Journal of Applied Mathematics and Physics, 2023, 11, 135-146
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Fermi Function and Its Applications

Asma A. Elahwel[1*], Najat A. ALjalali[2], Mariam Barbash[2], Ali M. Awin[2]

[1]Department of Mathematics, Faculty of Education, University of Tripoli, Tripoli, Libya
[2]Department of Mathematics, Faculty of Science, University of Tripoli, Tripoli, Libya
Email: *awinsus@yahoo.com

How to cite this paper: Elahwel, A.A., ALjalali, N.A., Barbash, M. and Awin, A.M. (2023) Fermi Function and Its Applications. *Journal of Applied Mathematics and Physics*, **11**, 135-146.
https://doi.org/10.4236/jamp.2023.111010

Received: December 21, 2022
Accepted: January 17, 2023
Published: January 20, 2023

Abstract

In this paper, we give a definition of the Fermi function, or the so-called Woods-Saxon potential, a well-known potential in nuclear physics; then, we give a few of its applications as examples. Some important integrals, which involve this function, are computed discussing the integrability and convergence of these integrals. Following, we derive formulae that encounter the above-mentioned function to get nuclear and generalized moments; the radial Fourier transformation is also exposed. Some related applications are then given that use such important integrals; in particular, we give the computation in conjunction with the problem of getting the optical-model potential for heavy-ion interactions at intermediate energies. Finally, we conclude with important remarks to do with the evolution of the subject.

Keywords

Fermi Function, Integrals, Optical Model, Potential, Nuclear Interactions, Convergence

1. Introduction

Fermi Function (FF) is one of the important functions used in nuclear and solid state physics; it is also called the Woods-Saxon function (or potential) and it is interpreted as a probability of occupation of energy levels by electrons at a certain temperature subject to thermo-dynamical equilibrium conditions [1].

Note that the motive behind writing this paper is the importance of the subject of the Fermi function due to the vast number of applications of it that are used in physics. However, we will concentrate, here, on the mathematical details involved with this function [1] [2].

Hence, in the next section, we give the definition of the Fermi function and some of its important properties, and its relation with Fermi-Dirac statistics; moreover, we discuss the method of its derivation [3] [4].

In Section 3, based on the three-parameter Fermi function, important integrals are presented. The three-parameter FF is written in terms of a power series using the binomial theorem. The convergence of this power series is discussed and the possibility of evaluating some integrals term-by-term is presented [1].

The nuclear moments are evaluated in Section 4 where integration by parts is used with the Riemann-Zeta function to get a final result. In Section 5, generalized moments are evaluated, followed by the use of radial Fourier transform with the three-parameter FF; and in Section 6, we present a few applications which include the method of obtaining the optical-model potential using inverse scattering and Fermi energy for metals [5] [6] [7].

In Section 7, symmetrized FF is presented and studied [8]. Finally, we give a concluding discussion in the last section.

2. Fermi Function

The Probability Function (PF) which describes the thermo-dynamical behavior in quantum mechanics for a group of particles is called the Fermi function and is given by

$$f(E) = \frac{1}{1+e^{(E-E_f)/KT}} \tag{1}$$

where $f(E)$ is the PF for the existence of states with energy E occupied by an electron at temperature T; K and E_f are Boltzmann constant and Fermi Energy (FE) respectively [1]. **Figure 1** shows FF at $T = 0$ and $T > 0$.

Not that for $|E - E_f| \gg KT$, $f(E)$ can be written as

$$f(E) \cong e^{-(E-E_f)}/KT \tag{2}$$

FF is related to Fermi-Dirac distribution which is used to study the behavior of electrons in metals, where the average number of these particles in the s^{th} state while diffusing is given by

$$n_s = \frac{1}{1+e^{\beta(E_s-E_f)}} \tag{3}$$

E_f is FE and can be calculated from the condition

$$\sum_s n_s = \sum_s \frac{1}{1+e^{\beta(E_s-E_f)}} = N \tag{4}$$

N is the number of particles in the volume V.

Equation (4) implies that E_f is a function of temperature [3].

