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A Flux Ratio Independent of the Permanent Charge in PNP Models

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Abstract

PNP models with an arbitrary number of positively charged ion species and one negatively charged ion species are studied in this paper under the assumption that positively charged ion species have the same valence and the permanent charge is a piecewise constant function. The permanent charge plays the key role in many functions of an ion channel, such as selectivity and gating. In this paper, using the geometric singular perturbation theory, a flux ratio independent of the permanent charge is proved.

Keywords

PNP Model, The Permanent Charge, Ionic Flow

1. Introduction

The cell membrane is a biological membrane that separates the interior of all cells from the outside environment and protects the cell from its environment. The cell membrane consists of a lipid bilayer that is semipermeable. It regulates the transport of materials entering and exiting the cell. Ion channels are large proteins embedded in cell membranes that have holes open to the inside and the outside of cells. The charged ions flow through the open channels and represent an electric current. These currents alter the distribution of charge and the voltage across the membrane is changed. Ionic flow through ion channels can be described mathematically by the Poisson-Nernst-Planck model [1] [2]. A stationary one-dimensional Poisson-Nernst-Planck model [3] [4] [5] is

$$\begin{aligned} &\frac{1}{h(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\varepsilon_r\varepsilon_0 h(x)\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right) = -e\left(\sum_{j=1}^{n} z_j c_j(x) + Q(x)\right), \\ &\frac{\mathrm{d}\mathcal{J}_i}{\mathrm{d}x} = 0, \quad -\mathcal{J}_i = \frac{1}{kT}D_i h(x) c_i(x)\frac{\mathrm{d}\mu_i}{\mathrm{d}x}, \quad i = 1,2,\cdots,n, \end{aligned} \tag{1.1}$$

where Φ is the electric potential, c_i is the concentration for the ith ion species, z_i is the valence, $Q(x)$ is the permanent charge of the channel, $\mu_i(x)$ is the electrochemical potential, $h(x)$ is the area of the cross-section of the channel, $\mathcal{J}_i$ is the flux density, D_i is the diffusion coefficient, ε_r is the relative dielectric coefficient, ε_0 is the vacuum permittivity, k is the Boltzmann constant, T is the absolute temperature, and e is the elementary charge.

The boundary conditions are, for $i = 1, 2, \cdots, n$,

$$\Phi(0) = V,\ c_i(0) = L_i;\ \Phi(1) = 0,\ c_i(1) = R_i. \tag{1.2}$$

$\mu_i(x)$ in the classical Poisson-Nernst-Planck model takes the following form

$$\mu_i(x) = z_i e\phi(x) + kT\ln\frac{c_i(x)}{c_0} \tag{1.3}$$

which c_0 is a constant.

The Poisson-Nernst-Planck model (1.1) is actually a simplified model which is derived from the Maxwell-Boltzmann equations [6] [7] and the Langevin-Poisson equations [8] [9] by capturing key features. Recently, the Poisson-Nernst-Planck model (1.1) has been studied [10]-[17] greatly. In [18], under the assumption $Q(x)$ is a piecewise constant function, the boundary value problems (1.1) and (1.2) have been analyzed based on the geometric singular perturbation theory [19] [20] [21]. However, due to the lack of the explicit formula for individual flux, it is difficult to analyze the properties of individual flux. In this paper, a property of individual flux, that is, a flux ratio is independent of the permanent charge, is identified under the following assumptions.

(A1) $z_1 = \cdots = z_{n-1} = z > 0$ and $z_n < 0$.

(A2) For $0 < a_1 < b_1 < \cdots < a_i < b_i < \cdots < a_m < b_m < 1$, let $Q(x) = 0$ for $0 < x < a_1$; $Q(x) = Q_1$ for $a_1 < x < b_1$; $Q(x) = 0$ for $b_{i-1} < x < a_i$; $Q(x) = Q_i$ for $a_i < x < b_i$; $Q(x) = 0$ for $b_i < x < a_{i+1}$; $Q(x) = Q_m$ for $a_m < x < b_m$; $Q(x) = 0$ for $b_m < x < 1$; where $Q_i, i = 1, \cdots, m$, are constants and m is an arbitrary positive integer.

By re-scaling,

$$\phi = \frac{e}{kT}\Phi,\ \bar{V} = \frac{e}{kT}V,\ \varepsilon^2 = \frac{\varepsilon_r\varepsilon_0 kT}{e^2},\ J_i = \frac{\mathcal{J}_i}{D_i}.$$

The model (1.1) is reduced to a standard singularly perturbed system of the following

$$\begin{aligned}
&\frac{\varepsilon^2}{h(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)\frac{\mathrm{d}}{\mathrm{d}x}\phi\right) = -\left[zc_1 + \cdots + zc_{n-1} + z_n c_n + Q(x)\right],\\
&h(x)\left(\frac{\mathrm{d}c_1}{\mathrm{d}x} + zc_1\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = -J_1,\\
&\qquad\vdots\\
&h(x)\left(\frac{\mathrm{d}c_{n-1}}{\mathrm{d}x} + zc_{n-1}\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = -J_{n-1},\\
&h(x)\left(\frac{\mathrm{d}c_n}{\mathrm{d}x} + z_n c_n\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = -J_n,
\end{aligned}$$

$$\frac{\mathrm{d}J_1}{\mathrm{d}x} = \cdots = \frac{\mathrm{d}J_n}{\mathrm{d}x} = 0, \tag{1.4}$$

with the boundary condition, for $j = 1, \cdots, n$.

