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Hölder Derivative of the Koch Curve

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Abstract

In this paper, we introduce a K Hölder *p*-adic derivative that can be applied to fractal curves with different Hölder exponent K. We will show that the Koch curve satisfies the Hölder condition with exponent $\frac{\ln 3}{\ln 4}$ and has a 4-adic arithmetic-analytic representation. We will prove that the Koch curve has exact $\frac{\ln 3}{\ln 4}$-Hölder 4-adic derivative.

Keywords

Fractal, Koch Curve, Hölder Inequality, Hölder Derivative

1. Introduction

Fractals are extremely non-smooth sets. It is very difficult to define derivatives on fractals. Kigami [1] said "Since fractals like the Sierpinski gasket and Koch curve do not have any structures, to define differential operator like the Laplacian is not possible from the classical viewpoint of analysis. To overcome such difficulty is a new challenge in mathematics".

In fact, Kigami made it open that the problem of finding the derivative of a fractal was a very difficult one. In this work, we will focus on the everywhere continuous and nowhere differentiable Koch curve in our series of studies of the "derivative" of fractals. It is known that one characteristic of continuous but not differentiable functions $f(x)$ is for some $\alpha, 0<\alpha<1$, the Hölder equality is satisfied

$$\left|f(x+h)-f(x)\right|<c|h|^{\alpha}. \quad (1)$$

For example, Hardy proved [2] that for Weierstrass function

$$f(x)=\sum_{n=0}^{\infty} b^n \cos\left(\alpha^n \pi x\right), b>1, a\geq b.$$

(1) is satisfied for $\alpha=\frac{\ln b}{\ln a}$. Yang *et al.* [3] [4] [5] also proved (1) for the Koch curve, the Levy curve, and the Kiesswetter functions respectively with different α values.

Traditional differentiability can be almost considered as a result of (1) for $\alpha=1$, *i.e.* the Lipschitz inequality holds. Naturally, one guesses that the α-Hölder derivative of fractals should be

$$\lim_{y\to x}\frac{f(y)-f(x)}{|y-x|^{\alpha}}.$$

Besicovitch was one of the mathematicians who were interested in this [6]. However, there has not been any example of fractal that possesses such Hölder derivative. Thomson introduced [7] the concept of upper α-Hölder derivative, proposed as

$$D^{\alpha}(f,x)=\overline{\lim_{\substack{y,z\to x\\ y<x<z}}}\sup\frac{f(z)-f(y)}{(z-y)^{\alpha}}.$$

There are many generalizations of traditional derivatives, but none of which is helpful for our problem. Based on the dyadic derivatives [8] [9] introduced by Kahane and Pecguement respectively, Fu [10] proved the Cantor function on the Cantor set had ternary Hölder derivative with $\alpha=\frac{\ln 2}{\ln 3}$:

$$\lim_{n\to\infty}\frac{f\left(\frac{l_{n+1}}{2^n}\right)-f\left(\frac{l_n}{2^n}\right)}{\frac{l_{n+1}-l_n}{2^n}},\quad l=0,1,\cdots,3^n-1.$$

Even though the Cantor set is a special case with zero Lebesque measure, this result provides some helpful ideas.

There have not been many activities in the area of differentialbility of fractals. We have published some results not long ago [5]. Recently, Prodanov obtain differentiability of a continuous function in terms of fractals [11] and Scott analyzed differentiable Iterated Function Systems [12] with computer technology and Arif [13] analyzed a fractal-fractional derivative of a stress fluid through a porous medium. In this paper, we introduce an α-Hölder p-adic derivative that can be applied to fractal curves with different Hölder exponent α. As for the Koch curve, it should satisfy the Hölder condition with exponent $\frac{\ln 3}{\ln 4}$ and has a 4-adic arithmetic-analytic representation. In fact, in this article, we can prove that the Koch curve has an exact order of $\frac{\ln 3}{\ln 4}$-Hölder 4-adic derivative. The contents of this paper are organized as follows: in Section 2, the α-Hölder p-adic derivatives of fractals are defined; in Section 3, analytic properties of the Koch curve are analyzed and, in Section 4, the α-Hölder of the Koch curve are proven.

2. α-Hölder p-Adic Derivatives

In the p-adic system, the expansion of $t \in [0,1]$ is

$$t = \frac{t_1}{p} + \frac{t_2}{p^2} + \cdots + \frac{t_n}{p^n} + \cdots \qquad t_n \in \{0,1,\cdots,p-1\},$$

where p is a positive integer and t_n is the n-th digit of real number t in the decimal system. All the p-adic rational numbers

$$\frac{l}{p^m} = \frac{t_1}{p} + \frac{t_2}{p^2} + \cdots + \frac{t_m}{p^m} \quad l = 0,1,\cdots,p^m - 1, m = 1,2,\cdots$$

make up of a countable set D, which is a subset of $[0,1]$. So, a sequence of nested intervals $U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$ exists in the form of

$$U_n = \left(\frac{l_n}{p^n}, \frac{l_{n+1}}{p^n}\right)$$

such that $t = \bigcap_{n=1}^{\infty} U_n$.

