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Existence and Upper Semi-Continuity of Random Attractors for Nonclassical Diffusion Equation with Multiplicative Noise on $\mathbb{R}^n$

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Abstract

This paper is concerned with the existence and upper semi-continuity of random attractors for the nonclassical diffusion equation with arbitrary polynomial growth nonlinearity and multiplicative noise in $H^1(\mathbb{R}^n)$. First, we study the existence and uniqueness of solutions by a noise arising in a continuous random dynamical system and the asymptotic compactness is established by using uniform tail estimate technique, and then the existence of random attractors for the nonclassical diffusion equation with arbitrary polynomial growth nonlinearity. As a motivation of our results, we prove an existence and upper semi-continuity of random attractors with respect to the nonlinearity that enters the system together with the noise.

Keywords

Random Attractors, Nonclassical Diffusion Equations, Asymptotic Compactness, Upper Semi-Continuity

1. Introduction

In this paper, we investigate the asymptotic behavior of solution to the following stochastic nonclassical diffusion equations with arbitrary polynomial growth nonlinearity and multiplicative noise defined in the entire space $\mathbb{R}^n$:

$$du + (\alpha u - \Delta u - \Delta u_t)dt = (g(u) + f(x))dt + bu \circ dW(t), \tag{1.1}$$

with the initial value condition

$$u(x,0)=u_0(x),\quad x\in\mathbb{R}^n, \tag{1.2}$$

where $-\Delta$ is the Laplacian operator with respect to the variable $x\in\mathbb{R}^n$, $u=u(x,t)$ is a real function of $x\in\mathbb{R}^n$ and $t\geq 0$; α,b are proper positive constants; $f\in L^2(\mathbb{R}^n)$; g is a nonlinear function satisfying certain conditions; $W(t)$ is a two-sided real-valued Wiener process on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, where $\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\omega(0)=0\}$, $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$; $\circ$ denotes the Stratonovich sense in the stochastic term. We identify $\omega(t)$ with $W(t)$, *i.e.*, $W(t)=W(t,\omega)=\omega(t)$, $t\in\mathbb{R}$.

The nonclassical diffusion equation is an important mathematical model which depicts such physical phenomena as non-Newtonian flows, solid mechanics, and heat conduction, where the viscidity, the elasticity and the pressure of medium are taken into account. Equations (1.1) is known as the nonclassical diffusion equation when ($\alpha>0$) and the reaction-diffusion equation when ($\alpha=0$), Equations (1.1) this kind of equation has been studied by many researchers and several excellent results have been obtained in the recent twenty years, see Refs. [1] [2] [3] [4].

Since Equations (1.1) contains the term $-\Delta u_t$, it's different from the usual reaction-diffusion equation essentially. For example, the reaction-diffusion equation has some smoothing effect, e.g., although the initial data only belongs to a weaker topology space, the solution with initial conditions will belong to a stronger topology space with higher regularity. The existence, long-time behavior and regularity of solutions of Equations (1.1) have been considered by some recent related works [5]-[16] and the references therein. However, for Equations (1.1), if the initial data u_0 belongs to $H^1(\mathbb{R}^n)$, then the solution $u(x,t)$ is always in $H^1(\mathbb{R}^n)$ and has no higher regularity because of $-\Delta u_t$. There are a great number of results concerning the existence of random attractor involving stochastic partial differential equations, we refer the readers to [17]-[32]. In [20], the author has proved the existence of random attractor for the nonclassical diffusion equation with memory in $\mathcal{M}_1=\mathcal{D}\left(A^{\frac{1}{2}}\right)\times L^2_\mu\left(\mathbb{R}^+,\mathcal{D}\left(A^{\frac{1}{2}}\right)\right)$ on bounded domain.

In the case of unbounded domains established the existence of pullback attractor for the stochastic nonclassical diffusion equation in [17], and existence of random attractor with additive noise in [18]. For the upper semicontinuity of corresponding attractors between autonomous and perturb non-autonomous systems, we can refer to [16] [22] [23]. However, there are fewer results on the existence and upper semi-continuity of pullback attractors for stochastic nonclassical diffusion equation with multiplicative noise on unbounded domain also gives some difficulties since the embedding is no longer compact. Consequently, for Equations (1.1), we cannot use the compact Sobolev embedding to verify the asymptotic compactness of the solutions. Most recently, by using the tail-estimates method, and some omega-limit compactness argument and useful estimates of nonlinearity of the random dynamical system, which shows that the solutions are uniformly asymptotically small when space and time variables approach infinity, the reader can refer to [23] [24] [25] [26].

This paper is organized as follows. In Section 2, we recall some basic concepts and properties for general random dynamics system. In Section 3, we provide some basic settings about Equations (1.1) and show that it generates a random dynamical system on $H^1(\mathbb{R}^n)$. In Section 4, we prove the uniform estimates of solutions, which include the uniform estimates on the tails of solutions. In Section 5, we first establish the asymptotic compactness of the solution operator by given uniform estimates on the tails of solutions, and then prove the existence of a random attractor. The existence and upper semicontinuity (in $H^1(\mathbb{R}^n)$) of random attractors are given in the last section.

2. Preliminaries

As mentioned in the introduction, our main purpose is to prove the existence of the random attractor. For that matter, first, we will recapitulate basic concepts related to random attractors for stochastic dynamical systems. The reader is referred to [19] [22] [26] [29] for more details. Let $(X, \|\cdot\|_X)$ be separable Hilbert space with the Borel σ-algebra $\mathcal{B}(X)$, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, in the sequel, we use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and inner product of $L^2(\mathbb{R}^n)$, respectively.

Definition 2.1 $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\vartheta: \mathbb{R}\times\Omega\to\Omega$ is $(\mathcal{B}(\mathbb{R})\times\mathcal{F}, \mathcal{F})$-measurable, ϑ_0 is the identity on Ω, $\vartheta_{s+t} = \vartheta_t \circ \vartheta_s$ for all $s, t\in\mathbb{R}$ and $\vartheta_t P = P$ for all $t\in\mathbb{R}$.

Definition 2.2 A continuous random dynamical system (RDS) on X over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$ is a mapping

$$\phi: \mathbb{R}^+\times\Omega\times X\to X,\ (t,\omega,x)\mapsto\phi(t,\omega,x),$$

which is $(\mathcal{B}(\mathbb{R}^+)\times\mathcal{F}\times\mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies, for $\mathbb{P}$-a.e. $\omega\in\Omega$,

1) $\phi(0,\omega,\cdot)$ is the identity on X,
2) $\phi(t+s,\omega,\cdot) = \phi(t,\vartheta_s\omega,\cdot)\circ\phi(s,\omega,\cdot)$ for all $t,s\in\mathbb{R}^+$,
3) $\phi(t,\omega,\cdot): X\to X$ is continuous for all $t\in\mathbb{R}^+$.

Hereafter, we always assume that ϕ is continuous RDS on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\vartheta_t)_{t\in\mathbb{R}})$.

Definition 2.3 A set-valued mapping $\{D(\omega)\}: \Omega\to 2^X, \omega\to D(\omega)$, is said to be a random set if the mapping $\omega\mapsto d(u, D(\omega))$ is measurable for every $u\in X$. If $D(\omega)$ is also closed (compact) for each $\omega\in\Omega$, $\{D(\omega)\}$ is called a random closed (compact) set. A random set $\{D(\omega)\}$ is said to be bounded if there exist $u_0\in X$ and a random variable $R_1(\omega)>0$ such that $D(\omega)\subset\{u\in X: \|u-u_0\|_X\le R_1(\omega)\}$ for all $\omega\in\Omega$.

Definition 2.4 A random bounded set $\{D(\omega)\}$ is called tempered provided for $\mathbb{P}$-a.e, $\omega \in \Omega$,

$$\lim_{t\to+\infty} \mathrm{e}^{-\beta t} d\left(D\left(\vartheta_{-t}\omega\right)\right) = 0 \quad \text{for all} \quad \beta > 0,$$

where $d(D) = \sup\left\{\|b\|_X : b \in D\right\}$.

Definition 2.5 Let $\mathfrak{D}$ be a collection of random subset of X and $\{K(\omega)\} \in \mathfrak{D}$. Then $\{K(\omega)\}$ is called a random absorbing set for ϕ in $\mathfrak{D}$ for every $D \in \mathfrak{D}$ and $\mathbb{P}$-a.e, $\omega \in \Omega$, there exist $t_0(\omega)$ such that $\phi\left(t, \vartheta_{-t}\omega, D\left(\vartheta_{-t}\omega\right)\right) \subseteq K(\omega)$ for all $t \ge t_0(\omega)$.