$$\phi(0) = \bar{V},\ c_j(0) = L_j,\ \phi(1) = 0,\ c_j(1) = R_j. \tag{1.5}$$

Under the assumption that the permanent charge $Q(x)$ is small, the effects of small permanent charges on individual flux are investigated in [22]. On the other hand, under the assumption that the permanent charge $Q(x)$ is large, the effects of large permanent charges on individual flux have been also analyzed in [23] [24]. Actually, due to the assumption that the permanent charge $Q(x)$ is small or large, the solutions of (1.4) and (1.5) can be expanded with respect to $Q(x)$, therefore, the explicit formulae for the zeroth order approximation and the first order approximation of individual flux can be obtained. Based on these explicit formulae, the effects of small or large permanent charges on individual flux can be analyzed in [22] [23] [24].

In this paper, under the assumptions A_1 and A_2 and without the assumption the permanent charge $Q(x)$ is small or large, although it seems that there is no methods to get the explicit formula for J_k, but it still can be verified that a flux ratio is independent of $Q(x)$, that is,

$$\frac{J_k}{J_1 + \cdots + J_{n-1}} = \frac{R_k - L_k \mathrm{e}^{z\bar{V}}}{R_1 + \cdots + R_{n-1} - (L_1 + \cdots + L_{n-1})\mathrm{e}^{z\bar{V}}}. \tag{1.6}$$

The rest of this paper is organized as follows. In Section 2, limiting fast and slow orbits for (1.4) and (1.5) are constructed. In Section 3, limiting fast and slow orbits for (1.4) and (1.5) in Section 2 are matched, which results in a series of very complicated algebraic equations. The main results in this paper are presented in Section 4. Some conclusions are given in Section 5.

2. Limiting Fast and Slow Orbits for (4)-(5) over [0, 1]

Let $u = \varepsilon \frac{\mathrm{d}}{\mathrm{d}x}\phi$, $\tau = x$, system (1.4) becomes

$$\begin{aligned}
&\varepsilon\dot{\phi} = u, \quad \varepsilon\dot{u} = -\left[zc_1 + \cdots + zc_{n-1} + z_n c_n + Q(x)\right] - \varepsilon h^{-1}(\tau) h_\tau(\tau) u,\\
&\varepsilon\dot{c}_1 = -zc_1 u - \varepsilon h_\tau(\tau) J_1,\\
&\qquad \vdots\\
&\varepsilon\dot{c}_{n-1} = -zc_{n-1} u - \varepsilon h_\tau(\tau) J_{n-1},\\
&\varepsilon\dot{c}_n = -z_n c_n u - \varepsilon h_\tau(\tau) J_n,\\
&\dot{J}_1 = 0, \cdots, \dot{J}_n = 0, \quad \dot{\tau} = 1.
\end{aligned} \tag{2.7}$$

By using the rescaling $x = \varepsilon\xi$, one has

$$\begin{aligned}
&\phi' = u, \quad u' = -\left[zc_1 + \cdots + zc_{n-1} + z_n c_n + Q(x)\right] - \varepsilon h^{-1}(\tau) h_\tau(\tau) u,\\
&c_1' = -zc_1 u - \varepsilon h_\tau(\tau) J_1,\\
&\qquad \vdots\\
&c_{n-1}' = -zc_{n-1} u - \varepsilon h_\tau(\tau) J_{n-1},
\end{aligned}$$

$$c_n' = -z_n c_n u - \varepsilon h_\tau(\tau) J_n,$$
$$J_1' = 0, \cdots, J_n' = 0, \quad \tau' = \varepsilon. \tag{2.8}$$

Define

$$B_L = \left\{ \left(\bar{V}, u, L_1, \cdots, L_n, J_1, \cdots, J_n, 0 \right) \in \mathbb{R}^{2n+3} : \text{arbitrary } u, J_1, \cdots, J_n \right\},$$
$$B_R = \left\{ \left(0, u, R_1, \cdots, R_n, J_1, \cdots, J_n, 1 \right) \in \mathbb{R}^{2n+3} : \text{arbitrary } u, J_1, \cdots, J_n \right\}. \tag{2.9}$$

Then a solution to Equations (1.4) and (1.5) is to finding an orbit of Equation (2.7) or (2.8) from B_L to B_R.

By letting $\varepsilon = 0$, we analyze the limiting fast and limiting slow orbits of Equations (2.7) and (2.8) on intervals $[0, a_1]$, $[a_1, b_1]$, $[b_{i-1}, a_i]$, $[a_i, b_i]$, $[b_i, a_{i+1}]$, $[a_{i+1}, b_{i+1}]$ and $[b_m, 1]$ respectively due to the fact that $Q(x)$ is a piecewise constant function.