Definition 2.1 *Let* $f(x)$ *be a function defined on* $[0,1]$, $t \in [0,1] \setminus D$ *and* $t \in \bigcap_{n=1}^{\infty} \left(\frac{l_n}{p^n}, \frac{l_{n+1}}{p^n}\right)$, *if*

$$\lim_{n\to\infty} \frac{f\left(\frac{l_{n+1}}{p^n}\right) - f\left(\frac{l_n}{p^n}\right)}{\left(\frac{l_{n+1} - l_n}{p^n}\right)^{\alpha}}$$

exists, then $f(x)$ is called α-Hölder p-adic differentiable at point t or simply $\alpha - p$ differentiable, and the derivative is denoted by $f'_{\alpha-p}(t)$, called Hölder-adic derivative.

We can further define the upper and lower Hölder adic derivative as

$$f'^{+}_{\alpha-p}(t) = \overline{\lim_{n\to\infty}} \frac{f\left(\frac{l_{n+1}}{p^n}\right) - f\left(\frac{l_n}{p^n}\right)}{\left(\frac{l_{n+1} - l_n}{p^n}\right)^{\alpha}},$$

$$f'^{-}_{\alpha-p}(t) = \underline{\lim_{n\to\infty}} \frac{f\left(\frac{l_{n+1}}{p^n}\right) - f\left(\frac{l_n}{p^n}\right)}{\left(\frac{l_{n+1} - l_n}{p^n}\right)^{\alpha}}.$$

It is easy to see that the α-Hölder p-adic derivatives have some simple properties:

1) If $f(x)$ is α-Hölder p-adic differentiable at t, λ is a real constant, then

$$\left(\lambda f(t)\right)'_{\alpha-p} = \lambda f'_{\alpha-p}(t).$$

2) If $f(x)$ and $g(x)$ are $\alpha - p$ differentiable at t, then

$$\left(f(t) + g(t)\right)'_{\alpha-p} = f'_{\alpha-p}(t) + g'_{\alpha-p}(t).$$

3) If $f(x)$ is $\beta-p-d$ differentiable at *t*, $0<\alpha<\beta<1$, then

$$f'_{\alpha-p}(t)=0.$$

If $f(x)$ is $\alpha-p-d$ differentiable at *t*, $f'_{\alpha-p}(t)\neq 0$, then

$$f'_{\beta-p}(t)=\pm\infty.$$

In particular, if $f(x)$ is differentiable at *t*, then $f(x)$ is $\alpha-p$ differentiable at *t*, and

$$f'_{\alpha-p}(t)=0,\quad f'_{1-p}(t)=f'(t).$$

If $f(x)$ is $\beta-p$ differentiable at *t* and $f'_{\beta-p}(t)\neq 0$, then

$$f'(t)=\pm\infty.$$

However, if $f(x)$ is $1-p$ differentiable at *t*, $f'(t)$ does not necessarily exist.

Definition 2.2 *Let* $t=\frac{l}{p_m}$ *be a p-adic rational number with* $m=1,2,\cdots$, $l=0,1,\cdots,p^m-1$, *if*

$$\lim_{n\to\infty}\frac{f\left(\frac{l}{p^m}+\frac{1}{p^n}\right)-f\left(\frac{l}{p^m}\right)}{\left(\frac{l}{p^n}\right)^{\alpha}}\left(or\ \lim_{n\to\infty}\frac{f\left(\frac{l}{p^m}\right)-f\left(\frac{l}{p^m}-\frac{1}{p^n}\right)}{\left(\frac{l}{p^n}\right)^{\alpha}}\right)$$

exists, then the limit is called the right (*resp. left*) *α-Hölder p-adic derivative of* $f(x)$ *at t and denote it by* $f'_{\alpha-p^+}(t)$ (*resp.* $f'_{\alpha-p^-}(t)$).

The left and right *α*-Hölder *p*-adic derivatives of $f(x)$ at *t* also have the similar properties as the *α*-Hölder *p*-adic derivatives of $f(x)$ at *t*.

3. The Analytic Properties of the Koch Curve

It is well known that the Koch curve is a typical example of fractal curves. Discussing its analytic properties is obviously important. Von Koch initially constructed the curve with a recursive way of using pure geometric descriptions [14]. Recently, works of analytic representation of the curve have made some progresses. An arithmetic-analytic representation based on 4-adic expansion is obtained [3], which will be used in our investigation in this paper.

