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Solution of Laguerre's Differential Equations via Modified Adomian Decomposition Method

Mariam Al-Mazmumy, Aishah A. Alsulami

Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia
Email: mhalmazmumy@uj.edu.sa, aalsulami1183.stu@uj.edu.sa

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Abstract

This paper presents a technique for obtaining an exact solution for the well-known Laguerre's differential equations that arise in the modeling of several phenomena in quantum mechanics and engineering. We utilize an efficient procedure based on the modified Adomian decomposition method to obtain closed-form solutions of the Laguerre's and the associated Laguerre's differential equations. The proposed technique makes sense as the attitudes of the acquired solutions towards the neighboring singular points are correctly taken care of.

Keywords

Modification Method, Singular Ordinary Differential Equations, Laguerre's Equation, Associated Laguerre's Equation

1. Introduction

Mathematical models featuring both the ordinary and partial differential equations are realized in modeling different scenarios arising from physical and social sciences among others. Various efforts have been undertaken in recent decades to come up with dissimilar computational procedures for solving such models in numerous sectors of research, including modern technological situations. In particular, Laguerre's differential equation is a type of differential equation that is found in a variety of engineering problems [1], and in quantum mechanics, because it is one of several equations that appear in the quantum mechanical description of the hydrogen atom [2].

More explicitly, let us consider the equation,

$$xv''(x)+(1-x)v'(x)+nv(x)=0. \tag{1}$$

This equation is called Laguerre's differential equation of order n, where n is a real-valued constant. One crucial attribute of the present model is having a solution in form of a polynomial, which is usually referred to as Laguerre's polynomial. The Laguerre's polynomial being the solution of the Laguerre's differential equation is denoted by $L_n(x)$, and further expressed as

$$L_n(x)=\frac{\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\mathrm{e}^{-x}x^n\right). \tag{2}$$

Moreover, the generalized version of Equation (1) is the so-called associated Laguerre's differential equation, that is expressed as

$$xv''(x)+(k+1-x)v'(x)+nv(x)=0, \tag{3}$$

where k and n are real numbers [1] [3]; clearly, Equation (3) reduces to (1) when $k=0$.

Further, Laguerre's differential equation was solved in [1] by using the Haar wavelet method; while [4] solved the same model by using G-transform, a generalized Laplace-typed transform method. Also, the differential transformation method [5] was applied to solve Laguerre's differential equation. Furthermore, in recent times, research activities regarding second-order differential equations with initial data have drawn the inquisitiveness of different researchers. Various methods have been proposed in the past and present literature towards devising promising unified techniques to tackle a variety of differential equations; one could easily find the Adomian decomposition method [6] [7], and its related modifications and extensions to attract so many minds in this regard, see the following references [8]-[14] and the cited references therein to explore numerous scientific models in the presence of the method. Moreover, certain Adomian-based approaches have equally been utilized in the literature while treating various classes of linear and nonlinear integer and non-integer order differential equations, read [15]-[20] as an instance.

However, the current research focuses on the relevance of the modification of Adomian's approach in tackling Laguerre's and the associated Laguerre's equations, respectively. These equations are well-known models that arise in the modeling of several phenomena in engineering and quantum mechanics to state a few. Our main goal here is to demonstrate the advantages and the efficiency of using this modified version of the Adomian's method to obtain an optimal approximate solution of the mentioned Laguerre's equations. Additionally, we organize the paper in the following manner: Section 2 gives the classical Adomian's method; while its modification to be utilized in this study is presented in Section 3. Section 4 makes consideration to certain numerical applications as illustrative examples, and lastly, we give certain concluding remarks in Section 5.

2. Adomian Decomposition Method

Let us begin by giving a general survey of the Adomian's method by considering the following generalized second-order differential equation

$$Lv+Rv+Nv=g, \tag{4}$$

with L denoting the second-order linear differential operator, R is also a linear operator, but less than L, and N is the nonlinear differential operator, while g is a source term. Thus, we rewrite the above equation as follows

$$Lv = g - Rv - Nv. \tag{5}$$

Next, premultiplying each term of Equation (5) with the inverse operator L^{-1}, we obtain

$$L^{-1}Lv = L^{-1}g - L^{-1}Rv - L^{-1}Nv. \tag{6}$$

Therefore, making use of an infinite series via the Adomian's method to represent the solution $v(x)$ as

$$v(x) = \sum_{m=0}^{\infty} v_m(x), \tag{7}$$

and the nonlinear term $N(v)$ through

$$Nv = \sum_{m=0}^{\infty} A_m, \tag{8}$$

with A_m's representing the polynomials by Adomian. Therefore, we substitute Equations (7) and (8) into Equation (6) to obtain

$$\sum_{m=0}^{\infty} v_m = \varphi_0 + L^{-1}g - L^{-1}R\sum_{m=0}^{\infty} v_m - L^{-1}\sum_{m=0}^{\infty} A_m, \tag{9}$$

where

$$\varphi_0 = \begin{cases} v_0(0), & \text{if } L = \dfrac{\mathrm{d}}{\mathrm{d}x}, \\ v_0(0) + xv_0'(0), & \text{if } L = \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}, \\ v_0(0) + xv_0'(0) + \dfrac{x^2}{2!}v_0''(0), & \text{if } L = \dfrac{\mathrm{d}^3}{\mathrm{d}x^3}, \\ \quad\vdots & \\ v_0(0) + xv_0'(0) + \dfrac{x^2}{2!}v_0''(0) + \cdots + \dfrac{x^m}{m!}v_0^{(m)}(0), & \text{if } L = \dfrac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}}. \end{cases} \tag{10}$$