Definition 2.6 A random set $\{K_1(\omega)\}$ is said to be a random attracting set if for every tempered random set $\{D(\omega)\}$, and $\mathbb{P}$-a.e, $\omega \in \Omega$, we have

$$\lim_{t\to+\infty} d_H\left(\phi\left(t, \vartheta_{-t}\omega, D\left(\vartheta_{-t}\omega\right), K_1(\omega)\right)\right) = 0,$$

where d_H is the Hausdorff semi-distance given by

$d_H(E, F) = \sup_{u\in E} \inf_{v\in F} \|u - v\|_X$ for every $E, F \subset X$.

Definition 2.7 Let $\mathfrak{D}$ be the set of all random tempered sets in X. Then ϕ is said to be asymptotically compact in X if for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\left\{\phi\left(t_n, \vartheta_{-t_n}\omega, X_n\right)\right\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $X_n \in B\left(\vartheta_{-t_n}\omega\right)$ with $\{B(\omega)\} \in \mathfrak{D}$.

Definition 2.8 A random compact set $\{\mathcal{A}(\omega)\}$ is said to be a random attractor if it is a random attracting set and $\phi\left(t, \omega, \mathcal{A}(\omega)\right) = \mathcal{A}\left(\vartheta_{-t}\omega\right)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $t \ge 0$.

Theorem 2.9 Let ϕ be a continuous random dynamical system on X over $\left(\Omega, \mathcal{F}, \mathbb{P}, \left(\vartheta_t\right)_{t\in\mathbb{R}}\right)$. If there is a closed random tempered absorbing set $\{K(\omega)\}$ of ϕ and ϕ is asymptotically compact in X, then $\{\mathcal{A}(\omega)\}$ is a random attractor of ϕ, where

$$\mathcal{A}(\omega) = \bigcap_{t>0} \overline{\bigcup_{\tau\ge t} \phi\left(\tau, \vartheta_{-\tau}\omega, K\left(\vartheta_{-\tau}\omega\right)\right)}, \ \omega \in \Omega.$$

Moreover, $\{\mathcal{A}(\omega)\}$ is the unique attractor of ϕ.

Lemma 2.10 ([21]) Let $\left(X, \|\cdot\|_X\right)$ be a Banach space and ϕ_0 be an autonomous dynamical system with the global attractor $\mathcal{A}_0$ in X. Given $b > 0$, suppose that ϕ_0 is the perturbed random dynamical system with a random attractor $\mathcal{A}_b \in \mathcal{D}$ and a random absorbing set $E_b \in \mathcal{D}$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\operatorname{dist}\left(\mathcal{A}_b(\omega), \mathcal{A}_0\right) \to 0, \quad \text{as } b \downarrow 0$$

if the following conditions are satisfied:

1) For $\mathbb{P}$-a.e. $\omega \in \Omega$, $t \ge 0$, $b_n \to 0$, and $x_n, x \in X$ with $x_n \to x$, it hold that $\lim_{n\to\infty} \phi_{b_n}(t, \omega) x_n = \phi_0(t) x$;

2) There exists some deterministic constant c such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\limsup_{b\to 0^+} \left\|E_b(\omega)\right\|_X \le c,$$

where $\left\|E_b(\omega)\right\|_X = \sup_{x\in E_b(\omega)} \|x\|_X$;

3) There exists a $b_0 > 0$ such that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\bigcup_{0<b<b_0} \mathcal{A}_b(\omega)$ is precompact in X.

3. Random Dynamical System

In this section, we show that there is a continuous random dynamical system generated by the stochastic nonclassical diffusion equation defined on $\mathbb{R}^n$ with arbitrary polynomial growth nonlinearity and multiplicative noise:

$$\mathrm{d}u+\left(\alpha u-\Delta u-\Delta u_t\right)\mathrm{d}t=\left(g\left(u\right)+f\left(x\right)\right)\mathrm{d}t+bu\circ\mathrm{d}W\left(t\right), \tag{3.1}$$

with the initial value condition

$$u\left(x,0\right)=u_0\left(x\right),\quad x\in\mathbb{R}^n, \tag{3.2}$$

where α,b are proper positive constants, $f\in L^2\left(\mathbb{R}^n\right)$ and $g\left(u\right)$ is a nonlinear function satisfying the following conditions are the same as those in [24]:

$$-\beta_2\left|s\right|^p-\delta_2\left|s\right|^2\le g\left(s\right)s\le-\beta_1\left|s\right|^p+\delta_1\left|s\right|^2 \text{ for } s\in\mathbb{R} \text{ and } p\ge 2, \tag{3.3}$$

$$-\beta_4\left|s\right|^{p-1}-\delta_4\left|s\right|\le g\left(s\right)\le-\beta_3\left|s\right|^{p-1}+\delta_3\left|s\right| \text{ for } s\in\mathbb{R} \text{ and } p\ge 2, \tag{3.4}$$

$$g'\left(s\right)\le L \text{ for } s\in\mathbb{R}, \tag{3.5}$$

where $L,\beta_i,\delta_i\left(i=1,2,3,4\right)$ are a non-negative constant.

To model the random noise in Equation (3.1), we need to define a shift operator $\left\{\vartheta_t\right\}_{t\in\mathbb{R}}$ on Ω (where Ω is defined in the introduction) by

$$\vartheta_t\omega\left(\cdot\right)=\omega\left(\cdot+t\right)-\omega\left(t\right),\quad t\in\mathbb{R}, \tag{3.6}$$

then $\left(\Omega,\mathcal{F},\mathbb{P},\left(\vartheta_t\right)_{t\in\mathbb{R}}\right)$ is an ergodic metric dynamical system, see [20] [24].

For our purpose, it is convenient to convert Equation (3.1) into a deterministic system with a random parameter, and then show that it generates a random dynamical system.

We now introduce an Ornstein-Uhlenbeck process given by the Brownian motion. Put

$$z\left(\vartheta_t\omega\right):=-\int_{-\infty}^{0}\mathrm{e}^s\left(\vartheta_t\omega\right)\left(s\right)\mathrm{d}s,\ t\in\mathbb{R}, \tag{3.7}$$

which is called the Ornstein-Uhlenbeck process and solves the Itô equation

$$\mathrm{d}z+z\mathrm{d}t=\mathrm{d}W\left(t\right). \tag{3.8}$$

From [19] [25] [27] [28], it is known that the random variable $z\left(\omega\right)$ is tempered, and there is a ϑ_t-invariant set $\tilde{\Omega}\subset\Omega$ of full $\mathbb{P}$ measure such that for every $\omega\in\tilde{\Omega}$, $t\mapsto z\left(\vartheta_t\omega\right)$ is continuous in *t*; $\lim_{t\to\pm\infty}\frac{\left|z\left(\vartheta_t\omega\right)\right|}{\left|t\right|}=0$; and $\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z\left(\vartheta_s\omega\right)\mathrm{d}s=0$.

To show that Equation (3.1) generates a random dynamical system, we let

$$v\left(t\right)=\mathrm{e}^{-bz\left(\vartheta_t\omega\right)}u\left(t\right), \tag{3.9}$$

where u is a solution of Equation (3.1). Then we can consider the following evolution equation with random coefficients but without white noise:

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\alpha v-\Delta v-\Delta v_t=\mathrm{e}^{-bz\left(\vartheta_t\omega\right)}\left(g\left(\mathrm{e}^{bz\left(\vartheta_t\omega\right)}v\right)+f\left(x\right)\right)+bz\left(\vartheta_t\omega\right)v, \tag{3.10}$$

with the initial value condition

$$v(x,0) = v_0(x) = \mathrm{e}^{-bz(\vartheta_t\omega)}u_0(x),\ \ x \in \mathbb{R}^n. \tag{3.11}$$

Definition 3.1. A function υ is called a weak solution of Equations (3.10) and (3.11) on the interval $[0,\infty)$ if

$$\upsilon(\cdot,\omega,\upsilon_0) \in \mathcal{C}\left([0,\infty);H^1(\mathbb{R}^n)\right)\cap L^2\left([0,\infty);H^2(\mathbb{R}^n)\right),$$

$$\upsilon_t(\cdot,\omega,\upsilon_0) \in L^2\left([0,\infty);H^1(\mathbb{R}^n)\right),$$

and

$$(v_t + \alpha v - \Delta v - \Delta v_t, \varphi) = \left(\mathrm{e}^{-bz(\vartheta_t\omega)}\left(g\left(\mathrm{e}^{bz(\vartheta_t\omega)}v\right) + f(x)\right) + bz(\vartheta_t\omega)v, \varphi\right),$$

for all test function $\varphi \in \mathcal{C}\left([0,\infty)\times\mathbb{R}^n\right)$.