Due to the geometric properties of Koch curve [15], the 4-adic expansion

$$t=\frac{c_1}{4}+\frac{c_2}{4^2}+\cdots+\frac{c_n}{4^n}+\cdots$$

is appropriate, where $c_n\ (n=1,2,\cdots)$ takes the value of 0, 1, 2 or 3. (6) can also be represented by the 2-adic form

$$t=\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_{2n-1}}{2^{2n-1}}+\frac{a_{2n}}{2^{2n}}+\cdots,$$

where $a_n\ (n=1,2,\cdots)$ takes values 0 or 1. Both c_n and a_n satisfy the relations

$$c_n = 2a_{2n-1} + a_{2n} = \begin{cases} 0 & \text{if } a_{2n-1} = a_{2n} = 0 \\ 1 & \text{if } a_{2n-1} = 0, a_{2n} = 1 \\ 2 & \text{if } a_{2n-1} = 1, a_{2n} = 0 \\ 3 & \text{if } a_{2n-1} = a_{2n} = 1 \end{cases}.$$

Lemma 3.1 [4] *Suppose the parametric equation of the Koch curve with argument* $t \in [0,1]$ *is*

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}.$$

Then, the arithmetic-analytic representation of Koch curve from the Iterated Function System (IFS) [16] is expressed as

$$\varphi(t) = \sum_{k=1}^{\infty} \frac{a_k}{\left(\sqrt{3}\right)^k} \cos b_k \frac{\pi}{6}, \quad (2)$$

$$\psi(t) = \sum_{k=1}^{\infty} \frac{a_k}{\left(\sqrt{3}\right)^k} \sin b_k \frac{\pi}{6}, \quad (3)$$

where $b_k = a_1' - a_2' + \cdots + a_{2i-1}' - a_{2i}' + \cdots + (-1)^k a_{k-1}' - (-1)^k$, $a_k' = 1 - 2a_k$.

It is worthy noticing that the arguments of sinusoidal and cosine function are standard symbolic sequences determined by the expansion coefficient $a_n(0,1)$ or $a_k'(-,+)$ of binary system. The sharp-angled vertices without derivative in almost everywhere can be completely described by the symbolic sequences. So instead of being directly determined by the value of *t*, (3.1), (3.2) are a special type of parametric functions defined by the binary system. The following propositions present some of their properties that will be used for later discussions.

Lemma 3.2 [4] *The Koch curve* $(\varphi(t), \psi(t))$ *satisfies the Hölder condition with exponent* $\alpha = \frac{\ln 3}{\ln 4}$, *i.e.*

$$\left|\varphi(t') - \varphi(t)\right| \le c\left|t' - t\right|^{\alpha}, \quad (4)$$

$$\left|\psi(t') - \psi(t)\right| \le c\left|t' - t\right|^{\alpha}, \quad c = const. \quad (5)$$

Lemma 3.3 *The arithmetic-analytic representation of the Koch curve* (2), (3) *is uniquely determined at a* 4-*adic rational point* $t = \frac{l}{4^m}$, $l = 0, 1, 2, \cdots, 4^m - 1$, $m = 0, 1, 2, \cdots$.

Proof. 1) If

$$t = \sum_{k=1}^{2m-2} \frac{a_k}{2^k} + \frac{0}{2^{2m-1}} + \frac{1}{2^m},$$

i.e. $a_{2m-1} = 0, a_{2m} = 1$, then from (2),

$$\varphi(t) = \sum_{k=1}^{\infty} \frac{a_k}{\left(\sqrt{3}\right)^k} \cos\left(a_1' - a_2' + \cdots + a_{2i-1}' - a_{2i}' + \cdots + (-1)^k a_{k-1}' - \left(-1^k\right)\right)\frac{\pi}{6}\Bigg),$$

where $a_k' = 1 - 2a_k$. So

$$\varphi(t) = \varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k} + \frac{1}{2^{2m}}\right)$$
$$= \varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k} + \frac{0}{2^{2m-1}} + \frac{1}{2^{2m}} + \frac{0}{2^{2m+1}} + \frac{0}{2^{2m+2}} + \cdots\right)$$
$$= \sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1' - a_2' + \cdots + (-1)^k a_{k-1}' - (-1)^k\right)\frac{\pi}{6}$$
$$+ \frac{1}{3^m}\cos\left(a_1' - a_2' + \cdots + a_{2m-3}' - a_{2m-2}'\right)\frac{\pi}{6}.$$

If t takes the other form:

$$t = \sum_{k=1}^{2m-2}\frac{a_k}{2^k} + \frac{a_{2m-1}}{2^{2m-1}} + \frac{a_{2m}-1}{2^{2m}} + \sum_{k=2m+1}^{\infty}\frac{1}{2^k}, \tag{6}$$

where $a_{2m-1} = a_{2m} - 1 = 0$, $a_{2m+1} = a_{2m+2} = \cdots = 1$, $a_{2m-1}' = 1$, $a_{2m}' - 1 = 1$, $a_{2m+1}' = a_{2m+2}' = \cdots = -1$, then