Let $\phi(a_i) = \phi^{a_i}$, $c_1(a_i) = c_1^{a_i}$, $\cdots$, $c_n(a_i) = c_n^{a_i}$, where ϕ^{a_i}, $c_1^{a_i}$, $\cdots$, $c_n^{a_i}$ are unknowns to be determined. Let

$$B_{a_i} = \left\{ \left(\phi^{a_i}, u, c_1^{a_i}, \cdots, c_n^{a_i}, J_1, \cdots, J_n, a_i \right) \in \mathbb{R}^{2n+3} : \text{arbitrary } u, J_1, \cdots, J_n \right\}.$$

Let $\phi(b_i) = \phi^{b_i}$, $c_1(b_i) = c_1^{b_i}$, $\cdots$, $c_n(b_i) = c_n^{b_i}$, where ϕ^{b_i}, $c_1^{b_i}$, $\cdots$, $c_n^{b_i}$ are unknowns to be determined. Let

$$B_{b_i} = \left\{ \left(\phi^{b_i}, u, c_1^{b_i}, \cdots, c_n^{b_i}, J_1, \cdots, J_n, b_i \right) \in \mathbb{R}^{2n+3} : \text{arbitrary } u, J_1, \cdots, J_n \right\}.$$

Then limiting fast and slow orbits of Equation (2.7) or (2.8) from B_L to B_R will consists of several parts: limiting fast and slow orbits over the interval $[0, a_1]$ connecting orbit from B_L to B_{a_1}, limiting fast and slow orbits over the interval $[a_1, b_1]$ connecting orbit from B_{a_1} to B_{b_1}, limiting fast and slow orbits over the interval $[b_{i-1}, a_i]$ connecting orbit from $B_{b_{i-1}}$ to B_{a_i}, limiting fast and slow orbits over the interval $[a_i, b_i]$ connecting orbit from B_{a_i} to B_{b_i}, limiting fast and slow orbits over the interval $[b_i, a_{i+1}]$ connecting orbit from B_{b_i} to $B_{a_{i+1}}$, limiting fast and slow orbits over the interval $[a_{i+1}, b_{i+1}]$ connecting orbit from $B_{a_{i+1}}$ to $B_{b_{i+1}}$, and limiting fast and slow orbits over the interval $[b_m, 1]$ connecting orbit from B_{b_m} to B_R.

For convenience, let $H(x) = \int_0^x h^{-1}(s) \mathrm{d}s$.

2.1. Limiting Fast and Slow Orbits on $[0, a_1]$ Where $Q(x) = 0$

In this section, we will construct limiting fast and slow orbits that connects B_L to B_{a_1} by letting $\varepsilon = 0$ in Equations (2.7) and (2.8). As shown in [25], limiting fast and slow orbits that connect B_L to B_{a_1} are satisfied by:

$$J_1 + \cdots + J_{n-1}$$
$$= \frac{c_1^L + \cdots + c_{n-1}^L - \left(c_1^{a_1,l} + \cdots + c_{n-1}^{a_1,l} \right)}{H(a_1)} \left[1 - \frac{z\left(\phi^L - \phi^{a_1,l} \right)}{\ln \dfrac{c_1^{a_1,l} + \cdots + c_{n-1}^{a_1,l}}{c_1^L + \cdots + c_{n-1}^L}} \right],$$

$$J_n = \frac{z\left[c_1^{a_1,l} + \cdots + c_{n-1}^{a_1,l} - \left(c_1^L + \cdots + c_{n-}^L\right)\right]}{z_n H\left(a_1\right)}\left[1 - \frac{z_n\left(\phi^L - \phi^{a_1,l}\right)}{\ln \frac{c_1^{a_1,l} + \cdots + c_{n-1}^{a_1,l}}{c_1^L + \cdots + c_{n-1}^L}}\right],$$

$$\frac{J_k}{J_1 + \cdots + J_{n-1}} = \frac{c_k^{a_1,l} - c_k^L e^{z\left(\phi^L - \phi^{a_1,l}\right)}}{c_1^{a_1,l} + \cdots + c_{n-1}^{a_1,l} - \left(c_1^L + \cdots + c_{n-1}^L\right) e^{z\left(\phi^L - \phi^{a_1,l}\right)}}, \tag{2.10}$$

where $k = 1, \cdots, n-1$ and

$$c_1^L = L_1\left[\frac{-z_n L_n}{z\left(L_1 + \cdots + L_{n-1}\right)}\right]^{\frac{z}{z - z_n}},$$

$$\vdots$$

$$c_{n-1}^L = L_{n-1}\left[\frac{-z_n L_n}{z\left(L_1 + \cdots + L_{n-1}\right)}\right]^{\frac{z}{z - z_n}},$$

$$c_n^L = L_n\left[\frac{-z_n L_n}{z\left(L_1 + \cdots + L_{n-1}\right)}\right]^{\frac{z_n}{z - z_n}},$$

$$\phi^L = \bar{V} - \frac{1}{z - z_n} \ln \frac{-z_n L_n}{z\left(L_1 + \cdots + L_{n-1}\right)},$$

$$c_1^{a_1,l} = c_1^{a_1}\left[\frac{-z_n c_n^{a_1}}{z\left(c_1^{a_1} + \cdots + c_{n-1}^{a_1}\right)}\right]^{\frac{z}{z - z_n}},$$

$$\vdots$$

$$c_{n-1}^{a_1,l} = c_{n-1}^{a_1}\left[\frac{-z_n c_n^{a_1}}{z\left(c_1^{a_1} + \cdots + c_{n-1}^{a_1}\right)}\right]^{\frac{z}{z - z_n}},$$

$$c_n^{a_1,l} = c_n^{a_1}\left[\frac{-z_n c_n^{a_1}}{z\left(c_1^{a_1} + \cdots + c_{n-1}^{a_1}\right)}\right]^{\frac{z_n}{z - z_n}},$$

$$\phi^{a_1,l} = \phi^{a_1} - \frac{1}{z - z_n} \ln \frac{-z_n c_n^{a_1}}{z\left(c_1^{a_1} + \cdots + c_{n-1}^{a_1}\right)},$$

$$u(0) = \operatorname{sgn}\left(\phi^L - \bar{V}\right)\sqrt{2\left[L_1 + \cdots + L_n - \left(c_1^L + \cdots + c_n^L\right)\right]},$$

$$u_l\left(a_1\right) = \operatorname{sgn}\left(\phi^{a_1} - \phi^{a_1,l}\right)\sqrt{2\left[c_1^{a_1} + \cdots + c_n^{a_1} - \left(c_1^{a_1,l} + \cdots + c_n^{a_1,l}\right)\right]}. \tag{2.11}$$