Theorem 3.2. Under the assumptions (3.3)-(3.5), $f \in L^2(\mathbb{R}^n)$ for P-a.e. $\omega\in\Omega$ and any $\upsilon_0 \in H^1(\mathbb{R}^n)$, there is a unique solution $\upsilon(\cdot,\omega,\upsilon_0)$ satisfying

$$\upsilon(\cdot,\omega,\upsilon_0) \in \mathcal{C}\left([0,\infty);H^1(\mathbb{R}^n)\right)\cap L^2\left([0,\infty);H^2(\mathbb{R}^n)\right).$$

From Theorem 3.2 above, we now define a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n)$$

by

$$\phi(t,\omega,u_0) = u(t,\omega,u_0) = \mathrm{e}^{bz(\vartheta_t\omega)}v(t,\omega,v_0),$$

for all

$$(t,\omega,u_0) \in \mathbb{R}^+ \times \Omega \times H^1(\mathbb{R}^n).$$

Then ϕ satisfies conditions (1) and (2) in Definition 2.2. Therefore, ϕ is a continuous random dynamical system associated with Equation (3.1) on $\mathbb{R}^n$.

4. Uniform Estimates of Solutions

In this section, we derive uniform estimates on the solutions of (3.10) and (3.11) defined on $\mathbb{R}^n$ when $t\to\infty$ with the purpose of proving the existence of a bounded random absorbing set and the asymptotic compactness of the random dynamical system associated with the equation. In particular, we will show that the tails of the solutions for large space variable are uniformly small when time is sufficiently large. Some techniques about the unbounded case can be founded in [14] [24] [25] [26]. Here we always assume that $\mathfrak{D}$ is the collection of all tempered random subsets of $H^1(\mathbb{R}^n)$ with respect to $\left(\Omega,\mathcal{F},\mathbb{P},(\vartheta_t)_{t\in\mathbb{R}}\right)$. The next Lemma shows that ϕ has a random absorbing set in $\mathfrak{D}$.

Lemma 4.1 Assume that $f \in L^2(\mathbb{R}^n)$, and (3.3)-(3.5) hold. Then there exists a random ball $\{K(\omega)\}\in\mathfrak{D}$ centered at 0 with random radius $\rho(\omega)>0$ such that $\{K(\omega)\}$ is a random absorbing set for ϕ in $\mathfrak{D}$, that is, for any $\{B(\omega)\}\in\mathfrak{D}$ and $\mathbb{P}$-a.e. $\omega\in\Omega$, there is $T_B(\omega)>0$ such that

$$\phi\left(t,\vartheta_{-t}\omega,B\left(\vartheta_{-t}\omega\right)\right)\subseteq K\left(\omega\right)\text{ for all } t>T_B\left(\omega\right). \tag{4.1}$$

Proof We first derive uniform estimates on $v(t)=\mathrm{e}^{-bz(\vartheta_t\omega)}u(t)$ from which the uniform estimates on $u(t)$. Multiplying Equation (3.10) with v and then integrating over $\mathbb{R}^n$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+\|\nabla v\|^2\right)+\alpha\|v\|^2+\|\nabla v\|^2\\&=\mathrm{e}^{-bz(\vartheta_t\omega)}\int_{\mathbb{R}^n}g\left(\mathrm{e}^{bz(\vartheta_t\omega)}v\right)v\mathrm{d}x+\mathrm{e}^{-bz(\vartheta_t\omega)}\left(f,v\right)+bz\left(\vartheta_t\omega\right)\|v\|^2.\end{aligned} \tag{4.2}$$

By the Hölder inequality and the Young inequality, we conclude

$$\mathrm{e}^{-bz(\vartheta_t\omega)}\left(f,v\right)\le\frac{1}{2\alpha}\mathrm{e}^{-2bz(\vartheta_t\omega)}\|f\|^2+\frac{\alpha}{2}\|v\|^2. \tag{4.3}$$

By condition (3.3), we get

$$\begin{aligned}&\mathrm{e}^{-bz(\vartheta_t\omega)}\int_{\mathbb{R}^n}g\left(\mathrm{e}^{bz(\vartheta_t\omega)}v\right)v\mathrm{d}x\\&=\mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n}g\left(u\right)u\mathrm{d}x\\&\le\mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n}\left[-\beta_1|u|^p+\delta_1|u|^2\right]\mathrm{d}x\\&\le\mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n}\left[-\beta_1|u|^p+\delta_1\left|\mathrm{e}^{bz(\vartheta_t\omega)}v\right|^2\right]\mathrm{d}x\\&\le-\beta_1\mathrm{e}^{-2bz(\vartheta_t\omega)}\|u\|_p^p+\delta_1\|v\|^2.\end{aligned} \tag{4.4}$$

Then inserting (4.3) and (4.4) into (4.2), it yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+\|\nabla v\|^2\right)-\left(2bz\left(\vartheta_t\omega\right)-\alpha_0\right)\|v\|^2+2\|\nabla v\|^2+2\beta_1\mathrm{e}^{-2bz(\vartheta_t\omega)}\|u\|_p^p\\&\le\frac{1}{\alpha}\mathrm{e}^{-2bz(\vartheta_t\omega)}\|f\|^2,\end{aligned} \tag{4.5}$$

where $\alpha-2\delta_1=\alpha_0>0$.

Noticing that, from (4.5), let $\gamma=\min\left\{-\left(2bz\left(\vartheta_t\omega\right)-\alpha_0\right),1\right\}$, it follows that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+\|\nabla v\|^2\right)+\gamma\left(\|v\|^2+\|\nabla v\|^2\right)+\|\nabla v\|^2+2\beta_1\mathrm{e}^{-2bz(\vartheta_t\omega)}\|u\|_p^p\\&\le\frac{1}{\alpha}\mathrm{e}^{-2bz(\vartheta_t\omega)}\|f\|^2.\end{aligned} \tag{4.6}$$

Hence, we can rewrite the above equation as

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+\|\nabla v\|^2\right)+\gamma\left(\|v\|^2+\|\nabla v\|^2\right)+2\beta_1\mathrm{e}^{-2bz(\vartheta_t\omega)}\|u\|_p^p\\&\le\frac{1}{\alpha}\mathrm{e}^{-2bz(\vartheta_t\omega)}\|f\|^2.\end{aligned} \tag{4.7}$$

By applying Gronwall's lemma to (4.7), we find that

$$\begin{aligned}&\left\|v\left(t,\omega,v_0\left(\omega\right)\right)\right\|^2+\left\|\nabla v\left(t,\omega,v_0\left(\omega\right)\right)\right\|^2\\&+2\beta_1\mathrm{e}^{-\gamma t}\int_0^t\mathrm{e}^{-2bz(\vartheta_s\omega)+\gamma s}\left\|u\left(s,\omega,u_0\left(\omega\right)\right)\right\|_p^p\mathrm{d}s\\&\le\mathrm{e}^{-\gamma t}\left(\left\|v_0\left(\omega\right)\right\|^2+\left\|\nabla v_0\left(\omega\right)\right\|^2\right)+\frac{\|f\|^2}{\alpha}\mathrm{e}^{-\gamma t}\int_0^t\mathrm{e}^{-2bz(\vartheta_s\omega)+\gamma s}\mathrm{d}s.\end{aligned} \tag{4.8}$$

By replacing ω by $\vartheta_{-t}\omega$ in (4.8), we get

$$\begin{aligned}&\left\|v\left(t,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2+\left\|\nabla v\left(t,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2\\&+2\beta_1\mathrm{e}^{-\gamma t}\int_0^t\mathrm{e}^{-2bz\left(\vartheta_{s-t}\omega\right)+\gamma s}\left\|u\left(s,\vartheta_{-t}\omega,u_0\left(\vartheta_{-t}\omega\right)\right)\right\|_p^p\mathrm{d}s\\&\le\mathrm{e}^{-\gamma t}\left(\left\|v_0\left(\vartheta_{-t}\omega\right)\right\|^2+\left\|\nabla v_0\left(\vartheta_{-t}\omega\right)\right\|^2\right)+\frac{\|f\|^2}{\alpha}\int_{-\infty}^0\mathrm{e}^{-2bz\left(\vartheta_s\omega\right)+\gamma s}\mathrm{d}s.\end{aligned}\tag{4.9}$$

By the properties of Ornstein-Uhlenbeck process,

$$\int_{-\infty}^0\mathrm{e}^{-2bz\left(\vartheta_s\omega\right)+\gamma s}\mathrm{d}s<+\infty.\tag{4.10}$$

Notice that $\{B(\omega)\}\in\mathfrak{D}$ is tempered, then for any $v_0\left(\vartheta_{-t}\omega\right)\in B\left(\vartheta_{-t}\omega\right)$,

$$\lim_{t\to+\infty}\mathrm{e}^{-\gamma t}\left(\left\|v_0\left(\vartheta_{-t}\omega\right)\right\|^2+\left\|\nabla v_0\left(\vartheta_{-t}\omega\right)\right\|^2\right)=0.\tag{4.11}$$

We can choose

$$\rho(\omega)=1+\frac{\|f\|^2}{\alpha}\int_{-\infty}^0\mathrm{e}^{-2bz\left(\vartheta_s\omega\right)+\gamma s}\mathrm{d}s.\tag{4.12}$$