$$\varphi(t) = \varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k} + \frac{0}{2^{2m-1}} + \frac{0}{2^{2m}} + \sum_{k=2m+1}^{\infty}\frac{1}{2^k}\right)$$
$$= \sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1' - a_2' + \cdots + (-1)^k a_{k-1}' - (-1)^k\right)\frac{\pi}{6}$$
$$+ \frac{1}{3^m}\sum_{k=1}^{\infty}\left[\frac{1}{\left(\sqrt{3}\right)^{2k-1}}\cos\left(a_1' - a_2' + \cdots + a_{2m-3}' - a_{2m-2}' - (-1)^{2k-1}\right)\frac{\pi}{6}\right.$$
$$\left. + \frac{1}{\left(\sqrt{3}\right)^{2k}}\cos\left(a_1' - a_2' + \cdots + (-1)^{2k} a_{2k-1}' - (-1)^{2k}\right)\frac{\pi}{6}\right]$$
$$= \sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1' - a_2' + \cdots + (-1)^k a_{k-1}' - (-1)^k\right)\frac{\pi}{6}$$
$$+ \frac{1}{3^m}\sum_{k=1}^{\infty}\left[\frac{1}{\left(\sqrt{3}\right)^{2k-1}}\cos\left(a_1' - a_2' + \cdots + a_{2m-3}' - a_{2m-2}' + 1\right)\frac{\pi}{6}\right.$$
$$\left. + \frac{1}{\left(\sqrt{3}\right)^{2k}}\cos\left(a_1' - a_2' + \cdots + a_{2m-3}' - a_{2m-2}' - 2\right)\frac{\pi}{6}\right].$$

And from

$$\frac{1}{\sqrt{3}}\cos\left(a_1' - a_2' + \cdots + a_{2m-3}' - a_{2m-2}' + 1\right)\frac{\pi}{6}$$
$$+ \frac{1}{3}\cos\left(a_1' - a_2' + \cdots + a_{2m-3}' - a_{2m-2}' - 2\right)\frac{\pi}{6}$$
$$= \frac{2}{3}\cos\left(a_1' - a_2' + \cdots + a_{2m-3}' - a_{2m-2}'\right)\frac{\pi}{6},$$

we have:

$$\varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\sum_{k=2m+1}^{\infty}\frac{1}{2^k}\right)$$
$$=\sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1'-a_2'+\cdots+(-1)^k a_{k-1}'-(-1)^k\right)\frac{\pi}{6}$$
$$+\frac{1}{3^m}\left(\frac{2}{3}\sum_{k=1}^{\infty}\frac{1}{3^k}\right)\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'\right)\frac{\pi}{6}$$
$$=\varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\frac{1}{2^m}\right).$$

Fractal

2) If

$$t=\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\frac{1}{2^{2m-1}},$$

i.e. $a_{2m-1}=1, a_{2m}=0$, then from (2),

$$\varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\frac{1}{2^{2m-1}}\right)$$
$$=\sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1'-a_2'+\cdots+(-1)^k a_{k-1}'-(-1)^k\right)\frac{\pi}{6}$$
$$+\frac{1}{\left(\sqrt{3}\right)^{2m-1}}\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'+1\right)\frac{\pi}{6}.$$

If t takes the form of

$$t=\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\frac{a_{2m-1}-1}{2^{2m-1}}+\sum_{k=2m}^{\infty}\frac{1}{2^k},\quad \text{if } a_{2m}=0,$$

where $a_{2m-1}=0$, $a_{2m}=a_{2m+1}=\cdots=1$, $a_{2m-1}'-1=1$, $a_{2m}'=a_{2m+1}'=\cdots=-1$, then

$$\varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\sum_{k=2m}^{\infty}\frac{1}{2^k}\right)$$
$$=\sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1'-a_2'+\cdots+(-1)^k a_{k-1}'-(-1)^k\right)\frac{\pi}{6}$$
$$+\frac{1}{\left(\sqrt{3}\right)^{2m-1}}\left(\frac{2}{3}\sum_{k=0}^{\infty}\frac{1}{3^k}\right)\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'+1\right)$$
$$=\sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1'-a_2'+\cdots+(-1)^k a_{k-1}'-(-1)^k\right)\frac{\pi}{6}$$
$$+\frac{1}{\left(\sqrt{3}\right)^{2m-1}}\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'+1\right)\frac{\pi}{6}.$$
$$=\varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\frac{1}{2^{2m-1}}\right)$$

3) If

$$t=\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\frac{1}{2^{2m-1}}+\frac{1}{2^{2m}},$$

i.e. $a_{2m-1} = a_{2m} = 1$, then from (2),

$$\begin{aligned}
&\varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\frac{1}{2^{2m-1}}+\frac{1}{2^{2m}}\right)\\
&=\sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1'-a_2'+\cdots+(-1)^k a_{k-1}'-(-1)^k\right)\frac{\pi}{6}\\
&\quad+\frac{1}{\left(\sqrt{3}\right)^{2m-1}}\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'+1\right)\frac{\pi}{6}\\
&\quad+\frac{1}{3^m}\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'-2\right)\frac{\pi}{6}.
\end{aligned}$$