2.2. Limiting Fast and Slow Orbits on $[b_{i-1}, a_i]$ Where $Q(x) = 0$

In this section, we will construct limiting fast and slow orbits that connects $B_{b_{i-1}}$ to B_{a_i} by letting $\varepsilon = 0$ in Equations (2.7) and (2.8). Limiting fast and slow orbits that connects $B_{b_{i-1}}$ to B_{a_i} are satisfied by:

$$J_1 + \cdots + J_{n-1} = \frac{c_1^{b_{i-1},r} + \cdots + c_{n-1}^{b_{i-1},r} - \left(c_1^{a_i,l} + \cdots + c_{n-1}^{a_i,l}\right)}{H\left(a_i\right) - H\left(b_{i-1}\right)}\left[1 - \frac{z\left(\phi^{b_{i-1},r} - \phi^{a_i,l}\right)}{\ln \frac{c_1^{a_i,l} + \cdots + c_{n-1}^{a_i,l}}{c_1^{b_{i-1},r} + \cdots + c_{n-1}^{b_{i-1},r}}}\right],$$

$$J_n = \frac{z\left[c_1^{a_i,l} + \cdots + c_{n-1}^{a_i,l} - \left(c_1^{b_{i-1},r} + \cdots + c_{n-1}^{b_{i-1},r}\right)\right]}{z_n\left[H(a_i) - H(b_{i-1})\right]}\left[1 - \frac{z_n\left(\phi^{b_{i-1},r} - \phi^{a_i,l}\right)}{\ln\frac{c_1^{a_i,l} + \cdots + c_{n-1}^{a_i,l}}{c_1^{b_{i-1},r} + \cdots + c_{n-1}^{b_{i-1},r}}}\right],$$

$$\frac{J_k}{J_1 + \cdots + J_{n-1}} = \frac{c_k^{a_i,l} - c_k^{b_{i-1},r}\mathrm{e}^{z\left(\phi^{b_{i-1},r} - \phi^{a_i,l}\right)}}{c_1^{a_i,l} + \cdots + c_{n-1}^{a_i,l} - \left(c_1^{b_{i-1},r} + \cdots + c_{n-1}^{b_{i-1},r}\right)\mathrm{e}^{z\left(\phi^{b_{i-1},r} - \phi^{a_i,l}\right)}}, \tag{2.12}$$

where $k = 1, \cdots, n-1$, $i = 2, \cdots, m$ and

$$c_1^{b_{i-1},r} = c_1^{b_{i-1}}\left[\frac{-z_n c_n^{b_{i-1}}}{z\left(c_1^{b_{i-1}} + \cdots + c_{n-1}^{b_{i-1}}\right)}\right]^{\frac{z}{z-z_n}},$$

$$\vdots$$

$$c_{n-1}^{b_{i-1},r} = c_{n-1}^{b_{i-1}}\left[\frac{-z_n c_n^{b_{i-1}}}{z\left(c_1^{b_{i-1}} + \cdots + c_{n-1}^{b_{i-1}}\right)}\right]^{\frac{z}{z-z_n}},$$

$$c_n^{b_{i-1},r} = c_n^{b_{i-1}}\left[\frac{-z_n c_n^{b_{i-1}}}{z\left(c_1^{b_{i-1}} + \cdots + c_{n-1}^{b_{i-1}}\right)}\right]^{\frac{z_n}{z-z_n}},$$

$$\phi^{b_{i-1},r} = \phi^{b_{i-1}} - \frac{1}{z - z_n}\ln\frac{-z_n c_n^{b_{i-1}}}{z\left(c_1^{b_{i-1}} + \cdots + c_{n-1}^{b_{i-1}}\right)},$$

$$c_1^{a_i,l} = c_1^{a_i}\left[\frac{-z_n c_n^{a_i}}{z\left(c_1^{a_i} + \cdots + c_{n-1}^{a_i}\right)}\right]^{\frac{z}{z-z_n}},$$

$$\vdots$$

$$c_{n-1}^{a_i,l} = c_{n-1}^{a_i}\left[\frac{-z_n c_n^{a_i}}{z\left(c_1^{a_i} + \cdots + c_{n-1}^{a_i}\right)}\right]^{\frac{z}{z-z_n}},$$

$$c_n^{a_i,l} = c_n^{a_i}\left[\frac{-z_n c_n^{a_i}}{z\left(c_1^{a_i} + \cdots + c_{n-1}^{a_i}\right)}\right]^{\frac{z_n}{z-z_n}},$$