And let

$$K(\omega)=\left\{v\in H^1\left(\mathbb{R}^n\right):\|\nabla v\|^2\le\rho(\omega)\right\}.\tag{4.13}$$

Then $\{K(\omega)\}\in\mathfrak{D}$, and $\{K(\omega)\}$ is a random absorbing set for ϕ in $\mathfrak{D}$, which completes the proof. □

Lemma 4.2 Assume that $f\in L^2\left(\mathbb{R}^n\right)$, and (3.3)-(3.5) hold. Then there exists a tempered random variable $\tilde{R}_1(\omega)>0$, such that for any $\{B(\omega)\}\in\mathfrak{D}$ and $v_0(\omega)\in B(\omega)$, there exists a $T_B(\omega)>0$ such that the solution ϕ of (3.10) satisfies for $\mathbb{P}$-a.e. $\omega\in\Omega$, for all $t\ge T_B(\omega)$,

$$\int_t^{t+1}\left\|\nabla\phi\left(s,\vartheta_{-t-1}\omega,v_0\left(\vartheta_{-t-1}\omega\right)\right)\right\|^2\mathrm{d}s\le\tilde{R}_1(\omega).\tag{4.14}$$

Proof By substituting t by $\hat{T}$ and ω by $\vartheta_{-t}\omega$ in (4.8) for any $\hat{T}\ge0$, we find that

$$\begin{aligned}&\left\|v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2+\left\|\nabla v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2\\&\le\mathrm{e}^{-\gamma\hat{T}}\left(\left\|v_0\left(\vartheta_{-t}\omega\right)\right\|^2+\left\|\nabla v_0\left(\vartheta_{-t}\omega\right)\right\|^2\right)+\frac{\|f\|^2}{\alpha}\int_0^{\hat{T}}\mathrm{e}^{-2bz\left(\vartheta_{s-t}\omega\right)+\gamma\left(s-\hat{T}\right)}\mathrm{d}s.\end{aligned}\tag{4.15}$$

Multiplying two sides of Equation (4.15) by $\mathrm{e}^{\gamma\left(\hat{T}-t\right)}$, then simplifying it, we find that for all $t\ge\hat{T}$

$$\begin{aligned}&\mathrm{e}^{\gamma\left(\hat{T}-t\right)}\left(\left\|v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2+\left\|\nabla v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2\right)\\&\le\mathrm{e}^{-\gamma t}\left(\left\|v_0\left(\vartheta_{-t}\omega\right)\right\|^2+\left\|\nabla v_0\left(\vartheta_{-t}\omega\right)\right\|^2\right)+\frac{\|f\|^2}{\alpha}\int_0^{\hat{T}}\mathrm{e}^{-2bz\left(\vartheta_{s-t}\omega\right)+\gamma(s-t)}\mathrm{d}s.\end{aligned}\tag{4.16}$$

By the Gronwall lemma to (4.6), we get that for all $t\ge\hat{T}$,

$$\begin{aligned}&\left\|v\left(t,\omega,v_0(\omega)\right)\right\|^2+\left\|\nabla v\left(t,\omega,v_0(\omega)\right)\right\|^2\\&\le\mathrm{e}^{\gamma\left(\hat{T}-t\right)}\left(\left\|v\left(\hat{T},\omega,v_0(\omega)\right)\right\|^2+\left\|\nabla v\left(\hat{T},\omega,v_0(\omega)\right)\right\|^2\right)+\frac{\|f\|^2}{\alpha}\int_{\hat{T}}^t\mathrm{e}^{-2bz\left(\vartheta_s\omega\right)+\gamma(s-t)}\mathrm{d}s\\&-\int_{\hat{T}}^t\mathrm{e}^{\gamma(s-t)}\left\|\nabla v\left(s,\omega,v_0(\omega)\right)\right\|^2\mathrm{d}s,\end{aligned}\tag{4.17}$$

which obviously gives

$$\begin{aligned}&\int_{\hat{T}}^{t}\mathrm{e}^{\gamma(s-t)}\left\|\nabla v\left(s,\omega,v_0\left(\omega\right)\right)\right\|^2\mathrm{d}s\\&\le \mathrm{e}^{\gamma\left(\hat{T}-t\right)}\left(\left\|v\left(\hat{T},\omega,v_0\left(\omega\right)\right)\right\|^2+\left\|\nabla v\left(\hat{T},\omega,v_0\left(\omega\right)\right)\right\|^2\right)+\frac{\|f\|^2}{\alpha}\int_{\hat{T}}^{t}\mathrm{e}^{-2bz(\vartheta_s\omega)+\gamma(s-t)}\mathrm{d}s.\end{aligned}\tag{4.18}$$

By replacing ω by $\vartheta_{-t}\omega$ into (4.18), we get

$$\begin{aligned}&\int_{\hat{T}}^{t}\mathrm{e}^{\gamma(s-t)}\left\|\nabla v\left(s,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2\mathrm{d}s\\&\le \mathrm{e}^{\gamma\left(\hat{T}-t\right)}\left(\left\|v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2+\left\|\nabla v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|\right)^2\\&\quad+\frac{\|f\|^2}{\alpha}\int_{\hat{T}}^{t}\mathrm{e}^{-2bz(\vartheta_{s-t}\omega)+\gamma(s-t)}\mathrm{d}s.\end{aligned}\tag{4.19}$$

Together with (4.16) and (4.19), we have

$$\begin{aligned}&\int_{\hat{T}}^{t}\mathrm{e}^{\gamma(s-t)}\left\|\nabla v\left(s,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2\mathrm{d}s\\&\le \mathrm{e}^{-\gamma t}\left(\left\|v_0\left(\vartheta_{-t}\omega\right)\right\|^2+\left\|\nabla v_0\left(\vartheta_{-t}\omega\right)\right\|^2\right)+\frac{\|f\|^2}{\alpha}\int_{-t}^{0}\mathrm{e}^{-2bz(\vartheta_s\omega)+\gamma s}\mathrm{d}s.\end{aligned}\tag{4.20}$$

Replacing $\hat{T}$ by t and t by $t+1$ in (4.20), we have

$$\begin{aligned}&\int_{t}^{t+1}\mathrm{e}^{\gamma(s-t-1)}\left\|\nabla v\left(s,\vartheta_{-t-1}\omega,v_0\left(\vartheta_{-t-1}\omega\right)\right)\right\|^2\mathrm{d}s\\&\le \mathrm{e}^{-\gamma(t+1)}\left(\left\|v_0\left(\vartheta_{-t-1}\omega\right)\right\|^2+\left\|\nabla v_0\left(\vartheta_{-t-1}\omega\right)\right\|^2\right)+\frac{\|f\|^2}{\alpha}\int_{-t-1}^{0}\mathrm{e}^{-2bz(\vartheta_s\omega)+\gamma s}\mathrm{d}s.\end{aligned}\tag{4.21}$$

For $s\in[t,t+1]$, to yield that

$$\begin{aligned}&\int_{t}^{t+1}\mathrm{e}^{\gamma(s-t-1)}\left\|\nabla v\left(s,\vartheta_{-t-1}\omega,v_0\left(\vartheta_{-t-1}\omega\right)\right)\right\|^2\mathrm{d}s\\&\ge \int_{t}^{t+1}\mathrm{e}^{-\gamma}\left\|\nabla v\left(s,\vartheta_{-t-1}\omega,v_0\left(\vartheta_{-t-1}\omega\right)\right)\right\|^2\mathrm{d}s.\end{aligned}\tag{4.22}$$

By the property of $z(\omega)$ and temperedness of $\left\|v_0(\omega)\right\|$, there exists $T_B(\omega)>0$ such that for all $t\ge T_B(\omega)$, from (4.21) and (4.22), we find that

$$\int_{t}^{t+1}\left\|\nabla v\left(s,\vartheta_{-t-1}\omega,v_0\left(\vartheta_{-t-1}\omega\right)\right)\right\|^2\mathrm{d}s\le 1+\frac{\|f\|^2}{\alpha}\int_{-\infty}^{0}\mathrm{e}^{-2bz(\vartheta_s\omega)+\gamma(s+1)}\mathrm{d}s\le \tilde{R}_1(\omega).\tag{4.23}$$

It is easy to check that $\tilde{R}_1(\omega)$ is tempered. This completes the proof. □

