form $p'(x)=F(p(x))=kp(ax+b)-kp(ax-b)$ which with symmetry condition $p(x)=p(-x)$, normalization $p(0)=1$ and finiteness $p(x)=0,|x|>1$ lead to Atomic Function $p(x)=up(x)$, $up'(x)=2up(2x+1)-2up(2x-1)$ discovered in the 1970s [6] [7] [8] and described in §2. Using this spline function to compose the 3D polynomials, the geometry of spacetime $\tilde{x}_i(x_n)$ along with metric tensor $g_{ij}(x_n)$, Ricci tensor $R_{ij}(x_n)$ and Einsteinian tensor $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ lead to Theorems 11, 12 and the ability to express GR solutions via a series of Atomic Function pulses. Due to symmetry (2.5), the Atomic Function $up(x)$ can be represented via the sum of two simpler *AString* kink functions (3.8) $up(x)=AString(2x+1)-AString(2x-1)=AString'(x)$, so GR solutions can also be expressed via AStrings. AStrings can describe spacetime expansion via superpositions $x\equiv\sum_k AString(x-k)$ hence may be associated with some finite quanta of space.

Proof obtained. This conceptually important new theorem allows deducing finite Atomic and AString Functions from GR noticing a crucial property of GR operators to preserve analytic functions and polynomials and the unique ability of Atomic AString Functions to exactly represent them. But actually, there is not much of a surprise that spacetime and other smooth fields can be represented by some "mathematical atoms", as founders often called them [6] [7] [8] [9] [10]. The hard part, which took 6 years, was to formally work out how quite complex nonlinear Einstein's GR equations can yield simply looking Atomic Splines, and preservation of analyticity and atomization of polynomials were the key hints to achieve this.

6.4. Atomic Spacetime Model

Formulated Atomization Theorems 1 - 13 provide a theoretical foundation for atomization/quantization of spacetime field [1] [2] [3] [4] [5] based on Atomic and AString Functions when GR equations and solution, along with Ricci, curvature, and metric tensors, can be represented via Atomic Series over multidimensional Atomic and AString Functions (3.7):

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \\ &= \sum\nolimits_{\mu\nu i} \boldsymbol{UP}\left(x_i, a_{\mu\nu i}, b_{\mu\nu i}, c_{\mu\nu i}\right) \\ &= \sum\nolimits_{\mu\nu i} \boldsymbol{AString}\left(x_i, a_{\mu\nu i}, b_{\mu\nu i}, c_{\mu\nu i}\right), \end{aligned} \tag{6.7}$$

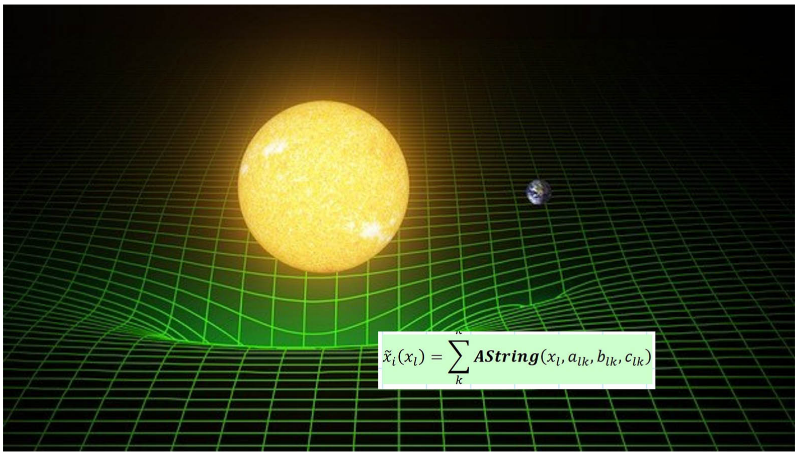
$$\begin{aligned} R_{ij}\left(x_n\right) &= \sum\nolimits_{ijnk} \boldsymbol{UP}\left(x_n, a_{ijnk}, b_{ijnk}, c_{ijnk}\right) \\ &= \sum\nolimits_{ijnl} \boldsymbol{AString}\left(x_n, a_{ijnk}, b_{ijnk}, c_{ijnk}\right), \end{aligned} \tag{6.8}$$

$$\begin{aligned} g_{ij}\left(x_n\right) &= \sum\nolimits_{ijnk} \boldsymbol{UP}\left(x_n, a_{ijnk}, b_{ijnk}, c_{ijnk}\right) \\ &= \sum\nolimits_{ijnl} \boldsymbol{AString}\left(x_n, a_{ijnk}, b_{ijnk}, c_{ijnk}\right), \end{aligned} \tag{6.9}$$