If t takes the form of (6), where $a_{2m-1} = 1$, $a_{2m} = 0$, $a_{2m+1} = a_{2m+2} = \cdots = 1$, $a_{2m-1}' = -1$, $a_{2m}' = 1$, $a_{2m+1}' = a_{2m+2}' = \cdots = -1$, then

$$\begin{aligned}
&\varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\frac{1}{2^{2m-1}}+\sum_{k=2m+1}^{\infty}\frac{1}{2^k}\right)\\
&=\sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1'-a_2'+\cdots+(-1)^k a_{k-1}'-(-1)^k\right)\frac{\pi}{6}\\
&\quad+\frac{1}{\left(\sqrt{3}\right)^{2m-1}}\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'+1\right)\frac{\pi}{6}\\
&\quad+\frac{1}{3^m}\sum_{k=1}^{\infty}\left[\frac{1}{\left(\sqrt{3}\right)^{2k-1}}\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'-1\right)\frac{\pi}{6}\right.\\
&\quad\left.+\frac{1}{\left(\sqrt{3}\right)^{2k}}\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'-4\right)\frac{\pi}{6}\right]\\
&=\sum_{k=1}^{2m-2}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos\left(a_1'-a_2'+\cdots+(-1)^k a_{k-1}'-(-1)^k\right)\frac{\pi}{6}\\
&\quad+\frac{1}{\left(\sqrt{3}\right)^{2m-1}}\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'+1\right)\frac{\pi}{6}\\
&\quad+\frac{1}{3^m}\cos\left(a_1'-a_2'+\cdots+a_{2m-3}'-a_{2m-2}'-2\right)\frac{\pi}{6}\\
&=\varphi\left(\sum_{k=1}^{2m-2}\frac{a_k}{2^k}+\frac{1}{2^{2m-1}}+\frac{1}{2^{2m}}\right).
\end{aligned}$$

Similarly, it can be proved that function $\psi(t)$ is also uniquely determined at a 4-adic rational point $t = \frac{l}{4^m}$.

Next, by using (2), (3) the values of $\varphi(t)$ and $\psi(t)$ at certain points can be calculated. In order to discuss them, we will proceed with various cases of $t \in [0,1]$. Partition the interval $[0,1]$ into 4 congruent segments and then partition each sub-segments similarly. After repeating these steps for m times, a family of sub-intervals is obtained:

$$\left(\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_{2m-1}}{2^{2m-1}}+\frac{a_{2m}}{2^{2m}},\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_{2m-1}}{2^{2m-1}}+\frac{a_{2m}}{2^{2m}}+\frac{1}{2^{2m}}\right).$$

Let $\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_{2m-1}}{2^{2m-1}}+\frac{a_{2m}}{2^{2m}}=\frac{l}{2^{2m}}$, where $l=0,1,\cdots,2^{2m}-1$, $m=0,1,\cdots$, and denote the interval

$$\left(\frac{l}{2^{2m}}+\frac{a_{2m+1}}{2^{2m+1}}+\cdots+\frac{a_{2m+2s}}{2^{2m+2s}},\frac{l}{2^{2m}}+\frac{a_{2m+1}}{2^{2m+1}}+\cdots+\frac{a_{2m+2s}}{2^{2m+2s}}+\frac{1}{2^{2m+2s+1}}\right)$$

as

$$U_{m,l}\left(a_{2m+1},a_{2m+2},\cdots,a_{2m+2s-1},a_{2m+2s}\right),$$

where $a_{2m+1},a_{2m+2},\cdots,a_{2m+2s-1},a_{2m+2s}\in\{0,1\}$. Then

$$U_{m,l}\left(a_{2m+1},a_{2m+2}\right)\supset U_{m,l}\left(a_{2m+1},a_{2m+2},a_{2m+3,}a_{2m+4}\right)\supset\cdots$$
$$\supset U_{m,l}\left(a_{2m+1},a_{2m+2},\cdots,a_{2m+2s-1},a_{2m+2s}\right)\supset\cdots$$

So, any point $t\in[0,1]\setminus D$ is the intersection of one of the above nest of intervals

$$\bigcap_{s=1}^{\infty}U_{m,l}\left(a_{2m+1},a_{2m+2},\cdots,a_{2m+2s-1},a_{2m+2s}\right).$$

All points $t\in[0,1]$ fall into three groups:

1) $t=\frac{l}{2^{2m}},\left(l=0,1,2,\cdots,2^{2m}-1,m=1,2,\cdots\right)$ is a 4-adic rational point. Altogether they form the countable set *D*. Its 4-adic decimal expansion contains finite terms (or an infinite cyclic decimal).

2) $t=\bigcap_{s=1}^{\infty}U_{m,l}\left(a_{2m+1},a_{2m+1},a_{2m+3},a_{2m+3},\cdots,a_{2m+2s-1},a_{2m+2s-1}\right)$. That is to say the 4-adic decimal expansion of *t* is infinite acylic, but from the $2m+1$ decimal place, $a_{2m+2s-1}=a_{2m+2s}$ for $s=1,2,\cdots$.

3) $t=\bigcap_{n=1}^{\infty}U_{m,l}\left(a_{2m+1},a_{2m+2},a_{2m+3},a_{2m+4},\cdots,a_{2m+2n-1},a_{2m+2n}\right)$ where $a_{2m+1}\neq 0,a_{2m+2n}$ for $n=1,2,\cdots$.