Lemma 4.3 Assume that $f\in L^2\left(\mathbb{R}^n\right)$, (3.3)-(3.5) hold. Then, there exists a tempered random variable $\tilde{R}_2(\omega),\tilde{R}(\omega)>0$, such that for any $\{B(\omega)\}\in\mathfrak{D}$ and $v_0(\omega)\in B(\omega)$, there exists a $T_B(\omega)>0$ such that the solution ϕ of (3.10) satisfies for $\mathbb{P}$-a.e. $\omega\in\Omega$, for all $t\ge T_B(\omega)$,

$$\int_{t}^{t+1}\left\|\Delta\phi\left(s,\vartheta_{-t-1}\omega,v_0\left(\vartheta_{-t-1}\omega\right)\right)\right\|^2\mathrm{d}s\le \tilde{R}_2(\omega).\tag{4.24}$$

$$\left\|\nabla\phi\left(t+1,\vartheta_{-t-1}\omega,v_0\left(\vartheta_{-t-1}\omega\right)\right)\right\|^2+\left\|\Delta\phi\left(t+1,\vartheta_{-t-1}\omega,v_0\left(\vartheta_{-t-1}\omega\right)\right)\right\|^2\le \tilde{R}(\omega).\tag{4.25}$$

Proof Taking the inner product of Equation (3.10) with Δv in $L^2\left(\mathbb{R}^n\right)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|^2+\|\Delta v\|^2\right)+\alpha\|\nabla v\|^2+\|\Delta v\|^2\\&=\mathrm{e}^{-bz(\vartheta_t\omega)}\int_{\mathbb{R}^n}g\left(\mathrm{e}^{bz(\vartheta_t\omega)}v\right)\Delta v\mathrm{d}x+\mathrm{e}^{-bz(\vartheta_t\omega)}(f,\Delta v)+bz(\vartheta_t\omega)\|\nabla v\|^2.\end{aligned}\tag{4.26}$$

Now, we estimate the first term on the right-hand side of (4.26) by the condition (3.5), we get

$$\begin{aligned}&\mathrm{e}^{-bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} g\left(\mathrm{e}^{bz(\vartheta_t\omega)}v\right)\Delta v\mathrm{d}x\\&=\mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} g(u)\Delta u\mathrm{d}x\le \mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n}\frac{\partial g}{\partial u}|\nabla u|^2\,\mathrm{d}x\le L\|\nabla v\|^2.\end{aligned}\tag{4.27}$$

On the other hand, in the second term on the right-hand side of (4.26) by Hölder' inequality and Young inequality, we conclude

$$\mathrm{e}^{-bz(\vartheta_t\omega)}(f,\Delta v)\le \mathrm{e}^{-bz(\vartheta_t\omega)}\|f\|\|\Delta v\|\le\frac{1}{2}\mathrm{e}^{-2bz(\vartheta_t\omega)}\|f\|^2+\frac{1}{2}\|\Delta v\|^2.\tag{4.28}$$

Then inserting (4.27) and (4.28) into (4.26), it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|^2+\|\Delta v\|^2\right)-\left(2bz(\vartheta_t\omega)-\alpha_1\right)\|\nabla v\|^2+\|\Delta v\|^2\le \mathrm{e}^{-2bz(\vartheta_t\omega)}\|f\|^2,\tag{4.29}$$

where $2(\alpha-L)=\alpha_1>0$.

Noticing that, from (4.29), let $\tilde{\gamma}=\min\left\{-\left(2bz(\vartheta_t\omega)-\alpha_1\right),\frac{1}{2}\right\}$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|^2+\|\Delta v\|^2\right)+\tilde{\gamma}\left(\|\nabla v\|^2+\|\Delta v\|^2\right)+\frac{1}{2}\|\Delta v\|^2\le \mathrm{e}^{-2bz(\vartheta_t\omega)}\|f\|^2.\tag{4.30}$$

Hence, we can rewrite the above equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|^2+\|\Delta v\|^2\right)+\tilde{\gamma}\left(\|\nabla v\|^2+\|\Delta v\|^2\right)\le \mathrm{e}^{-2bz(\vartheta_t\omega)}\|f\|^2.\tag{4.31}$$

By applying the Gronwall lemma to (4.30), we find that

$$\begin{aligned}&\left\|\nabla v(t,\omega,v_0(\omega))\right\|^2+\left\|\Delta v(t,\omega,v_0(\omega))\right\|^2+\frac{\mathrm{e}^{-\tilde{\gamma}t}}{2}\int_0^t \mathrm{e}^{\tilde{\gamma}s}\left\|\Delta v(s,\omega,v_0(\omega))\right\|^2\mathrm{d}s\\&\le \mathrm{e}^{-\tilde{\gamma}t}\left(\left\|\nabla v_0(\omega)\right\|^2+\left\|\Delta v_0(\omega)\right\|^2\right)+\|f\|^2\,\mathrm{e}^{-\tilde{\gamma}t}\int_0^t \mathrm{e}^{-2bz(\vartheta_s\omega)+\tilde{\gamma}s}\mathrm{d}s,\end{aligned}\tag{4.32}$$

which obviously gives

$$\begin{aligned}&\int_0^t \mathrm{e}^{\tilde{\gamma}(s-t)}\left\|\Delta v(s,\omega,v_0(\omega))\right\|^2\mathrm{d}s\\&\le \mathrm{e}^{-\tilde{\gamma}t}\left(\left\|\nabla v_0(\omega)\right\|^2+\left\|\Delta v_0(\omega)\right\|^2\right)+\|f\|^2\int_0^t \mathrm{e}^{-2bz(\vartheta_s\omega)+\tilde{\gamma}(s-t)}\mathrm{d}s,\end{aligned}\tag{4.33}$$

By replacing ω by $\vartheta_{-t}\omega$ and t by $t+1$ into (4.33), we get

$$\begin{aligned}&\int_0^{t+1}\mathrm{e}^{\tilde{\gamma}(s-t-1)}\left\|\Delta v(s,\vartheta_{-t-1}\omega,v_0(\vartheta_{-t-1}\omega))\right\|^2\mathrm{d}s\\&\le \mathrm{e}^{-\tilde{\gamma}(t+1)}\left(\left\|\nabla v_0(\vartheta_{-t-1}\omega)\right\|^2+\left\|\Delta v_0(\vartheta_{-t-1}\omega)\right\|^2\right)+\|f\|^2\int_0^{t+1}\mathrm{e}^{-2bz(\vartheta_{s-t-1}\omega)+\tilde{\gamma}(s-t-1)}\mathrm{d}s.\end{aligned}\tag{4.34}$$

Thanks to

$$\begin{aligned}&\int_0^{t+1}\mathrm{e}^{\tilde{\gamma}(s-t-1)}\left\|\Delta v(s,\vartheta_{-t-1}\omega,v_0(\vartheta_{-t-1}\omega))\right\|^2\mathrm{d}s\\&\ge\int_t^{t+1}\mathrm{e}^{\tilde{\gamma}(s-t-1)}\left\|\Delta v(s,\vartheta_{-t-1}\omega,v_0(\vartheta_{-t-1}\omega))\right\|^2\mathrm{d}s.\end{aligned}\tag{4.35}$$

Together with (4.34) and (4.35), we have

$$\int_t^{t+1} e^{\tilde{\gamma}(s-t-1)} \left\| \Delta v\left(s, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 ds \\ \le e^{-\tilde{\gamma}(t+1)} \left(\left\| \nabla v_0\left(\vartheta_{-t-1}\omega\right) \right\|^2 + \left\| \Delta v_0\left(\vartheta_{-t-1}\omega\right) \right\|^2 \right) + \|f\|^2 \int_0^{t+1} e^{-2bz\left(\vartheta_{s-t-1}\omega\right)+\tilde{\gamma}(s-t-1)} ds. \tag{4.36}$$

For $s \in [t, t+1]$, to yield that

$$\int_t^{t+1} e^{\tilde{\gamma}(s-t-1)} \left\| \Delta v\left(s, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 ds \\ \ge \int_t^{t+1} e^{-\tilde{\gamma}} \left\| \Delta v\left(s, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 ds. \tag{4.37}$$

Together with (4.36) and (4.37), we have

$$\int_t^{t+1} \left\| \Delta v\left(s, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 ds \\ \le e^{-\tilde{\gamma}t} \left(\left\| \nabla v_0\left(\vartheta_{-t-1}\omega\right) \right\|^2 + \left\| \Delta v_0\left(\vartheta_{-t-1}\omega\right) \right\|^2 \right) + \|f\|^2 \int_{-t-1}^{0} e^{-2bz\left(\vartheta_s\omega\right)+\tilde{\gamma}(s+1)} ds. \tag{4.38}$$

By the property of $z(\omega)$ and temperedness of $\left\| v_0(\omega) \right\|$, there exists $T_B(\omega) > 0$ such that for all $t \ge T_B(\omega)$, from (4.38), we find that

$$\int_t^{t+1} \left\| \Delta v\left(s, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 ds \le 1 + \|f\|^2 \int_{-\infty}^{0} e^{-2bz\left(\vartheta_s\omega\right)+\tilde{\gamma}(s+1)} ds \le \tilde{R}_2(\omega). \tag{4.39}$$

It is easy to check that $\tilde{R}_2(\omega)$ is tempered.