$$\phi^{a_i,l} = \phi^{a_i} - \frac{1}{z - z_n}\ln\frac{-z_n c_n^{a_i}}{z\left(c_1^{a_i} + \cdots + c_{n-1}^{a_i}\right)},$$

$$u_r(b_{i-1}) = \operatorname{sgn}\left(\phi^{b_{i-1},r} - \phi^{b_{i-1}}\right)\sqrt{2\left[c_1^{b_{i-1}} + \cdots + c_n^{b_{i-1}} - \left(c_1^{b_{i-1},r} + \cdots + c_n^{b_{i-1},r}\right)\right]}$$
$$u_l(a_i) = \operatorname{sgn}\left(\phi^{a_i} - \phi^{a_i,l}\right)\sqrt{2\left[c_1^{a_i} + \cdots + c_n^{a_i} - \left(c_1^{a_i,l} + \cdots + c_n^{a_i,l}\right)\right]}. \tag{2.13}$$

2.3. Limiting Fast and Slow Orbits on $[a_i, b_i]$ Where $Q(x) = Q_i$

In this section, we will construct limiting fast and slow orbits that connects B_{a_i} to B_{b_i} by letting $\varepsilon = 0$ in Equations (2.7) and (2.8). Limiting fast and slow orbits that connect B_{a_i} to B_{b_i} are satisfied by:

$$
\begin{aligned}
&J_1+\cdots+J_{n-1}+J_n \\
&=\frac{(z-z_n)\left[c_1^{b_i,l}+\cdots+c_{n-1}^{b_i,l}-\left(c_1^{a_i,r}+\cdots+c_{n-1}^{a_i,r}\right)\right]}{z_n\left[H(b_i)-H(a_i)\right]}+\frac{Q_i\left(\phi^{b_i,l}-\phi^{a_i,r}\right)}{H(b_i)-H(a_i)}, \\
&\phi^{b_i,l}=\phi^{a_i,r}-\frac{z(J_1+\cdots+J_{n-1})+z_nJ_n}{zz_n(J_1+\cdots+J_n)} \\
&\qquad\times\ln\frac{z(J_1+\cdots+J_n)\left(c_1^{b_i,l}+\cdots+c_{n-1}^{b_i,l}\right)+Q_i(J_1+\cdots+J_{n-1})}{z(J_1+\cdots+J_n)\left(c_1^{a_i,r}+\cdots+c_{n-1}^{a_i,r}\right)+Q_i(J_1+\cdots+J_{n-1})}, \\
&\frac{J_k}{J_1+\cdots+J_{n-1}}=\frac{c_k^{b_i,l}-c_k^{a_i,r}e^{z\left(\phi^{a_i,r}-\phi^{b_i,l}\right)}}{c_1^{b_i,l}+\cdots+c_{n-1}^{b_i,l}-\left(c_1^{a_i,r}+\cdots+c_{n-1}^{a_i,r}\right)e^{z\left(\phi^{a_i,r}-\phi^{b_i,l}\right)}},
\end{aligned}
\tag{2.14}
$$

where $k=1,\cdots,n-1$, $i=1,\cdots,m$ and

$$
\begin{aligned}
&c_1^{a_i,r}=c_1^{a_i}e^{z\left(\phi^{a_i}-\phi^{a_i,r}\right)},\cdots,c_{n-1}^{a_i,r}=c_{n-1}^{a_i}e^{z\left(\phi^{a_i}-\phi^{a_i,r}\right)},c_n^{a_i,r}=c_n^{a_i}e^{z_n\left(\phi^{a_i}-\phi^{a_i,r}\right)}, \\
&c_1^{b_i,l}=c_1^{b_i}e^{z\left(\phi^{b_i}-\phi^{b_i,l}\right)},\cdots,c_{n-1}^{b_i,l}=c_{n-1}^{b_i}e^{z\left(\phi^{b_i}-\phi^{b_i,l}\right)},c_n^{b_i,l}=c_n^{b_i}e^{z_n\left(\phi^{b_i}-\phi^{b_i,l}\right)}, \\
&zc_1^{a_i}e^{z\left(\phi^{a_i}-\phi^{a_i,r}\right)}+\cdots+zc_{n-1}^{a_i}e^{z\left(\phi^{a_i}-\phi^{a_i,r}\right)}+z_nc_n^{a_i}e^{z_n\left(\phi^{a_i}-\phi^{a_i,r}\right)}+Q_i=0, \\
&zc_1^{b_i}e^{z\left(\phi^{b_i}-\phi^{b_i,l}\right)}+\cdots+zc_{n-1}^{b_i}e^{z\left(\phi^{b_i}-\phi^{b_i,l}\right)}+z_nc_n^{b_i}e^{z_n\left(\phi^{b_i}-\phi^{b_i,l}\right)}+Q_i=0, \\
&u_r(a_i)=\operatorname{sgn}\left(\phi^{a_i,r}-\phi^{a_i}\right)\sqrt{2\left[c_1^{a_i}+\cdots+c_n^{a_i}-\left(c_1^{a_i,r}+\cdots+c_n^{a_i,r}\right)-Q_i\left(\phi^{a_i}-\phi^{a_i,r}\right)\right]}, \\
&u_l(b_i)=\operatorname{sgn}\left(\phi^{b_i}-\phi^{b_i,l}\right)\sqrt{2\left[c_1^{b_i}+\cdots+c_n^{b_i}-\left(c_1^{b_i,l}+\cdots+c_n^{b_i,l}\right)-Q_i\left(\phi^{b_i}-\phi^{b_i,l}\right)\right]}.
\end{aligned}
\tag{2.15}
$$