Therefore,

$$\begin{cases} v_0(x) = \varphi_0 + L^{-1}g, \\ v_1(x) = -L^{-1}Rv_0 - L^{-1}A_0, \\ v_2(x) = -L^{-1}Rv_1 - L^{-1}A_1, \\ v_3(x) = -L^{-1}Rv_2 - L^{-1}A_2, \\ \quad\vdots \\ v_{m+1}(x) = -L^{-1}Rv_m - L^{-1}A_m,\ m \ge 0, \end{cases} \tag{11}$$

where A_m's are the polynomials by Adomian, which are to be computed using the following formula

$$A_m = \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}\lambda^m}\left[N\left(\sum_{i=0}\lambda^i v_i(x)\right)\right]_{\lambda=0},\ m = 0,1,2,\cdots \tag{12}$$

Now, from Equation (12), the polynomials by Adomian are recurrently determined as

$$\begin{cases} A_0 = N(v_0), \\ A_1 = N'(v_0)v_1, \\ A_2 = N'(v_0)v_2 + \dfrac{1}{2}N''(v_0)v_1^2, \\ A_3 = N'(v_0)v_3 + N''(v_0)v_1v_2 + \dfrac{1}{3!}N'''(v_0)v_1^3, \\ \quad \vdots \end{cases} \tag{13}$$

Hence, the recurrent solution for the n-term scheme is, therefore, obtained as follows

$$\phi_m = \sum_{i=0}^{m-1} v_i(x), \tag{14}$$

where

$$v(x) = \lim_{m\to\infty} \phi_m(x) = \sum_{i=0}^{\infty} v_i(x).$$

Note also that, the convergence analysis of the classical Adomian's method was discussed by many researchers, including the famous research by Cherruault *et al.* [21] [22] [23] [24] and other notable works like Gabet [25] and Babolian and Biazar [26], to state a few.

3. Modified Adomian Decomposition Method

Here, we give two important algorithms based on certain modifications of Adomian's method to solve the governing equations under consideration.

3.1. Algorithm 1

The Adomian decomposition method has been slightly modified in the following way as suggested by Dita and Grama [27]. First, we make consideration to the following generalized second-order linear equation

$$L(x,\Delta)v(x) - R(x,v,\Delta v,\Delta^2 v) = 0, \tag{15}$$

where $L(x,\Delta)$ is the principal linear operator given by

$$L(x,\Delta) = h(x)\Delta p(x)\Delta, \tag{16}$$

and R is the outstanding linear operator tackling all the other operators less than L. More so, the functions $h(x)$ together with $p(x)$ are all continuously differentiable functions. The minus sign in Equation (15) is taken for convenience. At this point, for brevity, proper choices of the operator L and R should be made in such a way that the resultant pseudo-Volterra integral equation can be solved easily.

However, sometimes writing Equation (15) in its standard form complicates the matter. Recall that we are aiming at obtaining an inverse for the general form in Equation (16) and make use of it to look for different known formulae for the solutions of certain special ordinary differential equations. The second proposal entails writing Equation (15) in a form of an inhomogeneous equation by utilizing a modified version of the variation of parameters method to obtain a transformed equation as a pseudo-Volterra integral equation. This will then be considered to be the generalized version of Adomian's approach for tackling ordinary differential equations of the second-order. Most of the equations of the form given in Equation (15) are called the classical special functions, together with their principal linear part given in Equation (16) with a lot of these equations having $h(x)=1$. What is more, the functions $h(x)$ together with $p(x)$ are both assumed continuously differentiable, and further $\frac{1}{p(x)}$ is locally integrable near a certain point; such that without loss of generality, it can be chosen to be near $x=0$. The decomposition approach involves determining the inverse operator L^{-1} such that through it many cases of the exact solution of the governing model are constructed. Again, an inverse corresponding to the principal linear operator given in Equation (15) is thus readily suggested as follows

$$L^{-1}(x,\Delta)v(x)=\int_0^x \frac{\mathrm{d}t}{p(t)}\int_0^t \frac{\mathrm{d}y}{h(y)}v(y). \tag{17}$$