Using the arithmetic-analytic representation of the Koch curve of 4-adic rational points (see Lemma 3.3), we can prove the following two lemmas:

Lemma 3.4. *If* $t=\sum_{k=1}^{2n}\frac{a_k}{2^k}+\frac{a}{2^{2n}},a=0,1$ *then*

$$\varphi(t)=\sum_{k=1}^{2n}\frac{a_k}{\left(\sqrt{3}\right)^k}\cos b_k\frac{\pi}{6}+\frac{a}{3^n}\cos\left(b_{2n+1}-1\right)\frac{\pi}{6}$$

$$\psi(t)=\sum_{k=1}^{2n}\frac{a_k}{\left(\sqrt{3}\right)^k}\sin b_k\frac{\pi}{6}+\frac{a}{3^n}\sin\left(b_{2n+1}-1\right)\frac{\pi}{6}.$$

In particular, if $t=\sum_{k=1}^{2m}\frac{a_k}{2^k}+\sum_{k=1}^{n-m}\left(\frac{a_{2m+2k-1}}{2^{2m+2k-1}}+\frac{a_{2m+2k-1}}{2^{2m+2k}}\right)+\frac{a}{2^{2n}}$, *where* $n>m$, *then*

$$\varphi(t)=\sum_{k=1}^{2m}\frac{a_k}{(\sqrt{3})^k}\cos b_k\frac{\pi}{6}+\sum_{k=1}^{n-m}\left[\frac{a_{2m+2k-1}}{(\sqrt{3})^{2m+2k-1}}\cos b_{2m+1}\frac{\pi}{6}\right.$$

$$\left.+\frac{a_{2m+2k-1}}{(\sqrt{3})^{2m+2k}}\cos b_{2m+2}\frac{a}{6}\right]+\frac{a}{3^n}\cos(b_{2m+1}-1)\frac{\pi}{6},$$

$$\psi(t)=\sum_{k=1}^{2m}\frac{a_k}{(\sqrt{3})^k}\sin b_k\frac{\pi}{6}+\sum_{k=1}^{n-m}\left[\frac{a_{2m+2k-1}}{(\sqrt{3})^{2m+2k-1}}\sin b_{2m+1}\frac{\pi}{6}\right.$$

$$\left.+\frac{a}{(\sqrt{3})^{2m+2k}}\sin b_{2m+2}\frac{\pi}{6}\right]+\frac{a}{3^n}\sin(b_{2m+1}-1)\frac{\pi}{6}.$$

Lemma 3.5. *If* $t=\sum_{k=1}^{2m}\frac{a_k}{2^k}-\frac{1}{2^{2m+2n}}$ *then*

$$\varphi(t)=\sum_{k=1}^{2m}\frac{a_k}{(\sqrt{3})^k}\cos(b_k)\frac{\pi}{6}+\frac{1}{3^{m+n}}\cos(b_{2m+1}-1+c)\frac{\pi}{6},$$

$$\psi(t)=\sum_{k=1}^{2m}\frac{a_k}{(\sqrt{3})^k}\sin(b_k)\frac{\pi}{6}+\frac{1}{3^{m+n}}\sin(b_{2m+1}-1+c)\frac{\pi}{6},$$

where $c=-2$ when $a_{2m}=1$, $c=4$ when $a_{2m}=0$.

4. The $\frac{\ln 3}{\ln 4}$-Hölder 4-Adic Derivatives of the Koch Curve

The Koch curve with 4-adic decimal expansion satisfies the Hölder condition (4), (5), so it is reasonable to consider dyadic derivatives of $\alpha=\frac{\ln 3}{\ln 4}$ and $p=4$, which are defined in Section 3. Now we shall prove that for the Koch curve the exact $\frac{\ln 3}{\ln 4}$-Hölder 4-adic derivative exists which is the main result of this paper.

Theorem 4.1 If $t\in[0,1]\setminus D$ and

$$t=\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_{2m}}{2^{2m}}+\sum_{k=m+1}^{\infty}\left(\frac{a_{2k-1}}{2^{2k-1}}+\frac{a_{2k-1}}{2^{2k}}\right) \tag{7}$$

is point type 2), then

$$\varphi'_{\alpha-4}(t)=\cos(b_{2m+1}-1)\frac{\pi}{6},$$

$$\psi'_{\alpha-4}(t)=\sin(b_{2m+1}-1)\frac{\pi}{6},$$

where $b_{2m+1}=a'_1-a'_2+\cdots+a'_{2k-1}-a'_{2k}+\cdots-a'_{2m}+1$.