Using Equation (1), where $E \geq 0$, and considering Maxwell-Boltzmann distribution which is [1] [3]

$$n_s = N\frac{e^{-\beta E_s}}{\sum_r e^{-\beta E_r}} \tag{5}$$

and if $E \gg E_f$, we obtain

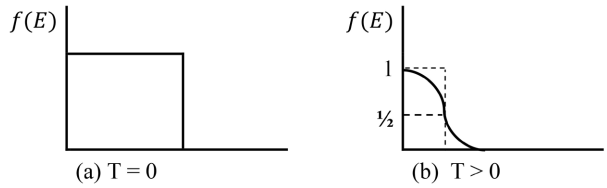

Figure 1. Fermi function at $T = 0$ (a) and $T > 0$ (b).

$$f(E) \cong \exp\left[\beta\left(E - E_f\right)\right] \tag{6}$$

While if $E = E_f$, then $f(E) = \frac{1}{2}$ [1]. Note that $f(E)$ can be derived on the basis of studying elastic scattering of two electrons with the same interpretation that $f(E)$ represents the probability of having the electron at a quantum state with energy E [1].

3. Three-Parameter FF and Important Integrals

The three-parameter FF is written as

$$f(r, R, a, w) = \frac{\rho_0\left(1 + wr^2/R^2\right)}{1 + \exp\left[(r - R)/a\right]} \tag{7}$$

where the three parameters are w, R, and r [1].

If $w = 0$, then we get FF with the two parameters r and R; R is taken as the density radius and a is determined from the surface thickness t given by $t = 4a\ln 3$ [1].

Now, consider the integral I where

$$I = \int_0^\infty \frac{f(r)\mathrm{d}r}{1 + \exp\left[(r - R)/a\right]} \tag{8}$$

where $f(r)$ is an analytic function. From Equation (8), we see that

$$I = \int_0^R \frac{f(r)\mathrm{d}r}{1 + \exp\left[(r - R)/a\right]} + \int_R^\infty \frac{f(r)\mathrm{d}r}{1 + \exp\left[(r - R)/a\right]} \tag{9}$$

I can, also, be written as

$$I = \int_0^R \frac{f(r)\mathrm{d}r}{1 + \exp\left[(r - R)/a\right]} + \int_R^\infty \frac{f(r)\exp\left[(R - r)/a\right]\mathrm{d}r}{1 + \exp\left[(R - r)/a\right]} \tag{10}$$

Using the binomial theorem, we obtain

$$\left(1 + \exp\left[(r - R)/a\right]\right)^{-1} = \sum_{m=0}^\infty (-1)^m \mathrm{e}^{mr/a}\mathrm{e}^{-mR/a} \tag{11}$$

Similarly, we have

$$\left(1 + \exp\left[(R - r)/a\right]\right)^{-1} = \sum_{m=0}^\infty (-1)^m \mathrm{e}^{mR/a}\mathrm{e}^{-mr/a} \tag{12}$$

Hence, we obtain

$$\frac{\mathrm{e}^{(R-r)/a}}{1 + \exp\left[(R - r)/a\right]} = \sum_{m=0}^\infty (-1)^m \mathrm{e}^{(m+1)R/a}\mathrm{e}^{-(m+1)r/a} \tag{13}$$

From which and from Equation (10) I is written as

$$I=\int_0^R\sum_{m=0}^{\infty}(-1)^m \mathrm{e}^{mr/a}\mathrm{e}^{-mR/a} f(r)\mathrm{d}r+\int_R^{\infty}\sum_{m=0}^{\infty}\mathrm{e}^{(m+1)R/a}\mathrm{e}^{-(m+1)r/a} f(r)\mathrm{d}r \quad (14)$$

The obtained series in Equation (14) does not satisfy the strong and sufficient condition for uniform convergence and this implies that we cannot integrate the series term-by-term [1]. However, it is possible to get some functions and series which satisfy the weak convergence criteria so as term-by-term integration can be attained.