2.4. Limiting Fast and Slow Orbits on $[b_m,1]$ Where $Q(x)=0$

In this section, we will construct limiting fast and slow orbits that connects B_{b_m} to B_R by letting $\varepsilon=0$ in Equations (2.7) and (2.8). Limiting fast and slow orbits that connect B_{b_m} to B_R are satisfied by:

$$
\begin{aligned}
&J_1+\cdots+J_{n-1}=\frac{c_1^{b_m,r}+\cdots+c_{n-1}^{b_m,r}-\left(c_1^R+\cdots+c_{n-1}^R\right)}{H(1)-H(b_m)}\left[1-\frac{z\left(\phi^{b_m,r}-\phi^R\right)}{\ln\frac{c_1^R+\cdots+c_{n-1}^R}{c_1^{b_m,r}+\cdots+c_{n-1}^{b_m,r}}}\right], \\
&J_n=\frac{z\left[c_1^R+\cdots+c_{n-1}^R-\left(c_1^{b_m,r}+\cdots+c_{n-1}^{b_m,r}\right)\right]}{z_n\left[H(1)-H(b_m)\right]}\left[1-\frac{z_n\left(\phi^{b_m,r}-\phi^R\right)}{\ln\frac{c_1^R+\cdots+c_{n-1}^R}{c_1^{b_m,r}+\cdots+c_{n-1}^{b_m,r}}}\right], \\
&\frac{J_k}{J_1+\cdots+J_{n-1}}=\frac{c_k^R-c_k^{b_m,r}e^{z\left(\phi^{b_m,r}-\phi^R\right)}}{c_1^R+\cdots+c_{n-1}^R-\left(c_1^{b_m,r}+\cdots+c_{n-1}^{b_m,r}\right)e^{z\left(\phi^{b_m,r}-\phi^R\right)}},
\end{aligned}
\tag{2.16}
$$

where $k=1,\cdots,n-1$ and

$$
c_1^{b_m,r}=c_1^{b_m}\left[\frac{-z_nc_n^{b_m}}{z\left(c_1^{b_m}+\cdots+c_{n-1}^{b_m}\right)}\right]^{\frac{z}{z-z_n}},
$$

$$\vdots$$

$$c_{n-1}^{b_m,r} = c_{n-1}^{b_m}\left[\frac{-z_n c_n^{b_m}}{z\left(c_1^{b_m}+\cdots+c_{n-1}^{b_m}\right)}\right]^{\frac{z}{z-z_n}},$$

$$c_n^{b_m,r} = c_n^{b_m}\left[\frac{-z_n c_n^{b_m}}{z\left(c_1^{b_m}+\cdots+c_{n-1}^{b_m}\right)}\right]^{\frac{z_n}{z-z_n}},$$

$$\phi^{b_m,r} = \phi^{b_m} - \frac{1}{z-z_n}\ln\frac{-z_n c_n^{b_m}}{z\left(c_1^{b_m}+\cdots+c_{n-1}^{b_m}\right)},$$

$$c_1^R = R_1\left[\frac{-z_n R_n}{z\left(R_1+\cdots+R_{n-1}\right)}\right]^{\frac{z}{z-z_n}},$$

$$\vdots$$

$$c_{n-1}^R = R_{n-1}\left[\frac{-z_n R_n}{z\left(R_1+\cdots+R_{n-1}\right)}\right]^{\frac{z}{z-z_n}},$$

$$c_n^R = R_n\left[\frac{-z_n R_n}{z\left(R_1+\cdots+R_{n-1}\right)}\right]^{\frac{z_n}{z-z_n}},$$

$$\phi^R = -\frac{1}{z-z_n}\ln\frac{-z_n R_n}{z\left(R_1+\cdots+R_{n-1}\right)},$$

$$u_r\left(b_m\right) = \operatorname{sgn}\left(\phi^{b_m,r}-\phi^{b_m}\right)\sqrt{2\left[c_1^{b_m}+\cdots+c_n^{b_m}-\left(c_1^{b_m,r}+\cdots+c_n^{b_m,r}\right)\right]},$$
$$u(1) = \operatorname{sgn}\left(-\phi^R\right)\sqrt{2\left[R_1+\cdots+R_n-\left(c_1^R+\cdots+c_n^R\right)\right]}. \tag{2.17}$$

3. Matching Limiting Fast and Slow Orbits on $[0,1]$

Based on Sections 2.1-2.4, to obtain limiting fast and slow orbits from B_L to B_R, the following algebraic equations should holds simultaneously:

$$u_l\left(a_i\right) = u_r\left(a_i\right),$$
$$u_l\left(b_i\right) = u_r\left(b_i\right),$$