Now, let $T_B(\omega)$ be the non-negative constant in Lemma 4.2 and Equation (4.39), take $t \ge T_B(\omega)$ and $s \in (t, t+1)$. Then integrate (4.31) over $(s, t+1)$, we find that

$$\left\| \nabla v\left(t+1, \omega, v_0(\omega)\right) \right\|^2 + \left\| \Delta v\left(t+1, \omega, v_0(\omega)\right) \right\|^2 \\ \le e^{\tilde{\gamma}(s-t-1)} \left(\left\| \nabla v\left(s, \omega, v_0(\omega)\right) \right\|^2 + \left\| \Delta v\left(s, \omega, v_0(\omega)\right) \right\|^2 \right) \\ + e^{\tilde{\gamma}(-t-1)} \int_s^{t+1} e^{\tilde{\gamma}\tau - 2bz\left(\vartheta_\tau\omega\right)} \|f\|^2 d\tau. \tag{4.40}$$

Now integrating (4.40) with respect to s over $(t, t+1)$, we conclude that

$$\left\| \nabla v\left(t+1, \omega, v_0(\omega)\right) \right\|^2 + \left\| \Delta v\left(t+1, \omega, v_0(\omega)\right) \right\|^2 \\ \le \int_t^{t+1} e^{-\tilde{\gamma}} \left(\left\| \nabla v\left(s, \omega, v_0(\omega)\right) \right\|^2 + \left\| \Delta v\left(s, \omega, v_0(\omega)\right) \right\|^2 \right) ds \\ + \int_t^{t+1} e^{-\tilde{\gamma} - 2bz\left(\vartheta_\tau\omega\right)} \|f\|^2 d\tau. \tag{4.41}$$

Replacing ω by $\vartheta_{-t-1}\omega$

$$\left\| \nabla v\left(t+1, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 + \left\| \Delta v\left(t+1, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 \\ \le \int_t^{t+1} e^{-\tilde{\gamma}} \left(\left\| \nabla v\left(s, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 + \left\| \Delta v\left(s, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 \right) ds \\ + \int_t^{t+1} e^{-\tilde{\gamma} - 2bz\left(\vartheta_{\tau-t-1}\omega\right)} \|f\|^2 d\tau. \tag{4.42}$$

By Lemma 4.2 and Equation (4.39), it follows from (4.42) it yield that, for all $t \ge T_B(\omega)$

$$\left\| \nabla v\left(t+1, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 + \left\| \Delta v\left(t+1, \vartheta_{-t-1}\omega, v_0\left(\vartheta_{-t-1}\omega\right)\right) \right\|^2 \\ \le e^{-\tilde{\gamma}} \left(\tilde{R}_1(\omega) + \tilde{R}_2(\omega) \right) + \int_{-1}^{0} e^{-\tilde{\gamma} - 2bz\left(\vartheta_\tau\omega\right)} \|f\|^2 d\tau \le \tilde{R}(\omega). \tag{4.43}$$

This proof is concluded. □

Lemma 4.4 Assume that $f \in L^2(\mathbb{R}^n)$, and (3.3)-(3.5) hold. Let $\{B(\omega)\} \in \mathfrak{D}$ and $v_0(\omega) \in B(\omega)$. Then, for any $\zeta > 0$, there exist $\tilde{T} = \tilde{T}(\zeta, \omega, B) > 0$ and $\tilde{K} = \tilde{K}(\zeta, \omega) > 0$, such that the solution ϕ of Equation (3.10) satisfies for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\forall t \geq \tilde{T}$,

$$\int_{|x|\geq \tilde{R}} \left(\left| v\left(t, \vartheta_{-t}\omega, v_0\left(\vartheta_{-t}\omega\right)\right) \right|^2 + \left| \nabla v\left(t, \vartheta_{-t}\omega, v_0\left(\vartheta_{-t}\omega\right)\right) \right|^2 \right) \mathrm{d}x \leq \zeta. \tag{4.44}$$

Proof We first need to define a smooth function $\sigma(\cdot)$ from $\mathbb{R}^+$ into $[0,1]$ such that $\sigma(\cdot) = 0$ on $[0,1]$ and $\sigma(\cdot) = 1$ on $[2, +\infty)$, which evidently implies that there is a positive constant c such that the $|\sigma'(s)| \leq c$ for all $s \geq 0$. For convenience, we write $\sigma_\kappa = \sigma\left(\frac{|x|^2}{\kappa^2}\right)$.

Multiplying Equation (3.10) with $\sigma_\kappa v$ and integrating over $\mathbb{R}^n$, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \sigma_\kappa \left(|v|^2 + |\nabla v|^2\right)\mathrm{d}x + \alpha \int_{\mathbb{R}^n} \sigma_\kappa |v|^2 \mathrm{d}x \\ &= \int_{\mathbb{R}^n} (\Delta v)\sigma_\kappa v \mathrm{d}x + bz(\vartheta_t\omega)\int_{\mathbb{R}^n} \sigma_\kappa |v|^2 \mathrm{d}x \\ &\quad + \mathrm{e}^{-bz(\vartheta_t\omega)}\left(\int_{\mathbb{R}^n} \sigma_\kappa g\left(\mathrm{e}^{bz(\vartheta_t\omega)} v\right) v \mathrm{d}x + \int_{\mathbb{R}^n} \sigma_\kappa f v \mathrm{d}x\right), \end{aligned} \tag{4.45}$$

where

$$\begin{aligned} \int_{\mathbb{R}^n} (\Delta v)\sigma_\kappa v \mathrm{d}x &= -\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa \mathrm{d}x - \int_{\mathbb{R}^n} v\sigma'_\kappa \frac{2x}{\kappa^2}(\nabla v)\mathrm{d}x \\ &\leq -\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa \mathrm{d}x + \frac{\mathcal{C}_0}{\kappa}\left(\|v\|^2 + \|\nabla v\|^2\right), \end{aligned} \tag{4.46}$$

where $\mathcal{C}_0$ is a non-negative constant.

By condition (3.3), we get

$$\begin{aligned} &\mathrm{e}^{-bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} \sigma_\kappa g\left(\mathrm{e}^{bz(\vartheta_t\omega)} v\right) v \mathrm{d}x \\ &= \mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} \sigma_\kappa g(u) u \mathrm{d}x \\ &\leq -\beta_1 \mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} \sigma_\kappa |v|^p \mathrm{d}x + \delta_1 \int_{\mathbb{R}^n} \sigma_\kappa |v|^2 \mathrm{d}x. \end{aligned} \tag{4.47}$$

For the last term on the right-hand side of (4.45), we have that

$$\mathrm{e}^{-bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} \sigma_\kappa f v \mathrm{d}x \leq \frac{\alpha}{2}\int_{\mathbb{R}^n} \sigma_\kappa |v|^2 \mathrm{d}x + \frac{1}{2\alpha}\mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} \sigma_\kappa |f|^2 \mathrm{d}x. \tag{4.48}$$

Then inserting (4.46)-(4.48) into (4.45) to see that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \sigma_\kappa \left(|v|^2 + |\nabla v|^2\right)\mathrm{d}x - \left(2bz(\vartheta_t\omega) - \alpha_0\right)\int_{\mathbb{R}^n} \sigma_\kappa |v|^2 \mathrm{d}x + 2\int_{\mathbb{R}^n} |\nabla v|^2 \sigma_\kappa \mathrm{d}x \\ &\leq \frac{1}{\alpha}\mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} \sigma_\kappa |f|^2 \mathrm{d}x + \frac{2\mathcal{C}_0}{\kappa}\left(\|v\|^2 + \|\nabla v\|^2\right). \end{aligned} \tag{4.49}$$

Hence, we can rewrite (4.49) as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} \sigma_\kappa \left(|v|^2 + |\nabla v|^2\right)\mathrm{d}x + \gamma \int_{\mathbb{R}^n} \sigma_\kappa \left(|v|^2 + |\nabla v|^2\right)\mathrm{d}x \\ &\leq \frac{1}{\alpha}\mathrm{e}^{-2bz(\vartheta_t\omega)}\int_{\mathbb{R}^n} \sigma_\kappa |f|^2 \mathrm{d}x + \frac{2\mathcal{C}_0}{\kappa}\left(\|v\|^2 + \|\nabla v\|^2\right). \end{aligned} \tag{4.50}$$