Taking into account the functions $f_m(r)$, we get a monotonic decreasing series for $r\neq R$ for the sum $s_r=\sum_0^{\infty} f_m(r)$ which is bounded on the intervals $[0,R)$ and (R,∞); and since $\lim_{m\to\infty} f_m(r)=0$ and that the terms are consecutively with alternate signs, then the series will converge pointwise and from Lebesgue theorem for convergence, we have: if $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in $L[a,b]$ such that $\lim_{n\to\infty} f_n(x)=f(x)$ almost everywhere in $[a,b]$ and if $g\in L[a,b]$ such that $|f_n(x)|\le g(x)$ almost everywhere $(a\le x\le b; n\in I)$, then

$$\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x;\ f(x)\in L[a,b] \quad (15)$$

and integration term-by-term is possible; although some difficulty may arise [1] [5].

In the next section, we will consider the cases $f(r)=r^n, r^n\mathrm{e}^{-\alpha r}, \sin(qr), \cos(qr)$; where n is a positive integer, $\alpha>0$, and q is the parameter in Fourier transforms.

The infinite series then in Equation (14) may converge and can be integrated on the domains $[0,R]$ and $[R,\infty)$ [1].

4. Evaluation of Important Integrals

4.1. Integrals with $f(r)=r^n$

Considering the integral in Equation (14) with $f(r)=r^n$, we obtain

$$\begin{aligned} I_n={}&\frac{R^{n+1}}{n+1}+\sum_{k=1}^{\infty}(-1)^k \mathrm{e}^{-kR/a}\int_0^R r^n\mathrm{e}^{kr/a}\mathrm{d}r\\ &+\sum_{k=0}^{\infty}(-1)^k \mathrm{e}^{(k+1)R/a}\int_R^{\infty} r^n\mathrm{e}^{-(k+1)r/a}\mathrm{d}r \end{aligned} \quad (16)$$

Using integration by parts, we get

$$\begin{aligned} I_n={}&\frac{R^{n+1}}{n+1}-\sum_{m=0}^{\infty}\frac{(-1)^m n!a^{m+1}R^{n-m}}{(n-m)!}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{m+1}}\\ &-\sum_{k=1}^{\infty}\frac{(-1)^k n!\mathrm{e}^{-kR/a}(-1)^n a^{n+1}}{k^{n+1}}+\sum_{k=1}^{\infty}(-1)^{k+1}\sum_{m=0}^{n}\frac{n!a^{m+1}R^{n-m}}{k^{m+1}(n-m)!} \end{aligned} \quad (17)$$

With few mathematical manipulations, Equation (17) can be written as

$$\begin{aligned} I_n={}&\frac{R^{n+1}}{n+1}-\sum_{k=1}^{\infty}\frac{(-1)^{k+n} n!a^{n+1}\mathrm{e}^{-kR/a}}{k^{n+1}}\\ &+\sum_{m=0}^{n}\left[1-(-1)^n\right]\left(1-2^{-m}\right)^{-1}\frac{a^{m+1}R^{n-m}n!\zeta(m+1)}{(n-m)!} \end{aligned} \quad (18)$$

where $\zeta\left(m+1\right)$ is Riemann-Zeta function given by

$$\zeta\left(\mathcal{Z}\right)=\left(1-2^{1-}\right)\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k+1}}{K^{\mathcal{Z}}};Re\mathcal{Z}>0 \tag{19}$$

Now, to apply these results on the two-parameter Fermi function, we see that

$$\begin{aligned} D_{n,\beta} &\equiv \int\frac{r^{n}\mathrm{d}r}{1+\exp\left[\left(r-R\right)/a\right]^{\beta}} \\ &=\frac{R^{n+1}}{n+1}+n!a^{n+1}\sum_{\nu=0}^{\infty}\frac{x^{n-\nu}}{\left(n-\nu\right)!}\lambda_{\nu}\left(\beta\right) \\ &\quad+n!a^{n+1}\left(-1\right)^{n}\sum_{\nu=0}^{\infty}\frac{\Gamma\left(\beta+\nu\right)\left(-1\right)^{\nu-1}}{\Gamma\left(\beta\right)\nu!\nu^{n+1}}\mathrm{e}^{-\nu x} \end{aligned} \tag{20}$$

With $x=R/a$ and

$$\lambda_{\nu}\left(\beta\right)=\frac{\left(-1\right)^{\nu}}{\nu!}\int_{0}^{\infty}t^{\nu}\left[\frac{1+\left(-1\right)^{\nu}\mathrm{e}^{-\beta t}}{\left(1+\mathrm{e}^{-t}\right)\beta}-1\right]\mathrm{d}t \tag{21}$$

Putting $\beta=1$, we get the result in Equation (18) [6].