Proof. By (4.1), it can be seen that t is contained in the following sequence of intervals

$$U_n=\left\{\frac{l}{2^{2m}}+\sum_{k=m+1}^{n}\left(\frac{a_{2k-1}}{2^{2k-1}}+\frac{a_{2k}}{2^{2k}}\right),\frac{l}{2^{2m}}+\sum_{k=m+1}^{n}\left(\frac{a_{2k-1}}{2^{2k-1}}+\frac{a_{2k}}{2^{2k}}\right)+\frac{1}{2^{2n}}\right\}_{n=m+1}^{\infty}$$

and $U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$ Therefore $t = \bigcap_{n=1}^{\infty} U_n$. According to Lemma 3.4,

$$\varphi\left(\frac{l}{2^{2m}} + \sum_{k=m+1}^{n}\left(\frac{a_{2k-1}}{2^{2k-1}} + \frac{a_{2k}}{2^{2k}}\right) + \frac{1}{2^{2n}}\right) - \varphi\left(\frac{l}{2^{2m}} + \sum_{k=m+1}^{n}\left(\frac{a_{2k-1}}{2^{2k-1}} + \frac{a_{2k}}{2^{2k}}\right)\right)$$
$$= \frac{1}{3^n}\cos\left(a_1' - a_2' + \cdots + a_{2m-1}' - a_{2m}'\right)\frac{\pi}{6}.$$

So, by Definition 2.1,

$$\varphi_{\alpha-4}'(t) = \cos\left(a_1' - a_2' + \cdots + a_{2m-1}' - a_{2m}'\right)\frac{\pi}{6}.$$

Similarly

$$\psi_{\alpha-4}'(t) = \sin\left(a_1' - a_2' + \cdots + a_{2m-1}' - a_{2m}'\right)\frac{\pi}{6}.$$

Theorem 4.2 If

$$t = \frac{l}{2^{2m}} + \sum_{k=m+1}^{\infty}\left(\frac{a_{2k-1}}{2^{2k-1}} + \frac{a_{2k}}{2^{2k}}\right) \quad a_{2k-1} \neq a_{2k}, k = m+1, m+2, \cdots$$

is point type 3), then

$$\varphi_{\alpha-4}'(t) = \cos\left(\sum_{k=1}^{\infty}\left(a_{2k-1}' - a_{2k}'\right)\frac{\pi}{6}\right),$$
$$\psi_{\alpha-4}'(t) = \sin\left(\sum_{k=1}^{\infty}\left(a_{2k-1}' - a_{2k}'\right)\frac{\pi}{6}\right).$$

Proof. By (4.1) t is contained in the following nested intervals

$$V_n = \left(\frac{l}{2^{2m}} + \sum_{k=m+1}^{n}\left(\frac{a_{2k-1}}{2^{2k-1}} + \frac{a_{2k}}{2^{2k}}\right), \frac{l}{2^{2m}} + \sum_{k=m+1}^{n}\left(\frac{a_{2k-1}}{2^{2k-1}} + \frac{a_{2k}}{2^{2k}}\right) + \frac{1}{2^{2n}}\right),$$

where $n = m+1, m+2, \cdots$, and

$$V_{m+1} \supset V_{m+2} \supset V_{m+3} \supset \cdots$$

Therefore

$$t = \bigcap_{n=m+1}^{\infty} V_n.$$

According to Lemma 4,

$$\varphi\left(\frac{l}{2^{2m}} + \sum_{k=m+1}^{n}\left(\frac{a_{2k-1}}{2^{2k-1}} + \frac{a_{2k}}{2^{2k}}\right) + \frac{1}{2^{2n}}\right) - \varphi\left(\frac{l}{2^{2m}} + \sum_{k=m+1}^{n}\left(\frac{a_{2k-1}}{2^{2k-1}} + \frac{a_{2k}}{2^{2k}}\right)\right)$$
$$= \frac{1}{3^n}\cos\left(b_{2n+1} - 1\right)\frac{\pi}{6}.$$

By Definition 2.1,

$$\varphi_{\alpha-4}'(t) = \cos\sum_{k=1}^{\infty}\left(a_{2k-1}' - a_{2k}'\right)\frac{\pi}{6}.$$

Similarly

$$\psi_{\alpha-4}'(t) = \sin\sum_{k=1}^{\infty}\left(a_{2k-1}' - a_{2k}'\right)\frac{\pi}{6}.$$

The conclusions of Theorem 4.1 and Theorem 4.2 can be written as: for

$t \in [0,1] \setminus D$, then

$$\varphi'_{\alpha-4}(t) = \cos\left[\sum_{k=1}^{n}(a'_{2k-1} - a'_{2k})\frac{\pi}{6} + \sum_{k=m+1}^{\infty}(a'_{2k-1} - a'_{2k})\frac{\pi}{6}\right],$$

$$\psi'_{\alpha-4}(t) = \sin\left[\sum_{k=1}^{n}(a'_{2k-1} - a'_{2k})\frac{\pi}{6} + \sum_{k=m+1}^{\infty}(a'_{2k-1} - a'_{2k})\frac{\pi}{6}\right].$$

Note that when t is the point of type 2), $\sum_{k=m+1}^{\infty}(a'_{2k-1} - a'_{2k}) = 0$.

We have shown that the Koch curve $(x = \varphi(t), \psi(t))$ has $\frac{\ln 3}{\ln 4}$-Hölder 4-adic derivatives for every $t \in [0,1] \setminus D$. Next, we consider the case of a 4-adic rational point on *D*.