$$J_1+\cdots+J_{n-1} = \frac{c_1^L+\cdots+c_{n-1}^L-\left(c_1^{a_1,l}+\cdots+c_{n-1}^{a_1,l}\right)}{H\left(a_1\right)}\left[1-\frac{z\left(\phi^L-\phi^{a_1,l}\right)}{\ln\frac{c_1^{a_1,l}+\cdots+c_{n-1}^{a_1,l}}{c_1^L+\cdots+c_{n-1}^L}}\right]$$
$$= \frac{c_1^{b_{i-1},r}+\cdots+c_{n-1}^{b_{i-1},r}-\left(c_1^{a_i,l}+\cdots+c_{n-1}^{a_i,l}\right)}{H\left(a_i\right)-H\left(b_{i-1}\right)}\left[1-\frac{z\left(\phi^{b_{i-1},r}-\phi^{a_i,l}\right)}{\ln\frac{c_1^{a_i,l}+\cdots+c_{n-1}^{a_i,l}}{c_1^{b_{i-1},r}+\cdots+c_{n-1}^{b_{i-1},r}}}\right]$$
$$= \frac{c_1^{b_m,r}+\cdots+c_{n-1}^{b_m,r}-\left(c_1^R+\cdots+c_{n-1}^R\right)}{H(1)-H\left(b_m\right)}\left[1-\frac{z\left(\phi^{b_m,r}-\phi^R\right)}{\ln\frac{c_1^R+\cdots+c_{n-1}^R}{c_1^{b_m,r}+\cdots+c_{n-1}^{b_m,r}}}\right],$$

$$J_n = \frac{z\left[c_1^{a_1,l} + \cdots + c_{n-1}^{a_1,l} - \left(c_1^L + \cdots + c_{n-}^L\right)\right]}{z_n H(a_1)} \left[1 - \frac{z_n\left(\phi^L - \phi^{a_1,l}\right)}{\ln \frac{c_1^{a_1,l} + \cdots + c_{n-1}^{a_1,l}}{c_1^L + \cdots + c_{n-1}^L}}\right]$$

$$= \frac{z\left[c_1^{a_i,l} + \cdots + c_{n-1}^{a_i,l} - \left(c_1^{b_{i-1},r} + \cdots + c_{n-1}^{b_{i-1},r}\right)\right]}{z_n\left[H(a_i) - H(b_{i-1})\right]} \left[1 - \frac{z_n\left(\phi^{b_{i-1},r} - \phi^{a_i,l}\right)}{\ln \frac{c_1^{a_i,l} + \cdots + c_{n-1}^{a_i,l}}{c_1^{b_{i-1},r} + \cdots + c_{n-1}^{b_{i-1},r}}}\right]$$

$$= \frac{z\left[c_1^R + \cdots + c_{n-1}^R - \left(c_1^{b_m,r} + \cdots + c_{n-1}^{b_m,r}\right)\right]}{z_n\left[H(1) - H(b_m)\right]} \left[1 - \frac{z_n\left(\phi^{b_m,r} - \phi^R\right)}{\ln \frac{c_1^R + \cdots + c_{n-1}^R}{c_1^{b_m,r} + \cdots + c_{n-1}^{b_m,r}}}\right],$$

$$J_1 + \cdots + J_{n-1} + J_n$$

$$= \frac{(z - z_n)\left[c_1^{b_i,l} + \cdots + c_{n-1}^{b_i,l} - \left(c_1^{a_i,r} + \cdots + c_{n-1}^{a_i,r}\right)\right]}{z_n\left[H(b_i) - H(a_i)\right]} + \frac{Q_i\left(\phi^{b_i,l} - \phi^{a_i,r}\right)}{H(b_i) - H(a_i)},$$

$$\phi^{b_i,l} = \phi^{a_i,r} - \frac{z(J_1 + \cdots + J_{n-1}) + z_n J_n}{z z_n (J_1 + \cdots + J_n)} \times \ln \frac{z(J_1 + \cdots + J_n)\left(c_1^{b_i,l} + \cdots + c_{n-1}^{b_i,l}\right) + Q_i(J_1 + \cdots + J_{n-1})}{z(J_1 + \cdots + J_n)\left(c_1^{a_i,r} + \cdots + c_{n-1}^{a_i,r}\right) + Q_i(J_1 + \cdots + J_{n-1})}, \tag{3.18}$$

and

$$\frac{J_k}{J_1 + \cdots + J_{n-1}} = \frac{c_k^{a_1,l} - c_k^L \mathrm{e}^{z\left(\phi^L - \phi^{a_1,l}\right)}}{c_1^{a_1,l} + \cdots + c_{n-1}^{a_1,l} - \left(c_1^L + \cdots + c_{n-1}^L\right) e^{z\left(\phi^L - \phi^{a_1,l}\right)}}$$
$$= \frac{c_k^{a_i,l} - c_k^{b_{i-1},r} \mathrm{e}^{z\left(\phi^{b_{i-1},r} - \phi^{a_i,l}\right)}}{c_1^{a_i,l} + \cdots + c_{n-1}^{a_i,l} - \left(c_1^{b_{i-1},r} + \cdots + c_{n-1}^{b_{i-1},r}\right) \mathrm{e}^{z\left(\phi^{b_{i-1},r} - \phi^{a_i,l}\right)}}$$
$$= \frac{c_k^R - c_k^{b_m,r} \mathrm{e}^{z\left(\phi^{b_m,r} - \phi^R\right)}}{c_1^R + \cdots + c_{n-1}^R - \left(c_1^{b_m,r} + \cdots + c_{n-1}^{b_m,r}\right) \mathrm{e}^{z\left(\phi^{b_m,r} - \phi^R\right)}}$$
$$= \frac{c_k^{b_i,l} - c_k^{a_i,r} \mathrm{e}^{z\left(\phi^{a_i,r} - \phi^{b_i,l}\right)}}{c_1^{b_i,l} + \cdots + c_{n-1}^{b_i,l} - \left(c_1^{a_i,r} + \cdots + c_{n-1}^{a_i,r}\right) \mathrm{e}^{z\left(\phi^{a_i,r} - \phi^{b_i,l}\right)}}. \tag{3.19}$$

Note that the total number of the unknown parameters

$$\phi^{a_i}, c_1^{a_i}, \cdots, c_n^{a_i}, \phi^{b_i}, c_1^{b_i}, \cdots, c_n^{b_i}, i = 1, \cdots, m$$

and

$$J_1, \cdots, J_n$$

is $2m(n+1)+n$. Also, the total number of Equations (3.18) and (3.19) is exactly $2m(n+1)+n$, which matches the total number of the unknown parameters.