By applying the Gronwall's lemma to (4.50), for every $t \geq \hat{T}$, we find that

$$\begin{aligned}&\int_{\mathbb{R}^n} \sigma_\kappa \left(\left| v\left(t,\omega,v_0\left(\omega\right)\right)\right|^2 + \left|\nabla v\left(t,\omega,v_0\left(\omega\right)\right)\right|^2 \right) \mathrm{d}x \\ &\leq \mathrm{e}^{-\gamma\left(t-\hat{T}\right)} \int_{\mathbb{R}^n} \sigma_\kappa \left(\left| v\left(\hat{T},\omega,v_0\left(\omega\right)\right)\right|^2 + \left|\nabla v\left(\hat{T},\omega,v_0\left(\omega\right)\right)\right|^2 \right) \mathrm{d}x \\ &\quad + \frac{1}{\alpha}\int_{\hat{T}}^t \mathrm{e}^{\gamma(s-t)-2bz(\vartheta_s\omega)} \int_{\mathbb{R}^n} \sigma_\kappa |f|^2 \,\mathrm{d}x\mathrm{d}s \\ &\quad + \frac{2\mathcal{C}_0}{\kappa}\int_{\hat{T}}^t \mathrm{e}^{\gamma(s-t)} \left(\left\| v\left(s,\omega,v_0\left(\omega\right)\right)\right\|^2 + \left\|\nabla v\left(s,\omega,v_0\left(\omega\right)\right)\right\|^2 \right) \mathrm{d}s.\end{aligned} \tag{4.51}$$

Then, substituting ω by $\vartheta_{-t}\omega$ into (4.51), we have that

$$\begin{aligned}&\int_{\mathbb{R}^n} \sigma_\kappa \left(\left| v\left(t,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2 + \left|\nabla v\left(t,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2 \right) \mathrm{d}x \\ &\leq \mathrm{e}^{-\gamma\left(t-\hat{T}\right)} \int_{\mathbb{R}^n} \sigma_\kappa \left(\left| v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2 + \left|\nabla v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2 \right) \mathrm{d}x \\ &\quad + \frac{1}{\alpha}\int_{\hat{T}}^t \mathrm{e}^{\gamma(s-t)-2bz(\vartheta_{s-t}\omega)} \int_{\mathbb{R}^n} \sigma_\kappa |f|^2 \,\mathrm{d}x\mathrm{d}s \\ &\quad + \frac{2\mathcal{C}_0}{\kappa}\int_{\hat{T}}^t \mathrm{e}^{\gamma(s-t)} \left(\left\| v\left(s,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2 + \left\|\nabla v\left(s,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2 \right) \mathrm{d}s.\end{aligned} \tag{4.52}$$

Then, we estimate every term on the right-hand side of (4.52). Firstly by (4.8), and replacing t by $\hat{T}$ and ω by $\vartheta_{-t}\omega$, then we get

$$\begin{aligned}&\mathrm{e}^{-\gamma\left(t-\hat{T}\right)} \int_{\mathbb{R}^n} \sigma_\kappa \left(\left| v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2 + \left|\nabla v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2 \right) \mathrm{d}x \\ &\leq \mathrm{e}^{-\gamma\left(t-\hat{T}\right)} \left(\mathrm{e}^{-\gamma\hat{T}} \left(\left\| v_0\left(\vartheta_{-t}\omega\right)\right\|^2 + \left\|\nabla v_0\left(\vartheta_{-t}\omega\right)\right\|^2 \right) + \frac{\|f\|^2}{\alpha} \mathrm{e}^{-\gamma\hat{T}} \int_0^{\hat{T}} \mathrm{e}^{-2bz(\vartheta_{s-t}\omega)+\gamma s} \mathrm{d}s \right) \\ &\leq \mathrm{e}^{-\gamma t} \left(\left\| v_0\left(\vartheta_{-t}\omega\right)\right\|^2 + \left\|\nabla v_0\left(\vartheta_{-t}\omega\right)\right\|^2 \right) + \frac{\|f\|^2}{\alpha} \mathrm{e}^{-\gamma t} \int_0^{\hat{T}} \mathrm{e}^{-2bz(\vartheta_{s-t}\omega)+\gamma s} \mathrm{d}s.\end{aligned} \tag{4.53}$$

Then, there exists $\tilde{T}_1 = \tilde{T}_1\left(B,\zeta,\omega\right) > \hat{T}$, such that for all $t > \tilde{T}_1$, then

$$\mathrm{e}^{-\gamma\left(t-\hat{T}\right)} \int_{\mathbb{R}^n} \sigma_\kappa \left(\left| v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2 + \left|\nabla v\left(\hat{T},\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2 \right) \mathrm{d}x \leq \zeta. \tag{4.54}$$

For the second term on the right-hand side of (4.52), Since $f \in L^2\left(\mathbb{R}^n\right)$, there are $\tilde{T}_2 = \tilde{T}_2\left(\zeta,\omega\right) > \hat{T}$ and $\tilde{K}_1 = \tilde{K}_1\left(\zeta,\omega\right) > 0$, such that for all $t > \tilde{T}_2$ and $\kappa > \tilde{K}_1$, then

$$\frac{1}{\alpha}\int_{\hat{T}}^t \mathrm{e}^{\gamma(s-t)-2bz(\vartheta_{s-t}\omega)} \int_{\mathbb{R}^n} \sigma_\kappa |f|^2 \,\mathrm{d}x\mathrm{d}s \leq \frac{1}{\alpha}\int_{\hat{T}}^t \mathrm{e}^{\gamma(s-t)-2bz(\vartheta_{s-t}\omega)} \int_{|x|\geq\kappa} |f|^2 \,\mathrm{d}x\mathrm{d}s \leq \zeta. \tag{4.55}$$

For the last term on the right-hand side of (4.52). By replacing t by s and ω by $\vartheta_{-t}\omega$ in (4.8), we get

$$\begin{aligned}&\frac{2\mathcal{C}_0}{\kappa}\int_{\hat{T}}^t \mathrm{e}^{\gamma(s-t)} \left(\left\| v\left(s,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2 + \left\|\nabla v\left(s,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2 \right) \mathrm{d}s \\ &\leq \frac{2\mathcal{C}_0}{\kappa}\int_{\hat{T}}^t \mathrm{e}^{-\gamma t} \left(\left\| v_0\left(\vartheta_{-t}\omega\right)\right\|^2 + \left\|\nabla v_0\left(\vartheta_{-t}\omega\right)\right\|^2 \right) \mathrm{d}s \\ &\quad + \frac{2\mathcal{C}_0\|f\|^2}{\alpha\kappa}\int_{\hat{T}}^t \mathrm{e}^{\gamma(s-t)} \int_0^s \mathrm{e}^{\gamma(\tau-s)-2bz(\vartheta_{\tau-t}\omega)} \mathrm{d}\tau\mathrm{d}s\end{aligned}$$

$$\leq \frac{2\mathcal{C}_0}{\kappa} e^{-\gamma t}\left(t-\hat{T}\right)\left(\left\|v_0\left(\vartheta_{-t}\omega\right)\right\|^2+\left\|\nabla v_0\left(\vartheta_{-t}\omega\right)\right\|^2\right)ds$$
$$+\frac{2\mathcal{C}_0\|f\|^2}{\alpha\kappa}\int_{\hat{T}}^{t}\int_0^s e^{\gamma(\tau-t)-2bz(\vartheta_{\tau-t}\omega)} d\tau ds. \quad (4.56)$$

Then, by $f\in L^2\left(\mathbb{R}^n\right)$, there exist $\tilde{T}_3=\tilde{T}_3\left(B,\zeta,\omega\right)>\hat{T}$ and $\tilde{K}_2=\tilde{K}_2\left(\zeta,\omega\right)>0$, such that for all $t>\tilde{T}_3$ and $\kappa>\tilde{K}_2$, we find that

$$\frac{2\mathcal{C}_0}{\kappa}\int_{\hat{T}}^{t} e^{\gamma(s-t)}\left(\left\|v\left(s,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2+\left\|\nabla v\left(s,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right\|^2\right)ds\leq\zeta. \quad (4.57)$$

By letting $\tilde{T}=\max\left\{\tilde{T}_1,\tilde{T}_2,\tilde{T}_3\right\}$, and $\tilde{K}=\max\left\{\tilde{K}_1,\tilde{K}_2\right\}$.

Then, inserting (4.54) and (4.55) and (4.57) into (4.52), for all $t>\tilde{T}$ and $\kappa>\tilde{K}$, we obtain that

$$\int_{\mathbb{R}^n}\sigma_\kappa\left(\left|v\left(t,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2+\left|\nabla v\left(t,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2\right)dx\leq 3\zeta, \quad (4.58)$$

which shows that

$$\int_{|x|\geq\tilde{K}}\left(\left|\phi\left(t,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2+\left|\nabla\phi\left(t,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)\right|^2\right)dx\leq 3\zeta. \quad (4.59)$$

This proof is completed. □

5. Random Attractors

In this section, we prove the existence of a global random attractor for the random dynamical system ϕ associated with the stochastic reaction-diffusion Equations (3.1) and (3.2) on $\mathbb{R}^n$. The main result of this section can now be stated as follows.