Additionally, one sees that

$$\left(LL^{-1}\right)v(x)=I\cdot v(x). \tag{18}$$

However,

$$\left(L^{-1}L\right)v(x)\neq I\cdot v(x), \tag{19}$$

with I denoting the identity operator. Equation (19) also shows the inverse operator L^{-1}, indicating that it is not a true inversion of the principal operator; rather, it is only so when the prescribed initial data are taken into account, and that is similar to providing the respective values of $v(0)$ and $v'(0)$. Hence,

$$\begin{aligned}\left(L^{-1}L\right)v(x)&=\int_0^x \frac{\mathrm{d}t}{p(t)}\int_0^t \frac{\mathrm{d}y}{h(y)}h(y)\frac{\mathrm{d}}{\mathrm{d}y}p(y)\frac{\mathrm{d}v(y)}{\mathrm{d}y},\\ &=\int_0^x \frac{\mathrm{d}t}{p(t)}\left[p(y)\frac{\mathrm{d}v}{\mathrm{d}y}\right]_0^t=\int_0^x \frac{\mathrm{d}v(t)}{\mathrm{d}t}-p(0)v'(0)\int_0^x \frac{\mathrm{d}t}{p(t)},\\ &=v(x)-v(0)-p(0)v'(0)\int_0^x \frac{\mathrm{d}t}{p(t)}.\end{aligned} \tag{20}$$

Therefore, the following result is obtained from Equation (17),

$$v(x)=v(0)+p(0)v'(0)\int_0^x \frac{\mathrm{d}t}{p(t)}+\int_0^x \frac{\mathrm{d}y}{p(t)}\int_0^t \frac{\mathrm{d}t}{h(y)}R(y,\Delta v). \tag{21}$$

Further, Equation (21) is a Volterra integral equation, which assumes the following series solution

$$v(x)=\sum_{k=0}^{\infty}v_k(x), \tag{22}$$

where

$$v_0 = v(0) + p(0)v'(0)\int_0^x \frac{\mathrm{d}t}{p(t)}, \tag{23}$$

and $v_k(x)$ is thus determined via the Picard approach of successive approximation as follows

$$v_k = L^{-1}(x,\Delta)v_{k-1}. \tag{24}$$

3.2. Algorithm 2

Consider the generalized second-order linear differential equation [27]

$$v'' + a(x)v' + b(x)v = h(x), \tag{25}$$

with $\varphi(x) \neq 0$ presumed to be a regular solution (of Equation (25)) admitted by the homogeneous part of the equation, which would be obtained by using a modified version of the variation of parameters method. Thus, the following solution is obtained via a more general integral representation as follows

$$v(x) = s_1\varphi(x) + s_2\varphi(x)\int\frac{\mathrm{d}x}{I(x)\varphi^2(x)} + \varphi(x)\int\frac{\mathrm{d}x}{I(x)\varphi^2(x)}\int I(x)\varphi(x)h(x)\mathrm{d}x, \tag{26}$$

with s_1 and s_2 denoting the real constants; while $I(x)$ is the integrating factor given by

$$I(x) = \mathrm{e}^{\int a(x)\mathrm{d}x}.$$

Moreover, the inverse differential operator L^{-1} becomes an indefinite integral in the above; that is, for each problem it has to be transformed into a definite integral according to the required solution. Additionally, Equation (26) can be considered as the beginning step with regard to the implementation of the decomposition approach in most general cases. More so, a linear homogeneous differential equation is transformed to a pseudo-Volterra integral equation via Equation (25). An important issue here is the separation of the convenient component on the left-hand side of Equation (25) such that the solution of this component can easily be found. The behavior of the solution around the point where the solution is expected to have to be taken into account, in order to maximize the chances of obtaining the complete explicit infinite series form. This easy separation procedure is carried out through shifting the $b(x)v$ term to the right-hand side of the equation, such that the solution corresponding to the left-hand side only is $v(x) = 1$.

4. Illustrative Examples

Here, the applicability of the devised schemes is exhibited in the current section on Laguerre's and the associated Laguerre's differential equations. Besides, it is an attempt to deploy the proposed algorithms in acquiring the known solutions of the governing models.

4.1. Laguerre's Differential Equation

Let us make a reconsideration of Laguerre's differential equation expressed as

$$xv''(x)+(1-x)v'(x)+nv(x)=0. \tag{27}$$

Algorithm 1: Rewrite Equation (27) as

$$Lv(x)=xv''(x)+v'(x)=\frac{\mathrm{d}}{\mathrm{d}x}\left[xv'(x)\right]=xv'(x)-nv(x). \tag{28}$$