Note that one can neglect the infinite sum, except for the first few terms. e.g. in the case of oxygen atom (O^{16}) and if we take $n=2$ with $k=2.0608$ fm and $a=0.513$, then neglecting high order terms will lead to an error equal to 0.02% only [6].

Furthermore, to get the mean square-radius $\left\langle r^{2}\right\rangle$, we see that

$$\left\langle r^{2}\right\rangle=4\pi\rho_{0}\int_{0}^{\infty}\frac{r^{2}\left(1+\omega r^{2}/R^{2}\right)r^{2}\mathrm{d}r}{1+\exp\left[\left(r-R\right)/a\right]} \tag{22}$$

Where we took the three-parameter Fermi function into consideration. Using Equation (18), we obtain

$$\left\langle r^{2}\right\rangle=4\pi\rho_{0}\left[I_{4}+\frac{\omega}{R^{2}}I_{6}\right] \tag{23}$$

with

$$\rho_{0}=4\pi\int_{0}^{\infty}\frac{\left(1+\omega r^{2}/R^{2}\right)r^{2}\mathrm{d}r}{1+\exp\left[\left(r-R\right)/a\right]} \tag{24}$$

Which, also, with the use of Equation (18) becomes

$$\rho_{0}=\left[4\pi\left[I_{2}+\frac{\omega}{R^{2}}I_{4}\right]\right]^{-1} \tag{25}$$

Hence, from Equation (23) and Equation (25), we get

$$\left\langle r^{2}\right\rangle=\frac{I_{4}+\frac{\omega}{R^{2}}I_{6}}{I_{2}+\frac{\omega}{R^{2}}I_{4}} \tag{26}$$

Going back to Equation (18) and keeping few terms in the infinite sum with the knowledge that $\zeta\left(1\right)=0$ and $\zeta\left(2\right)=\pi^{2}/6$, one gets

$$I_2 \cong \frac{R^2}{3} + \frac{1}{3} a^2 R \pi^3 \tag{27}$$

In the same manner, I_4 and I_6 are calculated so as to get

$$I_4 \cong \frac{R^5}{5} + \frac{2}{3} a^2 R^3 \pi^2 + \frac{7}{15} R a^4 \pi^4 \tag{28}$$

and

$$I_6 \cong \frac{R^7}{7} + a^2 R^5 \pi^2 + \frac{7}{3} R^3 a^4 \pi^4 + \frac{31}{21} R a^6 \pi^6 \tag{29}$$

Therefore, we have

$$\langle r^2 \rangle \cong \frac{3}{5} R^2 + \frac{7}{5} a^2 \pi^2 + \cdots \tag{30}$$

This equation has its impact on the relation between the mass number A and radius R [6].

4.2. Integrals with $f(r) = r^n e^{-\alpha r}$

Here, we consider the function $f(r) = r^n e^{-\alpha r}$, where α is a parameter such that $0.04\ \text{fm}^{-1} \le \alpha \le 0.15\ \text{fm}^{-1}$, then

$$I_n = \int_0^\infty \frac{r^n e^{-\alpha r} \mathrm{d}r}{1 + \exp\left[(r-R)/a\right]} \tag{31}$$