It can be seen that Equations (3.18) and (3.19) are very complicated nonlinear algebraic equations, which are intricately difficult to be solved, however, algebraic Equations (3.19) can be solved in the next section.

4. Main Results

In this section, the main results of this paper, that is, the ratio of J_k to $J_1+\cdots+J_{n-1}$ is independent of the permanent charge $Q(x)$, will be proved. As shown in the following, to justify the main results, it is sufficient that only Equations (3.19) are used.

Theorem 4.1. *Under the assumptions A_1 and A_2, let $\varepsilon=0$ in Equations* (2.7) *and* (2.8), *then one has*

$$\frac{J_k}{J_1+\cdots+J_{n-1}}=\frac{R_k-L_k\mathrm{e}^{z\bar{V}}}{R_1+\cdots+R_{n-1}-\left(L_1+\cdots+L_{n-1}\right)\mathrm{e}^{z\bar{V}}}, \tag{4.20}$$

where $k=1,\cdots,n-1$, which indicates that the ratio of J_k to $J_1+\cdots+J_{n-1}$ is independent of the permanent charge $Q(x)$.

Proof. Substitute the formulae for

$$c_k^L, c_k^{a_i,l}, c_k^{a_i,r}, c_k^{b_i,l}, c_k^{b_i,r}, c_k^R, \phi^{a_i,l}, \phi^{a_i,r}, \phi^{b_i,l}, \phi^{b_i,r}, i=1,\cdots,m; k=1,\cdots,n-1$$

in Equations (2.11), (2.13), (2.15) and (2.17) into algebraic Equations (3.19), then direct calculations show that

$$\begin{aligned}
c_k^{a_i}&=\frac{\left(c_1^{a_i}+\cdots+c_{n-1}^{a_i}\right)\left(R_k-L_k\mathrm{e}^{z\bar{V}}\right)}{R_1+\cdots+R_{n-1}-\left(L_1+\cdots+L_{n-1}\right)\mathrm{e}^{z\bar{V}}}\\
&\quad+\frac{\left(R_1+\cdots+R_{n-1}\right)L_k-\left(L_1+\cdots+L_{n-1}\right)R_k}{R_1+\cdots+R_{n-1}-\left(L_1+\cdots+L_{n-1}\right)\mathrm{e}^{z\bar{V}}}\mathrm{e}^{z\left(\bar{V}-\phi^{a_i}\right)},\\
c_k^{b_i}&=\frac{\left(c_1^{b_i}+\cdots+c_{n-1}^{b_i}\right)\left(R_k-L_k\mathrm{e}^{z\bar{V}}\right)}{R_1+\cdots+R_{n-1}-\left(L_1+\cdots+L_{n-1}\right)\mathrm{e}^{z\bar{V}}}\\
&\quad+\frac{\left(R_1+\cdots+R_{n-1}\right)L_k-\left(L_1+\cdots+L_{n-1}\right)R_k}{R_1+\cdots+R_{n-1}-\left(L_1+\cdots+L_{n-1}\right)\mathrm{e}^{z\bar{V}}}\mathrm{e}^{z\left(\bar{V}-\phi^{b_i}\right)},
\end{aligned} \tag{4.21}$$

where $k=1,\cdots,n-1$ and $i=1,\cdots,m$.

Again, substitute the formulae for

$$c_k^{a_i}, c_k^{b_i}, i=1,\cdots,m; k=1,\cdots,n-1$$

in Equations (4.21) into algebraic Equations (3.19), then the statement in Theorem 4.1 can be obtained. □

Remark 4.2. *The remain parameters $c_1^{a_i}+\cdots+c_{n-1}^{a_i}$, $c_n^{a_i}$, $c_1^{b_i}+\cdots+c_{n-1}^{b_i}$, $c_n^{b_i}$, $J_1+\cdots+J_{n-1}$, J_n, ϕ^{a_i}, $\phi^{b_i}, i=1,\cdots,m$, are determined by nonlinear algebraic Equations* (3.18), *and it seems that it is extremely difficult to get the explicit formulae for these parameters.*

5. Conclusion

In this paper, PNP models with an arbitrary number of positively charged ion species and one negatively charged ion species are investigated under the assumptions A_1 and A_2. By using the geometric singular perturbation theory and solving Equations (3.19), it is proved that the ratio of J_k to $J_1+\cdots+J_{n-1}$ is independent of the permanent charge $Q(x)$. Also, it can be seen that although Equations (3.18) are not used in the proof of Theorem 4.1, the number of solutions to the boundary value problems (1.1) and (1.2) is determined by Equations (3.18). However, due to the fact that Equations (3.18) are very sophisticated algebraic equations, it is very challenging to solve Equations (3.18).

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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