Here, we consider $h(x)=1$ and $p(x)=x$; and also decompose the solution function $v(x)$ via the stated infinite series of the form

$$v(x)=\sum_{m=0}^{\infty}v_m(x), \tag{29}$$

together with the following initial data

$$v_0=1, v'(0)=0. \tag{30}$$

Then, upon applying the form of Equation (21), a recurrence scheme is obtained of the following form

$$\begin{aligned}&v_0=1\\&v_{r+1}=L^{-1}\left[xv_r'-nv_r(x)\right]=\int_0^x x^{-1}\int_0^x\left[xv_r'-nv_r(x)\right]\mathrm{d}x\mathrm{d}x,\end{aligned} \tag{31}$$

such that

$$\begin{aligned}
v_1&=L^{-1}\left[xv_0'-nv_0(x)\right]=\int_0^x x^{-1}\int_0^x[-n]\mathrm{d}x\mathrm{d}x=-nx=-\frac{n!}{(n-1)!}x,\\
v_2&=L^{-1}\left[xv_1'-nv_1(x)\right]=\int_0^x x^{-1}\int_0^x\left[n^2-n\right]x\mathrm{d}x\mathrm{d}x\\
&=\frac{n(n-1)}{4}x^2=\frac{n!}{(n-2)!(2!)^2}x^2,\\
v_3&=L^{-1}\left[xv_2'-nv_2(x)\right]=\int_0^x x^{-1}\int_0^x\left[\frac{n(n-1)}{2}-\frac{n^2(n-1)}{4}\right]x^2\mathrm{d}x\mathrm{d}x\\
&=-\frac{n(n-1)(n-2)}{36}x^3=-\frac{n!}{(n-3)!(3!)^2}x^3,\\
&\vdots
\end{aligned} \tag{32}$$

Hence, upon taking the net sum of the above components, we obtain following known series solution

$$v(x)=L_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{n!}{(n-r)!(r!)^2}x^r. \tag{33}$$

Algorithm 2: Rewrite the Laguerre's equation as follows

$$v''(x)+\frac{1}{x}v'(x)=v'(x)-\frac{n}{x}v(x). \tag{34}$$

Now, considering the right-hand side entirely as a normal inhomogeneous term, we then solve for the other side of the equation as a normal homogeneous component.

Next, if the solutions are $\varphi_+ = 1$ and $\varphi_- = \ln x$, we start off by considering φ_+ and fix the constants $s_1 = 1$ and $s_2 = 0$. Then, since an exact solution is searched at $x = 0$, a two-fold integral operator is adopted as the inverse operator L^{-1}, occupying the area from 0 to *x*, and thus determine from Equation (25) the following Volterra integral equation

$$v(x) = 1 + \int_0^x x^{-1} \int_0^x x \left[v'(x) - \frac{n}{x} v(x) \right] \mathrm{d}x\mathrm{d}x. \tag{35}$$

We, therefore, solve Equation (35) iteratively by considering the zeroth-order approximating the inhomogeneous term $v(0) = 1 = v_0$ and thus get

$$v_0(x) = 1, \tag{36}$$

and

$$v_{r+1}(x) = \int_0^x x^{-1} \int_0^x x \left[v_r'(x) - \frac{n}{x} v_r(x) \right] \mathrm{d}x\mathrm{d}x. \tag{37}$$

More iteratively, we find that

$$\begin{aligned}
v_1 &= \int_0^x x^{-1} \int_0^x x \left[v_0'(x) - \frac{n}{x} v_0(x) \right] \mathrm{d}x\mathrm{d}x = \int_0^x x^{-1} \int_0^x x \left[\frac{-n}{x} \right] \mathrm{d}x\mathrm{d}x \\
&= -nx = -\frac{n!}{(n-1)!} x, \\
v_2 &= \int_0^x x^{-1} \int_0^x x \left[v_1'(x) - \frac{n}{x} v_1(x) \right] \mathrm{d}x\mathrm{d}x = \int_0^x x^{-1} \int_0^x x \left[n^2 - n \right] \mathrm{d}x\mathrm{d}x \\
&= \frac{n(n-1)}{4} x^2 = \frac{n!}{(n-2)!(2!)^2} x^2, \\
v_3 &= \int_0^x x^{-1} \int_0^x x \left[v_2'(x) - \frac{n}{x} v_2(x) \right] \mathrm{d}x\mathrm{d}x \\
&= \int_0^x x^{-1} \int_0^x x \left[\frac{n(n-1)}{2} x - \frac{n(n-1)}{4} x \right] \mathrm{d}x\mathrm{d}x \\
&= -\frac{n(n-1)(n-2)}{36} x^3 = \frac{n!}{(n-3)!(3!)^2} x^3, \\
&\vdots
\end{aligned} \tag{38}$$

Hence, upon taking the sum of the above components, we get the following well-known series solution

$$v(x) = L_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r. \tag{39}$$

Example 4.1

Let us consider Laguerre's differential equation of $n = 2$

$$\begin{gathered} xv''(x) + (1-x)v'(x) + 2v(x) = 0, \\ v(0) = 1, v'(0) = 0. \end{gathered} \tag{40}$$

Algorithm 1: Rewrite Equation (40) as

$$Lv(x) = xv''(x) + v'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[xv'(x) \right] = xv'(x) - 2v(x). \tag{41}$$