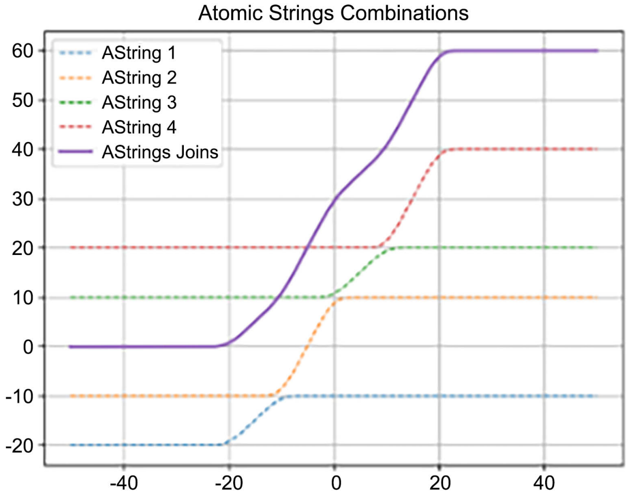
$$\tilde{x}_i\left(x_l\right) = \sum\nolimits_k \boldsymbol{AString}\left(x_l, a_{lk}, b_{lk}, c_{lk}\right). \tag{6.10}$$

These formulae express the mathematical fact that it is possible to compose analytical manifolds (**Figure 6**) by adjusting the parameters of Atomic Splines, or, like in the Lego game, composing a smooth shape from "elementary pieces" resembling quanta. If finite Atomics, for which derivatives are expressed via themselves, represent spacetime shape $\tilde{x}_i\left(x_l\right)$ (6.10), the series over Atomics would also describe spacetime deformations, curvatures, metrics, Ricci's, Einstein's, and energy-momentum tensors.

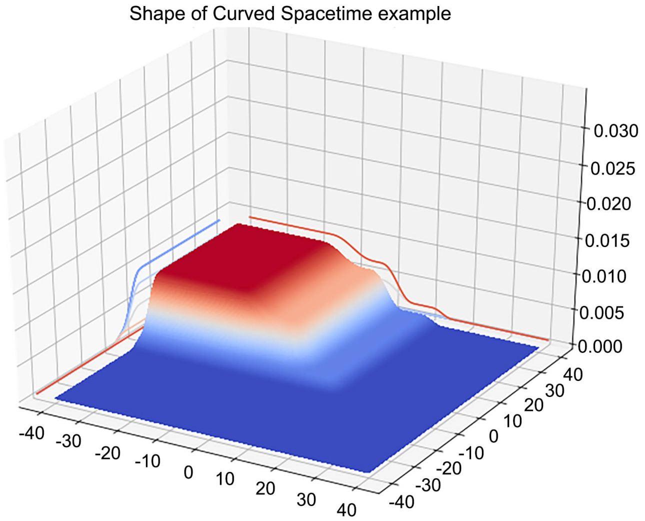
Because AString can compose a line and a curve from "elementary pieces" resembling quanta (§2, 6) one can envisage a spacetime field as a complex network of flexible spacetime quanta (**Figures 4-7**) on a lattice. Identical quanta produce flat uniform spacetime while shifting the location of one quantum causes the spaceline to curve (**Figure 7(b)** and **Figure 7(c)**). Let's note that the notion of "quantum" here is not directly related to Quantum Mechanics and Quantum Gravity [18] [19] [34]-[39] but rather to the finiteness of "solitonic atoms" capable to



(a)



(b)



(c)

Figure 7. (a) Curved spacetime composed of AStrings; (b) Joining AStrings of different heights simulates spacetime curving; (c) Curved spacetime geodesics represented via joints of 3D solitonic atoms.

compose shapes and fields.

A detailed description and evolution of Atomic Spacetime theory and properties of atomized spacetime are presented in [1] [2] [3] [4] [5] [23] [43].

6.5. Atomization of Known General Relativity Solutions

The idea of Atomic Spacetime quantization can be demonstrated for known GR solutions [13] [14] [15].