On the same lines followed in the last section and with a number of steps of integration, we reach the result

$$\begin{aligned} I_n = {} & e^{-\alpha R} \sum_{m=0}^{n} \frac{(-1)^m n! R^{n-m}}{(n-m)!} \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(\frac{k}{a} - \alpha\right)^{m+1}} \\ & + e^{-\alpha R} \sum_{m=0}^{n} \frac{n! R^{n-m}}{(n-m)!} \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(\alpha + \frac{(k+1)}{a}\right)^{m+1}} \\ & - \sum_{k=0}^{\infty} \frac{(-1)^{k+n} n! e^{-kR/a}}{\left(\frac{k}{a} - \alpha\right)^{n+1}} \end{aligned} \tag{32}$$

Introducing the poly gamma function defined as

$$\psi^n(\mathcal{Z}) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+\mathcal{Z})^n}; \mathcal{Z} \ne 0, -1, -2, \cdots \tag{33}$$

And from Equation (32), we see that I_n can be obtained, finally, as

$$\begin{aligned} I_n = {} & e^{-\alpha R} \sum_{m=0}^{n} \frac{n! R^{n-m}}{(n-m)!} \left[\frac{a^{m+1}}{m!} \left(\psi^m(-a\alpha) - 2^{-m} \psi^m\left(\frac{-a\alpha}{2}\right) \right) \right] \\ & + e^{-\alpha R} \sum_{m=0}^{\infty} \frac{n! R^{n-m}}{(n-m)!} \left[\frac{(-1)^{m+1} a^{m+1}}{m!} \left(\psi^m(a\alpha) - 2^{-m} \psi^m\left(\frac{a\alpha}{2}\right) \right) + \frac{1}{a^{m+1}} \right] \end{aligned} \tag{34}$$

Note that the poly gamma function can be written in terms of Riemann-Zeta and gamma functions as

$$\psi^{m}(\mathcal{Z}) = (-1)^{m+1}\zeta(n+1,\mathcal{Z})\Gamma(n+1) \tag{35}$$

Moreover, we observe that the lowest terms in the infinite series in Equation (32) are neglected [5].

5. Integrals with $f(r)=\sin(qr)$ and $f(r)=\cos(qr)$

In this section, we consider the radial Fourier transform of the three-parameter Fermi function, i.e. we calculate the integral

$$\begin{aligned} F(q) &= \frac{4\pi}{q}\int_0^\infty \sin(qr) f(r,R,a,w) r\mathrm{d}r \\ &= \frac{4\pi}{q}\int_0^\infty \frac{\rho_0\left(1+wr^2/R^2\right)\sin(qr) r\mathrm{d}r}{1+\exp\left[(r-R)/a\right]} \end{aligned} \tag{36}$$

Assume that

$$\eta(q) = -\frac{\mathrm{d}}{\mathrm{d}q}\int_0^\infty \frac{\cos(qr) r\mathrm{d}r}{1+\exp\left[(r-R)/a\right]} \tag{37}$$

then

$$\eta(q) = \int_0^\infty \frac{\sin(qr) r\mathrm{d}r}{1+\exp\left[(r-R)/a\right]} \tag{38}$$

In the same manner, we can calculate $\frac{\mathrm{d}\eta}{\mathrm{d}q}$ and $\frac{\mathrm{d}^2\eta}{\mathrm{d}q^2}$; with the obtained results and from Equation (36), one gets

$$F(q) = \frac{4\pi}{q}\left[\eta(q) - \frac{\omega}{R^2}\frac{\mathrm{d}^2\eta}{\mathrm{d}q^2}\right] \tag{39}$$

Now, if $I(q)$ is given by

$$I(q) = \int_0^\infty \frac{\cos(qr)\mathrm{d}r}{1+\exp\left[(r-R)/a\right]} \tag{40}$$

Then

$$\begin{aligned} I(q) &= \int_0^R \sum_{k=0}^\infty (-1)^k \mathrm{e}^{-kR/a}\mathrm{e}^{kr/a}\cos(qr)\mathrm{d}r \\ &+ \int_R^\infty \sum_{k=0}^\infty (-1)^k \mathrm{e}^{(k+1)R/a}\mathrm{e}^{-(k+1)r/a}\cos(qr)\mathrm{d}r \end{aligned} \tag{41}$$