Here, we consider $h(x)=1$ and $p(x)=x$; and also decompose the solution function $v(x)$ via the stated infinite series of the form

$$v(x)=\sum_{m=0}^{\infty} v_m(x), \tag{42}$$

Then, upon applying the form of Equation (21), a recurrence scheme is obtained of the following form

$$\begin{aligned} &v_0=1 \\ &v_{r+1}=L^{-1}\left[xv_r'-2v_r(x)\right]=\int_0^x x^{-1}\int_0^x\left[xv_r'-2v_r(x)\right]\mathrm{d}x\mathrm{d}x, \end{aligned} \tag{43}$$

such that

$$\begin{aligned} &v_1=L^{-1}\left[xv_0'-2v_0(x)\right]=\int_0^x x^{-1}\int_0^x[-2]\mathrm{d}x\mathrm{d}x=-2x, \\ &v_2=L^{-1}\left[xv_1'-2v_1(x)\right]=\int_0^x x^{-1}\int_0^x[-2x+4x]\mathrm{d}x\mathrm{d}x=\int_0^x x^{-1}\int_0^x[2x]\mathrm{d}x\mathrm{d}x=\frac{1}{2}x^2, \\ &v_3=L^{-1}\left[xv_2'-2v_2(x)\right]=\int_0^x x^{-1}\int_0^x[x-x]\mathrm{d}x\mathrm{d}x=0, \\ &v_{r+1}=0, r\geq 2. \end{aligned} \tag{44}$$

Hence, upon taking the net sum of the above components, we obtain following known series solution

$$v(x)=L_2(x)=1-2x+\frac{1}{2}x^2=\frac{1}{2!}\left(x^2-4x+2\right). \tag{45}$$

Algorithm 2: Rewrite the Laguerre's equation for $n=2$ as follows

$$v''(x)+\frac{1}{x}v'(x)=v'(x)-\frac{2}{x}v(x). \tag{46}$$

Now, considering the right-hand side entirely as a normal inhomogeneous term, we then solve for the other side of the equation as a normal homogeneous component.

Next, if the solutions are $\varphi_+=1$ and $\varphi_-=\ln x$, we start off by considering φ_+ and fix the constants $s_1=1$ and $s_2=0$. Then, since an exact solution is searched at $x=0$, a two-fold integral operator is adopted as the inverse operator L^{-1}, occupying the area from 0 to *x*, and thus determine from Equation (25) the following Volterra integral equation

$$v(x)=1+\int_0^x x^{-1}\int_0^x x\left[v'(x)-\frac{2}{x}v(x)\right]\mathrm{d}x\mathrm{d}x. \tag{47}$$

We, therefore, solve Equation (47) iteratively by considering the zeroth-order approximating the inhomogeneous term $v(0)=1=v_0$ and thus get

$$v_0(x)=1, \tag{48}$$

and

$$v_{r+1}(x)=\int_0^x x^{-1}\int_0^x x\left[v_r'(x)-\frac{2}{x}v_r(x)\right]\mathrm{d}x\mathrm{d}x. \tag{49}$$

More iteratively, we find that

$$\begin{aligned}
v_1 &= \int_0^x x^{-1}\int_0^x x\left[v_0'(x)-\frac{2}{x}v_0(x)\right]\mathrm{d}x\mathrm{d}x = \int_0^x x^{-1}\int_0^x x\left[\frac{-2}{x}\right]\mathrm{d}x\mathrm{d}x = -2x,\\
v_2 &= \int_0^x x^{-1}\int_0^x x\left[v_1'(x)-\frac{2}{x}v_1(x)\right]\mathrm{d}x\mathrm{d}x = \int_0^x x^{-1}\int_0^x x\left[-2+4\right]\mathrm{d}x\mathrm{d}x\\
&= \int_0^x x^{-1}\int_0^x 2x\mathrm{d}x\mathrm{d}x = \frac{1}{2}x^2,\\
v_3 &= \int_0^x x^{-1}\int_0^x x\left[v_2'(x)-\frac{2}{x}v_2(x)\right]\mathrm{d}x\mathrm{d}x = \int_0^x x^{-1}\int_0^x x\left[x-x\right]\mathrm{d}x\mathrm{d}x = 0,\\
v_{r+1} &= 0, r\geq 2
\end{aligned} \tag{50}$$

Hence, upon taking the sum of the above components, we get the following well-known series solution

$$v(x) = L_2(x) = 1-2x+\frac{1}{2}x^2 = \frac{1}{2!}\left(x^2-4x+2\right). \tag{51}$$

4.2. Associated Laguerre's Differential Equation

Let us make a reconsideration of the associated Laguerre's differential equation expressed as

$$xv''(x)+(k+1-x)v'(x)+nv(x)=0. \tag{52}$$

Algorithm 1: Rewrite Equation (52) as follows

$$Lv(x) = xv''(x)+(k+1)v'(x) = x^{-k}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{k+1}v'(x)\right] = xv'(x)-nv(x). \tag{53}$$