Einstein-Minkowski solution $T_{\mu\nu} = 0, g_{ij} = 1$ for homogeneous uniform spacetime/universe [13] [14] [15] [16] [20] is simply atomizable/quantizable via translations of identical overlapping AString quanta (§2, **Figure 1** and **Figure 8**) [1] [2] [3] [4] [5] in vector notation:

$$\begin{aligned} &\boldsymbol{AQuantum}(x_1, x_2, x_3, t, a, \rho, c_l) \\ &= AString(x_1, a, a, \rho a)\boldsymbol{e}_1 + AString(x_2, a, a, \rho a)\boldsymbol{e}_2 \\ &\quad + AString(x_3, a, a, \rho a)\boldsymbol{e}_3 + AString(t, a/c_l, a/c_l, \rho a/c_l)\boldsymbol{e}_t, \end{aligned} \tag{6.11}$$

or schematically (**Figure 3** and **Figure 4** and **Figure 8(c)**)

$$\boldsymbol{UniformSpace}(x_1, x_2, x_3, t) = \sum\nolimits_k \boldsymbol{AQuantum}(x_1, x_2, x_3, t, a, \rho, c_l). \tag{6.12}$$

Friedmann solution for expanding spatially homogeneous universe with metric [13] [14] [15] [16]

$$\mathrm{d}s^2 = a(t)^2 \mathrm{d}\bar{s}^2 - c^2 \mathrm{d}t^2; \mathrm{d}\bar{s}^2 = \mathrm{d}r^2 + S_k(r)^2 \mathrm{d}\Omega^2 \tag{6.13}$$

includes analytic function $S_k(r)$ representable via Atomic Series (5.4), (5.5) as per Theorems 3.4:

$$\begin{aligned} S_k(r) &= r\,sinc\left(r\sqrt{k}\right) = r - \frac{kr^3}{6} + \frac{kr^5}{120} - \cdots \\ &= \sum\nolimits_k c_k up\left(\frac{r - b_k}{a}\right) = \sum\nolimits_k AString(r, a_k, b_k, c_k). \end{aligned} \tag{6.14}$$

Scale factor $a(t)$ [13] [14] [15] [16] being an analytic power function [29] is also representable via Atomics:

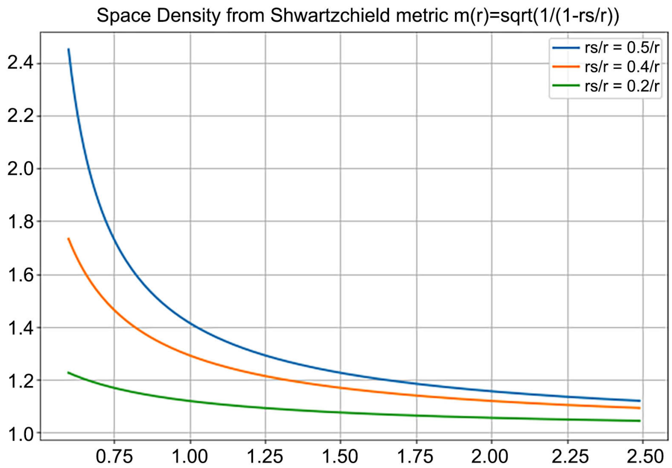
$$a(t) = a_0 t^{\frac{2}{3(w+1)}}; a(t) \sim t^{2/3}, w = 0; a(t) \sim t^{1/2}, w = 1/3, \tag{6.15}$$

$$a(t) = \sum\nolimits_k up(t, a_k, b_k, c_k) = \sum\nolimits_l AString(t, a_l, b_l, c_l). \tag{6.16}$$

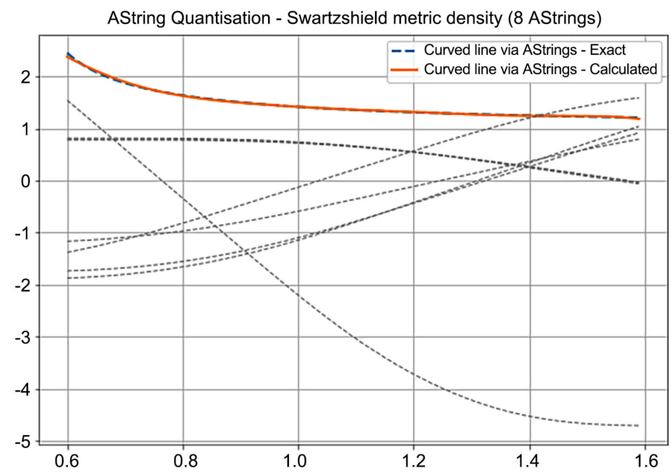
Schwarzschild solution (**Figure 8**) for radial bodies and black holes has spacetime metric [13] [14] [15] [16] [20]

$$\begin{aligned} &\mathrm{d}s^2 = -A(r)c^2 \mathrm{d}t^2 + B(r)\mathrm{d}r^2 + r^2 \mathrm{d}\Omega; \\ &A(r) = \left(1 - \frac{r_s}{r}\right); \; B(r) = \left(1 - \frac{r_s}{r}\right)^{-1}. \end{aligned} \tag{6.17}$$