Integrating by parts and following similar steps as before, we obtain

$$I(q) = \frac{a\pi\sin(kR)}{\sinh(a\pi q)} - \sum_{k=1}^\infty \frac{(-1)^k ak\mathrm{e}^{-kR/a}}{a^2q^2+k^2} \tag{42}$$

And hence, $\eta(q)$ will be given by

$$\eta(q) = \frac{-a\pi\cos(kR)}{\sinh(a\pi q)} + (a\pi)^2\frac{\coth(a\pi q)}{\sinh(a\pi q)}\sin(qR) + 2a^3q\sum_{k=1}^\infty \frac{(-1)^k k\mathrm{e}^{-kR/a}}{\left(a^2q^2+k^2\right)^2} \tag{43}$$

Similarly, we see that if

$$F(q)=\frac{4\pi}{q}\int_0^\infty \sin(qr)\, f(r,R,a,w)\, r\mathrm{d}r \tag{44}$$

Then, defining $I(q)$ as

$$I(q)=\int_0^\infty \frac{\cos(qr)\mathrm{d}r}{\left\{1+\exp\left[(r-R)/a\right]\right\}^2} \tag{45}$$

and carrying out a number of mathematical calculations, we get

$$\begin{aligned}I(q)=&\frac{-a}{1-\mathrm{e}^{-R/a}}-a\pi\left[\frac{aq\cos(qR)-\sin(qR)}{\sinh(a\pi q)}\right]\\&+aq^2\sum_{k=0}^{\infty}\frac{(-1)^k\mathrm{e}^{-kR/a}}{(k/a)^2+q^2}+\sum_{k=1}^{\infty}\frac{(-1)^k(k/a)\mathrm{e}^{-kR/a}}{(k/a)^2+q^2}\end{aligned} \tag{46}$$

Noting that

$$F(q)=\frac{-4\pi\rho_0^2}{q}\frac{\mathrm{d}}{\mathrm{d}q}\left(I(q)\right) \tag{47}$$

we get [6]

$$\begin{aligned}F(q)=&\frac{-4\pi\rho_0^2}{q}\frac{\sin(qR)}{\sinh(a\pi q)}\left(\pi a^2\right)\left[qR-\pi\coth(a\pi q)\right]\\&-\frac{4\pi\rho_0^2}{q}\frac{\cos(qR)}{\sinh(a\pi q)}(\pi a)\left[(R-a)-\pi qa^2\coth(a\pi q)\right]\\&+8\pi\rho_0^2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(k+1)\mathrm{e}^{-kR/a}}{\left[(k/a)^2+q^2\right]^2}\end{aligned} \tag{48}$$

6. Few Applications

In this section, we give few applications using Fermi function as follows.

6.1. Fermi Function and the Optical-Model Potential

Using Glauber eiquonal approximation, the elastic scattering $\mathcal{S}$-matrix is written as

$$S(b)=\exp\left[i\chi(b)\right] \tag{49}$$

where b is the impact parameter and the phase shift χ is

$$\chi(b)=-\frac{k}{E}\int_0^\infty \frac{rU(r)}{\sqrt{r^2-b^2}}\mathrm{d}r \tag{50}$$

$U(r)$ is the optical-model potential [7]. $\chi(b)$ has a real and an imaginary parts; the same thing with $U(r)$, where $U(r)=V(r)+iW(r)$; V and W are real. This means that Equation (50) can be decomposed into the two equations

$$\chi_R(b)=-\frac{k}{E}\int_0^\infty \frac{rV(r)}{\sqrt{r^2-b^2}}\mathrm{d}r \tag{51}$$

and

$$\chi_I(b) = -\frac{k}{E}\int_0^\infty \frac{rW(r)}{\sqrt{r^2-b^2}}\mathrm{d}r \tag{52}$$

These two equations are of the Abel type and the inverse solutions to them are

$$V(r) = \frac{2E}{k\pi}\cdot\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^\infty \frac{\chi_R(b)b}{\sqrt{r^2-b^2}}\mathrm{d}b \tag{53}$$

and

$$W(r) = \frac{2E}{k\pi}\cdot\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^\infty \frac{\chi_I(b)b}{\sqrt{r^2-b^2}}\mathrm{d}b \tag{54}$$

Note that V and W are given in a general form [7].