Here, we consider the following functions $h(x)=x^{-k}$, and $p(x)=x^{k+1}$. Next, we decompose the solution function $v(x)$ via infinite series expansion follows

$$v(x) = \sum_{m=0}^{\infty} v_m(x), \tag{54}$$

and further choose the following initial data

$$v(0) = \frac{(n+k)!}{n!k!}, v'(0) = 0. \tag{55}$$

Then, on employing the form of Equation (21), we get the recurrence relation below

$$\begin{aligned}
v_0 &= \frac{(n+k)!}{n!k!}\\
v_{r+1} &= L^{-1}\left[xv_r'-nv_r(x)\right] = \int_0^x x^{-k-1}\int_0^x x^k\left[xv_r'-nv_r(x)\right]\mathrm{d}x\mathrm{d}x,
\end{aligned} \tag{56}$$

such that

$$\begin{aligned}
v_1 &= L^{-1}\left[xv_0'-nv_0(x)\right] = \int_0^x x^{-k-1}\int_0^x x^k\left[-n\frac{(n+k)!}{n!k!}\right]\mathrm{d}x\mathrm{d}x\\
&= -n\frac{(n+k)!}{n!(k+1)!}x = -\frac{(n+k)!}{(n-1)!(k+1)!}x,
\end{aligned}$$

$$\begin{aligned}
v_2 &= L^{-1}\left[xv_1' - nv_1(x)\right] \\
&= \int_0^x x^{-k-1}\int_0^x x^k\left[-\frac{(n+k)!}{(n-1)!(k+1)!} + n\frac{(n+k)!}{(n-1)!(k+1)!}\right]x\mathrm{d}x\mathrm{d}x \\
&= \frac{(n-1)(n+k)!}{(n-1)!(k+2)!}\frac{x^2}{2} = \frac{(n+k)!}{(n-2)!(k+2)!}\frac{x^2}{2}, \\
v_3 &= L^{-1}\left[xv_2' - nv_2(x)\right] \\
&= \int_0^x x^{-k-1}\int_0^x x^k\left[\frac{(n+k)!}{(n-2)!(k+2)!} - n\frac{(n+k)!}{2(n-2)!(k+2)!}\right]x^2\mathrm{d}x\mathrm{d}x \\
&= -\frac{(n-2)(n+k)!}{(n-2)!(k+3)!}\frac{x^3}{6} = -\frac{(n+k)!}{(n-3)!(k+3)!}\frac{x^3}{3!}, \\
&\vdots
\end{aligned} \tag{57}$$

Finally, on summing the above components, the following documented series solution is yielded

$$v(x) = L_n^k(x) = \sum_{r=0}^{n}(-1)^r\frac{(n+k)!}{(n-r)!(k+r)!r!}x^r. \tag{58}$$

Algorithm 2: We firstly rewrite Equation (68) as

$$v''(x) + \frac{k+1}{x}v'(x) = v'(x) - \frac{n}{x}v(x). \tag{59}$$

Therefore, considering the right-hand side of the above equation as a normal inhomogeneous component, we then solve for the other side of the equation as a normal homogeneous component.

Additionally, with admitting the solutions $\varphi_+ = 1$ and $\varphi_- = x^{-k}$, we start off by considering φ_+ and fix the constants $s_1 = \frac{(n+k)!}{n!k!}$ and $s_2 = 0$. Then, since we are looking for the exact solution at $x = 0$, a two-fold integral operator is adopted as an inverse differential operator L^{-1} from 0 to x and thus determine from Equation (25) the following Volterra integral equation

$$v(x) = \frac{(n+k)!}{n!k!} + \int_0^x x^{-1}\int_0^x x\left[v'(x) - \frac{n}{x}v(x)\right]\mathrm{d}x\mathrm{d}x. \tag{60}$$

We then solve Equation (60) iteratively by considering the zeroth-order approximation as the inhomogeneous term $v(0) = \frac{(n+k)!}{n!k!} = v_0$ and thus obtain the remaining recurrent components from

$$\begin{aligned}
v_0 &= \frac{(n+k)!}{n!k!}, \\
v_{r+1} &= L^{-1}\left[v_r'(x) - \frac{n}{x}v_r(x)\right] \\
&= \int_0^x x^{-k-1}\int_0^x x^{k+1}\left[v_r'(x) - \frac{n}{x}v_r(x)\right]\mathrm{d}x\mathrm{d}x,
\end{aligned} \tag{61}$$

such that

$$
\begin{aligned}
v_1 &= L^{-1}\left[v_0' - \frac{n}{x}v_0(x)\right] = \int_0^x x^{-k-1}\int_0^x x^{k+1}\left[-\frac{n}{x}\frac{(n+k)!}{n!k!}\right]\mathrm{d}x\mathrm{d}x \\
&= -\frac{n(n+k)!}{n!(k+1)!}x = -\frac{(n+k)!}{(n-1)!(k+1)!}x, \\
v_2 &= L^{-1}\left[v_1' - \frac{n}{x}v_1(x)\right] \\
&= \int_0^x x^{-k-1}\int_0^x x^{k+1}\left[-\frac{(n+k)!}{(n-1)!(k+1)!} + n\frac{(n+k)!}{(n-1)!(k+1)!}\right]\mathrm{d}x\mathrm{d}x \\
&= \frac{(n-1)(n+k)!}{(n-1)!(k+2)!}\frac{x^2}{2} = \frac{(n+k)!}{(n-2)!(k+2)!}\frac{x^2}{2}, \\
v_3 &= L^{-1}\left[v_2' - \frac{n}{x}v_2(x)\right] \\
&= \int_0^x x^{-k-1}\int_0^x x^{k+1}\left[\frac{(n+k)!}{(n-2)!(k+2)!} - \frac{n(n+k)!}{2(n-2)!(k+2)!}\right]x\mathrm{d}x\mathrm{d}x \\
&= -\frac{(n-2)(n+k)!}{(n-2)!(k+3)!}\frac{x^3}{6} = -\frac{(n+k)!}{(n-3)!(k+3)!}\frac{x^3}{3!}, \\
&\vdots
\end{aligned}
\tag{62}
$$