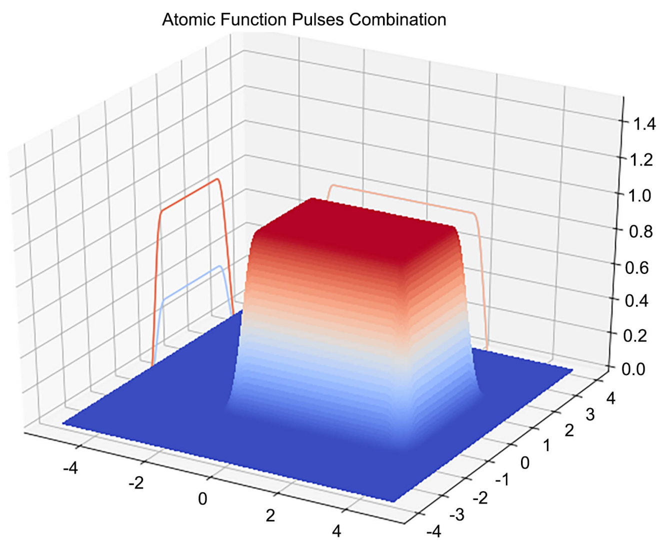
Analytic (outside of singularity) function $A(r)$ and its reciprocal $B(r)$ (also analytic as per Theorem 3) representable via converging Taylor's series is also representable via Atomics (5.4):



(a)



(b)



(c)

Figure 8. (a) Space density function from Schwarzschild GR solution; (b) Representing Schwarzschild metric via AStrings; (c) Uniform spacetime field as a superposition of solitonic atoms.

$$A(r) = \sum_k c_k up\left(\frac{r-b_k}{a}\right) = \sum_k AString\left(r, a_k, b_k, c_k\right), r \neq 0. \quad (6.18)$$

In summary, the atomization of known GR solutions confirms the main idea that analytic spacetime fields are representable via the superposition of finite AStrings and Atomic Functions, or Atomic Splines.

7. Atomic Splines vs Polynomial Splines and Discretization

In summary, formulated 13 Atomization Theorems provide a theoretical foundation for applying Atomic Series and Atomic Splines based on Atomic AString Functions to many theories of mathematical physics making the "atomization method" quite universal. It seems important to compare the method with traditional universal approaches, like polynomial spline approximations [22] [51] and finite elements discretization [23] [52].

7.1. Atomic Splines vs Polynomial Splines

Spline approximation, when a function is represented via the superposition of localized splines, is a universal method widely used in mathematical physics [21] [23] [51]. Typically, splines are based on polynomials [23] [51] (B-splines, Cubic Splines, Hermitian Splines), and there is a limitation problem with them—according to Strang-Fix condition [41] local polynomial splines of n-order can *exactly* represent/compose only a polynomial function up to n-order. For example, cubic splines can exactly compose only a cubic parabola, but not a polynomial of a higher order; it can *approximate* but not *exactly* represent. It means polynomial splines cannot be used for fundamental quantization formulations of spacetime and fields which should not be based on *approximations* of reality. Another limitation of polynomial splines is limited smoothness seemingly insufficient for fundamental theories. For example, building a spacetime quantization model based on cubic splines would imply that Einsteinian curvature tensor (6.7) based on second derivatives would be represented by linear functions leading to an unphysical unsmooth connection between curvature nodes. Increasing the order of basis splines leads to the so-called "polynomial trap" problem [1] [24] imposing artificial "polynomial order" constant to spacetime and field theories.

Atomic Splines based on smooth Atomic and AString Functions for which derivatives are expressed via the functions themselves (3.1), (3.8) are more advanced—they can exactly compose sections of polynomials of any order (Theorem 1, (4.1), (5.1), (§5.1) as well as provide smooth connections between Atomic Splines presumably modeling flexible finite quanta (§2, 6). However, this advancement comes with a price—nowadays, operating with polynomial splines seems easier than with relatively new nonanalytic Atomic Functions (3.1), (3.8). But the evolutional development of atomic methods can be further widened with multiple computer scripts [2] [4] [53] [54] available for Atomic and AString Functions, their elegant simplicity (**Figure 2** and **Figure 3**), and their convenient meaning of "mathematical atoms" [1]-[12] capable of composing complex functions.