The matrix elements S_l are written as

$$S_l = |S_l|\exp(2iS_l) \tag{55}$$

where $|S_l|$ and the phase sifts S_l are given by

$$|S_l| = \left[l+\exp\left(\frac{l_0-l}{a}\right)\right]^{-1} \tag{56}$$

with

$$|S_l| = \frac{\mu}{1+\exp\left[(l_0-l)/a\right]} \tag{57}$$

and

$$i\chi(b) = \ln|S_l| + 2iS_l \tag{58}$$

l_0 is the angular momentum and a is the amplitude.

From the last two equations, we obtain

$$\chi_R(b) = \frac{2i\mu}{1+\exp\left[\frac{b-b_0'}{a}\right]} \tag{59}$$

and

$$\chi_I(b) = \ln\left[1+\exp\left(\frac{b_0-b}{a}\right)\right] \tag{60}$$

$b_0 = R_0$ and $kb = l+\frac{1}{2}$ [1].

Writing $\chi_R(b)$ and $\chi_I(b)$ in series forms as

$$\chi_R(b) = 2\mu\sum_{n=1}^{N} c_n \exp\left(\frac{-nb^2}{\alpha^2}\right) \tag{61}$$

and

$$\chi_I(b) = \sum_{n=1}^{N} b_n \exp\left(\frac{-nb^2}{\beta^2}\right) \tag{62}$$

From Equation (61), Equation (62), and Equation (54), we get

$$V(r) = \frac{4\mu E}{k\pi\alpha} \sum_{n=1}^{N} c_n \sqrt{\pi} \exp\left(\frac{-nr^2}{\alpha^2}\right) \tag{63}$$

and

$$W(r) = \frac{-2\mu E}{k\pi\beta} \sum_{n=1}^{N} b_n \sqrt{\pi} n \exp\left(\frac{-nr^2}{\beta^2}\right) \tag{64}$$

Also, $V(r)$ and $W(r)$ can be written in terms of Woods-Saxon functions as

$$V(r) = \frac{-4\pi E}{h\pi\alpha} f(R', \Delta', r) \tag{65}$$

and

$$W(r) = \frac{-2E}{k\pi\beta} f(R, \Delta, r) \tag{66}$$

where

$$f(x_0, a, x) = \frac{1}{1 + \exp\left[(x - x_0)/a\right]} \tag{67}$$

Which is Fermi function [1] [7].

For more details on these calculations, and their use in obtaining optical-model potential for heavy-ion collisions at intermediate energies, one can refer to Reference [7].

6.2. Fermi Energy for Metals

Fermi energy is the maximum value of energy attained by the electron. This happens when the level is fully occupied by electrons. Note that the number of conduction electrons per unit volume n is given by

$$\frac{\mathrm{d}n}{\mathrm{d}E} = \frac{8\sqrt{2}\pi m^{3/2}}{h^3} \frac{\sqrt{E}}{\mathrm{e}^{(E - E_f)/Kt} + 1} \tag{68}$$

and hence, n can be obtained as [1]

$$n = \int_0^\infty \frac{\mathrm{d}n}{\mathrm{d}E} \mathrm{d}E = \frac{8\sqrt{2}\pi m^{3/2}}{h^3} \int_0^\infty \frac{\sqrt{E}\mathrm{d}E}{\mathrm{e}^{(E - E_f)/Kt} + 1} = \frac{8\sqrt{2}\pi m^{3/2}}{h^3} \left(\frac{2}{3} E_f^{3/2}\right) \tag{69}$$

From which, we get

$$E_f = \frac{(hc)^2}{8mc^2} \left(\frac{3n}{\pi}\right)^{2/3} \tag{70}$$

7. Symmetrized Fermi Function

Fermi function is used in many areas in physics; one of these areas is in quantum dots which describes the states occupied by fermions, where $R = E_f$ and $d = K_\beta T$ when f is written as

$$f\left[(r - R)/d\right] = \frac{1}{1 + \mathrm{e}^{(r - R)/d}} \tag{71}$$

f in Equation (71) is normally used in nuclear physics and it is generally used to represent the density radius distribution or to describe the energy function for the fermion and R is the density radius while d is the surface density parameter [1].