Thus, the overall series solution is obtained as follows

$$
v(x) = L_n^k(x) = \sum_{r=0}^{n}(-1)^r\frac{(n+k)!}{(n-r)!(k+r)!r!}x^r. \tag{63}
$$

Alternatively, if we choose the initial data as $s_1 = 0$ and $s_2 = \frac{k(n+k)!}{n!k!}$, and further choose $\varphi_- = x^{-k}$; we find from Equation (25) the following Volterra integral equation

$$
v(x) = \frac{k(n+k)!}{n!k!}x^{-k}\int_0^x x^{k-1}\mathrm{d}x + x^{-k}\int_0^x x^{k-1}\int_0^x\left[v'(x) - \frac{n}{x}v(x)\right]x\mathrm{d}x\mathrm{d}x. \tag{64}
$$

We then solve Equation (64) iteratively by considering the zeroth-order approximation as the inhomogeneous term

$v_0 = \frac{k(n+k)!}{n!k!}x^{-k}\int_0^x x^{k-1}\mathrm{d}x = \frac{(n+k)!}{n!k!} = v(0)$, and thus get

$$
\begin{aligned}
v_{r+1} &= L^{-1}\left[v_r'(x) - \frac{n}{x}v_r(x)\right] \\
&= x^{-k}\int_0^x x^{k-1}\int_0^x x\left[v_r'(x) - \frac{n}{x}v_r(x)\right]\mathrm{d}x\mathrm{d}x,
\end{aligned}
\tag{65}
$$

where the individual components are explicitly expressed as

$$
\begin{aligned}
v_1 &= L^{-1}\left[v_0' - \frac{n}{x}v_0(x)\right] = x^{-k}\int_0^x x^{k-1}\int_0^x x\left[-\frac{n}{x}\frac{(n+k)!}{n!k!}\right]\mathrm{d}x\mathrm{d}x \\
&= -x^{-k}\frac{n(n+k)!}{n!(k+1)!}x^{k+1} = -\frac{(n+k)!}{(n-1)!(k+1)!}x,
\end{aligned}
$$

$$
\begin{aligned}
v_2 &= L^{-1}\left[v_1' - \frac{n}{x} v_1(x)\right] \\
&= x^{-k}\int_0^x x^{k-1}\int_0^x x\left[-\frac{(n+k)!}{(n-1)!(k+1)!} + n\frac{(n+k)!}{(n-1)!(k+1)!}\right]\mathrm{d}x\mathrm{d}x \\
&= x^{-k}\frac{(n-1)(n+k)!}{(n-1)!(k+2)!}\frac{x^{2+k}}{2} = \frac{(n+k)!}{(n-2)!(k+2)!}\frac{x^2}{2}, \\
v_3 &= L^{-1}\left[v_2' - \frac{n}{x} v_2(x)\right] \\
&= x^{-k}\int_0^x x^{k-1}\int_0^x x\left[\frac{(n+k)!}{(n-2)!(k+2)!} - \frac{n(n+k)!}{2(n-2)!(k+2)!}\right]x\mathrm{d}x\mathrm{d}x \\
&= -x^{-k}\frac{(n-2)(n+k)!}{(n-2)!(k+3)!}\frac{x^{k+3}}{6} = -\frac{(n+k)!}{(n-3)!(k+3)!}\frac{x^3}{3!}, \\
&\vdots
\end{aligned}
\tag{66}
$$

Finally, as expected, similar series solution is obtained upon taking the net sum of the above components as follows

$$
v(x) = L_n^k(x) = \sum_{r=0}^{n}(-1)^r \frac{(n+k)!}{(n-r)!(k+r)!r!}x^r. \tag{67}
$$

Example 4.2

Let us consider the associated Laguerre's differential for $n=2, k=1$ equation expressed as

$$
xv''(x) + (2-x)v'(x) + 2v(x) = 0. \tag{68}
$$

Algorithm 1: Rewrite Equation (68) as follows

$$
Lv(x) = xv''(x) + 2v'(x) = x^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 v'(x)\right] = xv'(x) - 2v(x). \tag{69}
$$