Theorem 4.3 If $t \in D$ is 4-adic rational point on $[0,1]$

$$t = \frac{l}{2^m} \quad l = 0,1,\cdots,2^{2m}-1, m = 1,2,\cdots$$

Then, there exists $\alpha = \frac{\ln 3}{\ln 4}$-Hölder 4-adic left and right derivatives. The right derivatives are

$$\varphi'_{\alpha-4^+}\left(\frac{l}{2^{2m}}\right) = \cos\sum_{k=1}^{n}(a'_{2k-1} - a'_{2k})\frac{\pi}{6}.$$

Fractal

$$\psi'_{\alpha-4^+}\left(\frac{l}{2^{2m}}\right) = \sin\sum_{k=1}^{n}(a'_{2k-1} - a'_{2k})\frac{\pi}{6}.$$

The left derivatives are

$$\varphi'_{\alpha-4^-}\left(\frac{l}{2^{2m}}\right) = \cos\left[\sum_{k=1}^{n}(a'_{2k-1} - a'_{2k}) + c\right]\frac{\pi}{6},$$

$$\psi'_{\alpha-4^-}\left(\frac{l}{2^{2m}}\right) = \sin\left[\sum_{k=1}^{n}(a'_{2k-1} - a'_{2k}) + c\right]\frac{\pi}{6},$$

where $c = -2$ when $a_{2m} = 1$, $c = 4$ when $a_{2m} = 0$.

Proof. By Lemma 4,

$$\varphi\left(\frac{l}{2^{2m}} + \frac{1}{2^{2n}}\right) - \varphi\left(\frac{l}{2^{2m}}\right) = \frac{1}{3^n}\cos(b_{2m+1} - 1)\frac{\pi}{6}.$$

Then, by Definition 2.2,

$$\varphi'_{\alpha-4^+}(t) = \lim_{n\to\infty}\frac{\varphi\left(\frac{l}{2^{2m}} + \frac{1}{2^{2n}}\right) - \varphi\left(\frac{l}{2^{2m}}\right)}{\left(\frac{1}{2^{2n}}\right)^{\ln 4}\ln} = \cos\sum_{k=1}^{n}(a'_{2k-1} - a'_{2k})\frac{\pi}{6}.$$

Similarly

$$\psi'_{\alpha-4^+}(t) = \sin\sum_{k=1}^{n}(a'_{2k-1} - a'_{2k})\frac{\pi}{6}.$$

According to Lemma 5,

$$\varphi\left(\frac{l}{2^{2m}}\right) - \varphi\left(\frac{l}{2^{2m}} - \frac{1}{2^{2n}}\right) = \frac{1}{3^n}\cos(b_{2m+1} - 1 + c)\frac{\pi}{6},$$

where $c=-2$ when $a_{2m}=1$, $c=4$ when $a_{2m}=0$. So, by Definition 2.2,

$$\varphi'_{\alpha-4^-}(t)=\lim_{n\to\infty}\frac{\varphi\left(\frac{l}{2^{2m}}\right)-\varphi\left(\frac{l}{2^{2m}}-\frac{1}{2^{2n}}\right)}{\left(\frac{1}{2^{2n}}\right)^{\frac{\ln 3}{\ln 4}}}=\cos\left[\sum_{k=1}^{n}\left(a'_{2k-1}-a'_{2k}\right)+c\right]\frac{\pi}{6}.$$

Similarly

$$\psi'_{\alpha-4^-}(t)=\sin\left[\sum_{k=1}^{n}\left(a'_{2k-1}-a'_{2k}\right)+c\right]\frac{\pi}{6}.$$

Theorems 4.1 - 4.3 then have established the $\frac{\ln 3}{\ln 4}$-Hölder 4-adic derivatives of the Koch curve on [0, 1]. As He pointed out recently [17] that fractal derivative/calculus has very important applications in many applied fields including mathematics, engineering and fluid dynamics, and researchers have tried to define various derivatives of fractals. Our results will not only allow us to further investigate the differentiability of other fractals [4] [18], but also provide a new type of derivative for researchers in other fields to conduct their investigations.

Some observations: It is obvious that the right and left $\frac{\ln 3}{\ln 4}$-Hölder 4-adic derivatives of the Koch curve at the 4-adic rational point are not equal, as these points should be at the sharp-angled vertices. Furthermore, 1) There exists unequal left and right $\frac{\ln 3}{\ln 4}$-Hölder 4-adic derivatives for the Koch curve at countable points set (*i.e.* the 4-adic rational points) on $[0,1]$. This indicates that the knot point quality of a non-differentiable function seems not to be eliminated no matter how the derivative is defined. 2) For 4-adic irrational points, the $\frac{\ln 3}{\ln 4}$-Hölder 4-adic derivatives are determined at the second type of points, which are $\tan\left(\frac{0\pi}{6}\right)$ and $\tan\left(\pm\frac{\pi}{6}\right)$. 3) For the points of the third type, although the set of the $\frac{\ln 3}{\ln 4}$-Hölder 4-adic derivatives also contains $\tan\left(\frac{0\pi}{6}\right)$ and $\tan\left(\pm\frac{\pi}{6}\right)$, it is not definite. This reflects the oscillatory quality of non-differentiable function, *i.e.* it is also the case of the knot points without one-sided derivatives [19]. Of course, in this case, it might be better to consider the upper α-Hölder derivative.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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