The calculations, used in this case, require our acquaintance with symmetrized Fermi function defined as

$$\rho_s(r) = \frac{\sinh(R/d)}{\cosh(r/d) + \cosh(R/d)} = \rho_s\left(\frac{r}{d}, \frac{R}{d}\right) \tag{72}$$

For $R \gg d$, the symmetrized Fermi function is similar to the usual Fermi function [1].

8. Concluding Discussion

The Fermi function proved to be an important tool in calculating some important integrals which have their origin in physics; moreover, the Fermi function is of valuable use in obtaining optical-model potential by inversion in the case of heavy-ion collisions at intermediate energies [7].

One of the research articles written on the subject of the symmetrized Fermi function stresses the importance of the Fermi function and its usefulness in many fields of physics, from statistical to nuclear and astrophysics [8]. The article, also, pointed to the advantages of the symmetrized Fermi function, especially in the evaluation of some important integrals exactly.

Fermi function and its derivation from quantum field theory were presented and the benefits of this derivation were discussed in relation to the electromagnetic correction of beta decay [9]. Extra to its use in many fields of physics, the Fermi function is also useful in medical applications in the determination of myocardial blood flow and myocardial perfusion reserve [10].

In addition, the Fermi function is present in network structures, where its β-parameter can affect promoting information spreading on dynamic social networks [11].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Alahwal, A. (2005) Fermi Function and Important Integrals. MSc. Thesis, University of Tripoli, Tripoli, 1-76.

[2] Wert, C.A. and Thomson, R.M. (1964) Physics of Solids. 1st Edition, McGraw-Hill, New York.

[3] Reif, F. (1965) Fundamentals of Statistical and Thermal Physics. 1st Edition, McGraw-Hill, New York.

[4] Talley, H.E. and Dangherty, D.G. (1976) Physical Principles of Semiconductor Devices. 1st Edition, Iowa State University Press, Ames.

[5] Maung, K.M., Deutchman, P.A. and Royalty, W.D. (1989) Integrals Involving the

Three-Parameter Fermi Function. *Canadian Journal of Physics*, **67**, 95-99. https://doi.org/10.1139/p89-014

[6] Fernandez Velicia, F.J. (1987) On the Moments of a $(WS)^{\beta}$ Distribution. *Journal of Physics A*: *Mathematical and General*, **20**, 2293-2306. https://doi.org/10.1088/0305-4470/20/9/017

[7] Fayyad, H.M., Rihan, T.H. and Awin, A.M. (1996) Inversion Solution to Heavy-Ion Optical Model Potential at Intermediate Energies. *Physical Review* C, **53**, 2334-2340. https://doi.org/10.1103/PhysRevC.53.2334

[8] Sprung, D.W.L. and Martorell, J. (1997) The Symmetrized Fermi Function and Its Transforms. *Journal of Physics A*: *Mathematical and General*, **30**, 6525-6534. https://doi.org/10.1088/0305-4470/30/18/026

[9] Matsuzahi, A. and Tanaka, H. (2014) The Fermi Function beyond the Second Order Perturbation Theory. Arxiv: 1310.4082v3.

[10] Yamada, A., Ishida, M., Ichihara, T., Natsuma, T., Tsuge, S., Goto, Y., Uno, M., Nagata, M., Ichikawa, Y., Kitagawa, K. and Sakuma, H. (2016) Fermi Function Constrained Deconvolution Underestimates Myocardial and Myocardial Perfusion Reserve Regardless of Saturation Correction of Arterial Input Curve. *Journal of Cardiovascular Magnetic Resonance*, **18**, Article No. P87. https://doi.org/10.1186/1532-429X-18-S1-P87

[11] Ally, A. and Zhang, N. (2017) Effects of Network Structures and Fermi Function's Parameter β in Promoting Information Spreading on Dynamic Social Networks. *The International Arab Journal of Information Technology*, **14**, 267-273.