7.2. Discretization vs Atomization

Another aspect to discuss is the difference between traditional universal *discretization* methods and proposed "*atomization*" methods. Space discretization is the foundation of the most universal numerical methods like finite differences and finite elements [22] [52]. However, the separation of a volume, for example, spacetime fields, into finite elements does not explain the crucial feature of how the finite elements/quanta are "kept together", interact, and provide smoothness and "pass information" between them. Space "atomization", like a partition of a line (§2, 6), is more advanced—it tells that neighboring "finite elements" have "interaction zones" (**Figure 1**) within which it is possible to provide the desired smooth connection between elements, like in Lego game when perfectly adjusted elementary blocks are kept together in interaction zones by "connection layers" (**Figure 1(a)**). Moreover, Atomization Theorems (§5, 6) tell how stretches and shifts of overlapped finite elements can compose various physical fields giving rise to a novel atomic model of spacetime and fields [1] [2] [3] [4] [5] [23] [43] [55]. However, "atomization" as an "advanced discretization" is more complex than traditional discretization and finite element methods [21] [22] [52] and hence requires not only grasping of Atomic Functions theory but also building new "atomic" or "collocated" finite element methods [26] [33] especially complex for 3D problems.

In summary, Atomic Splines provide a more adequate description of smooth quantized fields composed of flexible overlapping solitonic atoms but require more complex methods of discretization and spline constructions.

8. Conclusions—Atomization Theorems in Theoretical Physics and Future Research

Atomization Theorems provide a theoretical formalism for applying Atomic Functions known since the 1970s [6]-[12] and their AString generalizations introduced in 2017 [1] [2] [3] [4] [5] to many equations of mathematical physics, including electromagnetism, elasticity, hydrodynamics, soliton theory, spacetime physics, quantum mechanics, and field theories [16]-[28] [30] [33] [35] [36] [47]-[55]. The theorems tell that fields describable by widely used differential equations and analytic functions are representable via Atomic Splines which, unlike conventional polynomial splines, can exactly reproduce (rather than approximate) polynomials of any order and hence represent widely used analytic functions and their superpositions including Taylor power and Fourier series. The "atomization method" applies to both linear and nonlinear differential equations as complex as General Relativity (GR) equations.

The physical meaning of an Atomic Function pulse should be adjusted to every physical theory but for spacetime, it describes an elementary finite object/spacetime distortion/spacetime quantum/metriant capable to compose fields from the superposition of flexible overlapping "solitonic atoms" (Atomic Solitons [2] [4]) $c_i up\left(\left(x-b_i\right)/a_i\right)$ made of kink-antikink pair of two AStrings

$up(x) = AString(2x+1) - AString(2x-1)$. Overlapping translation of AStrings describes spacetime expansion while two opposite AStrings form "solitonic atoms" resembling matter particles making spacetime and matter distributions expressed via the same set of basis functions. Also, the resulting Atomic Spacetime theory [1] [2] [3] [4] [5] [23] [44] evolving since 2017 correlates with A. Einstein's 1933 paper [17] predicting a "*perfectly thinkable*", "*atomic theory*" with "*simplest concepts and links between them*" resolving "*stumbling blocks*" of theories operating "… *exclusively with continuous functions of space*" and mentioning a "… *region of three-dimensional space at whose boundary electrical density vanishes everywhere*" naturally leading to finite Atomic Function.

Atomization Theorems applied to field theories [16]-[28] [30] [33] [35] [36] [47] [48] [49] [50] may also yield some novel interpretations and future directions under research now [1] [23] [44] [55]. If many analytic functions, fields, and equations from different related theories are representable via finite Atomics, it suggests the hypothesis [1] [2] [3] [4] [5] that they may express some common mathematical block/atom of fields leading to a unique unified theory based on a common "physical ancestor" like a string from string theory [50], or "elementary spacetime distortion/ripple" [16] [18] [19] [20] [23] [46] [47]. In Quantum Field Theory [35] [36], where fields are perceived to be the "building blocks" of the universe [46] [47], Atomics may describe those elementary mathematical "blocks", or "mathematical atoms" as the founders called them [6] [7] [8], composing different fields, with the theory in research now.

From the Generalized Thermodynamics point of view, AString describes a metriant [35]—conservable extensor/quantum of spacetime field (§2). The meaning of nonlinear solitary atomic function $up(x)$ (3.8) as a superposition of two AString kinks offers connections to soliton theory [2] [4] [27] [30] with Atomic Solitons [1] [2] [3] [4] [5]. The probabilistic meaning of atomic functions [1]-[12] offers connections to Quantum Mechanics [40]. AStrings and Atomic Functions can also be used in Atomic Machine Learning [44].

In summary, some novel mathematical constructs like Atomic and AString Functions, Atomic Solitons, Atomic Series, and Atomization Theorems can be useful for many physical theories extending 50 years of history of Atomic Functions [1]-[12] to new scientific domains.

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Conflicts of Interest

The author declares no conflicts of interest regarding this paper.

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