Here, we consider the following functions $h(x) = x^{-1}$, and $p(x) = x^2$. Next, we decompose the solution function $v(x)$ via infinite series expansion follows

$$
v(x) = \sum_{m=0}^{\infty} v_m(x), \tag{70}
$$

and further choose the following initial data

$$
v(0) = \frac{3!}{2!1!} = 3, v'(0) = 0. \tag{71}
$$

Then, on employing the form of Equation (21), we get the recurrence relation below

$$
\begin{aligned}
v_0 &= 3 \\
v_{r+1} &= L^{-1}\left[xv_r' - 2v_r(x)\right] = \int_0^x x^{-2}\int_0^x x\left[xv_r' - 2v_r(x)\right]\mathrm{d}x\mathrm{d}x,
\end{aligned}
\tag{72}
$$

such that

$$
\begin{aligned}
v_1 &= L^{-1}\left[xv_0' - 2v_0(x)\right] = \int_0^x x^{-2}\int_0^x x\left[-2(3)\right]\mathrm{d}x\mathrm{d}x = -3x, \\
v_2 &= L^{-1}\left[xv_1' - 2v_1(x)\right] = \int_0^x x^{-2}\int_0^x x\left[-3x + 6x\right]\mathrm{d}x\mathrm{d}x = \frac{x^2}{2},
\end{aligned}
$$

$$v_3 = L^{-1}\left[xv_2' - 2v_2(x)\right] = \int_0^x x^{-2}\int_0^x x\left[x^2 - x^2\right]\mathrm{d}x\mathrm{d}x = 0,$$
$$v_{r+1} = 0, r \geq 2. \tag{73}$$

Finally, on summing the above components, the following documented series solution is yielded

$$v(x) = L_2^1(x) = 3 - 3x + \frac{x^2}{2} = \sum_{r=0}^{2}(-1)^r \frac{3!}{(2-r)!(1+r)!r!}x^r. \tag{74}$$

Algorithm 2: We firstly rewrite Equation (68) as

$$v''(x) + \frac{2}{x}v'(x) = v'(x) - \frac{2}{x}v(x). \tag{75}$$

Therefore, considering the right-hand side of the above equation as a normal inhomogeneous component, we then solve for the other side of the equation as a normal homogeneous component.

Additionally, with admitting the solutions $\varphi_+ = 1$ and $\varphi_- = x^{-1}$, we start off by considering φ_+ and fix the constants $s_1 = 3$ and $s_2 = 0$. Then, since we are looking for the exact solution at $x = 0$, a two-fold integral operator is adopted as an inverse differential operator L^{-1} from 0 to x and thus determine from Equation (25) the following Volterra integral equation

$$v(x) = 3 + \int_0^x x^{-2}\int_0^x x^2\left[v'(x) - \frac{2}{x}v(x)\right]\mathrm{d}x\mathrm{d}x. \tag{76}$$

We then solve Equation (76) iteratively by considering the zeroth-order approximation as the inhomogeneous term $v(0) = 3 = v_0$ and thus obtain the remaining recurrent components from

$$v_0 = 3$$
$$v_{r+1} = L^{-1}\left[v_r'(x) - \frac{2}{x}v_r(x)\right] = \int_0^x x^{-2}\int_0^x x^2\left[v_r'(x) - \frac{2}{x}v_r(x)\right]\mathrm{d}x\mathrm{d}x, \tag{77}$$

such that

$$v_1 = L^{-1}\left[v_0' - \frac{2}{x}v_0(x)\right] = \int_0^x x^{-2}\int_0^x x^2\left[-\frac{6}{x}\right]\mathrm{d}x\mathrm{d}x = -3x,$$
$$v_2 = L^{-1}\left[v_1' - \frac{2}{x}v_1(x)\right] = \int_0^x x^2\int_0^x x^2\left[-3+6\right]\mathrm{d}x\mathrm{d}x = \frac{x^2}{2},$$
$$v_3 = L^{-1}\left[v_2' - \frac{2}{x}v_2(x)\right] = \int_0^x x^{-2}\int_0^x x^2\left[x - x\right]\mathrm{d}x\mathrm{d}x = 0,$$
$$v_{r+1} = 0, r \geq 2. \tag{78}$$

Thus, the overall series solution is obtained as follows

$$v(x) = L_2^1(x) = 3 - 3x + \frac{x^2}{2} = \sum_{r=0}^{2}(-1)^r \frac{(2+1)!}{(2-r)!(1+r)!r!}x^r. \tag{79}$$

5. Conclusion

In conclusion, we have successfully determined the well-known closed-form series solutions of the Laguerre's and the associated Laguerre's equations, respectively, by devising modification methodologies that are based upon the application of the Adomian decomposition technique. These modifications are highly accurate, efficient, and further converge rapidly within a few numbers of steps. Additionally, the obtained series of solutions affirm the available results in the literature. Besides, the resulting integration procedures that arise from the proposed inverse operators L^{-1} are calculated with the help of the Maple 18 package programmer. Therefore, the proposed schemes can be used to securitize different classes of both the ordinary and partial differential equations, which modeled diverse real-life circumstances.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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