Lemma 5.1 Assume that $f\in L^2\left(\mathbb{R}^n\right)$, and (3.3)-(3.5) hold. Then the random dynamical system ϕ generated by (3.10) is asymptotically compact in $H^1\left(\mathbb{R}^n\right)$, that is, for $\mathbb{P}$-a.e. $\omega\in\Omega$, the sequence $\left\{\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)\right\}$ has a convergent subsequence in $H^1\left(\mathbb{R}^n\right)$ provided $t_n\to+\infty$, $\left\{B\left(\omega\right)\right\}\in\mathfrak{D}$ and $v_{0,n}\left(\vartheta_{-t_n}\omega\right)\in B\left(\vartheta_{-t_n}\omega\right)$.

Proof Let $t_n\to+\infty$, $\left\{B\left(\omega\right)\right\}\in\mathfrak{D}$ and $v_{0,n}\left(\vartheta_{-t_n}\omega\right)\in B\left(\vartheta_{-t_n}\omega\right)$. Then by Lemma 4.1, for $\mathbb{P}$-a.e. $\omega\in\Omega$, we have that $\left\{\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)\right\}_{n=1}^{\infty}$ is bounded in $H^1\left(\mathbb{R}^n\right)$.

Hence, there exist $\xi\in H^1\left(\mathbb{R}^n\right)$ such that, up to a subsequence,

$$\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)\to\xi \quad \text{weakly in } H^1\left(\mathbb{R}^n\right). \quad (5.1)$$

Next, we prove the weak convergence of (5.1) is actually strong convergence. Given $\zeta>0$, by Lemma 4.4, there exist $\hat{T}_1=\hat{T}_1\left(B,\zeta,\omega\right)>0$, $\hat{\kappa}_1=\hat{\kappa}_1\left(\zeta,\omega\right)>0$ and $\hat{N}_1=\hat{N}_1\left(B,\zeta,\omega\right)>0$, such that $t_n\geq\hat{T}_1$ for every $n\geq\hat{N}_1$

$$\int_{|x|\geq\hat{\kappa}_1}\left|\nabla\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)\right|^2 dx\leq\zeta. \quad (5.2)$$

On the other hand, by Lemma 4.1 and 4.3, there exist $\hat{T}_2=\hat{T}_2\left(B,\omega\right)>0$, such that for all $t\geq\hat{T}_2$,

$$\left\|\phi\left(t,\vartheta_{-t}\omega,v_0\left(\vartheta_{-t}\omega\right)\right)-\xi\right\|_{H^2\left(\mathbb{R}^n\right)}^2\le R_1\left(\omega\right). \tag{5.3}$$

Let $\hat{N}_2=\hat{N}_2\left(B,\omega\right)$ be large enough such that $t_n\ge\hat{T}_2$ for $n\ge\hat{N}_2$. Then by (5.3) we find that, for all $n\ge\hat{N}_2$,

$$\left\|\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)-\xi\right\|_{H^2\left(\mathbb{R}^n\right)}^2\le R_1\left(\omega\right). \tag{5.4}$$

Denote by $Q_{\hat{\kappa}_1}=\left\{x\in\mathbb{R}^n:\left|x\right|\le\hat{\kappa}_1\right\}$. By the compactness of embedding $H^2\left(Q_{\hat{\kappa}_1}\right)\hookrightarrow H^1\left(Q_{\hat{\kappa}_1}\right)$. It follows from (5.4) that, up to a subsequence depending on $\hat{\kappa}_1$

$$\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)\to\xi \text{ strongly in } H^1\left(Q_{\hat{\kappa}_1}\right), \tag{5.5}$$

which shows that for the given $\zeta>0$, there exist $\hat{N}_3=\hat{N}_3\left(B,\omega\right)\left(B,\zeta,\omega\right)>0$, such that for all $n\ge\hat{N}_3$,

$$\left\|\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)-\xi\right\|_{H^1\left(Q_{\hat{\kappa}_1}\right)}^2\le\zeta. \tag{5.6}$$

Note that $\xi\in H^1\left(\mathbb{R}^n\right)$. Therefore, there exist $\hat{\kappa}_2=\hat{\kappa}_2\left(\zeta\right)>0$, such that

$$\int_{|x|\ge\hat{\kappa}_2}\left|\xi\left(x\right)\right|^2\mathrm{d}x\le\zeta. \tag{5.7}$$

By letting $\hat{N}=\max\left\{\hat{N}_1,\hat{N}_2,\hat{N}_3\right\}$, and $\hat{\kappa}=\max\left\{\hat{\kappa}_1,\hat{\kappa}_2\right\}$.

Then, by (5.2), (5.6) and (5.7), we find that for all $n\ge\hat{N}$,

$$\begin{aligned}
&\left\|\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)-\xi\right\|_{H^1\left(\mathbb{R}^n\right)}^2\\
&\le\int_{|x|\le\hat{\kappa}}\left|\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)-\xi\right|^2\mathrm{d}x\\
&\quad+\int_{|x|\ge\hat{\kappa}}\left|\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)-\xi\right|^2\mathrm{d}x\\
&\le\mathcal{C}\zeta.
\end{aligned} \tag{5.8}$$

which shows that

$$\phi\left(t_n,\vartheta_{-t_n}\omega,v_{0,n}\left(\vartheta_{-t_n}\omega\right)\right)\to\xi \quad \text{strongly in } H^1\left(\mathbb{R}^n\right). \tag{5.9}$$

This as desired. □

We are now in a position to present our main result, the existence of a global random attractor for ϕ in $H^1\left(\mathbb{R}^n\right)$.

Lemma 5.2 Assume that $f\in L^2\left(\mathbb{R}^n\right)$, and (3.3)-(3.5) hold. Then the random dynamical system ϕ generated by (3.10) has a unique global random attractor in $H^1\left(\mathbb{R}^n\right)$.

Proof Notice that the random dynamical system ϕ has a random absorbing set $\left\{K\left(\omega\right)\right\}$ in $\mathfrak{D}$ by Lemma 4.1. On the other hand, by Lemma 5.1, the random dynamical system ϕ is asymptotically compact in $H^1\left(\mathbb{R}^n\right)$. Then by Theorem 2.9, the random dynamical system ϕ generated by (3.10) has a unique global random attractor in $H^1\left(\mathbb{R}^n\right)$. □

6. Upper Semi-Continuity of Random Attractor in $H^1(\mathbb{R}^n)$

In this section, we investigate the existence and upper semi-continuity of random attractors for (3.1) and (3.2) by studying (3.10) and (3.11). To indicate the dependence of solutions on b, we respectively write the solutions of (3.1) and (3.2) and (3.10) and (3.11) as u^b and v^b. Let $v(x,t)$ be the solution of the following deterministic system corresponding to (3.10) and (3.11):

$$\begin{cases}\dfrac{\mathrm{d}v}{\mathrm{d}t}+\alpha v-\Delta v-\Delta v_t=g(v)+f(x),\\ v(x,0)=v_0(x)=u_0(x),\ x\in\mathbb{R}^n.\end{cases}\tag{6.1}$$

In fact, the system (6.1) is also equivalent to (3.1) and (3.2) provided $b=0$, that is, $v(t,x)\equiv u(t,x)$, where $u(t,x)$ is the solution of corresponding to (3.1) and (3.2).

Remark 6.1 Correspondingly, the deterministic and autonomous system ϕ_0 generated by (6.1) is readily verified to admit a global attractor $\mathcal{A}_0$ in $H^1(\mathbb{R}^n)$.

The next lemma shows the convergence $\phi_b(t,\omega)v_0^b(\omega)\to\phi_0(t)v_0(\omega)$ provided $v_0^b(\omega)\to v_0$ with $b\downarrow 0$, which is important for the upper semi-continuity of random attractors.

Lemma 6.2 Assume that $f\in L^2(\mathbb{R}^n), b\in(0,1]$ and (3.3)-(3.5) hold. Then, for each $t\ge 0$ and $\mathbb{P}$-a.e. $\omega\in\Omega$, there exist constants $m_1(t,\omega)$ and $l_1(t,\omega)$ independent of *b*, such that

$$\begin{aligned}&\left\|v^b\left(t,\omega,v_0^b(\omega)\right)-v\left(t,v_0(x)\right)\right\|^2+\left\|\nabla v^b\left(t,\omega,v_0^b(\omega)\right)-\nabla v\left(t,v_0(x)\right)\right\|^2\\ &\le \mathrm{e}^{\int_0^t m_1(\tau)\mathrm{d}\tau}\left(\left\|v_0^b(\omega)-v_0(x)\right\|^2+\left\|\nabla v_0^b(\omega)-\nabla v_0(x)\right\|^2\right)\\ &\quad+l(t,\omega)\sup_{0\le s\le t}\left(\left|1-\mathrm{e}^{-bz(\vartheta_s\omega)}\right|+b\left|2z(\vartheta_s\omega)\right|\right).\end{aligned}\tag{6.2}$$

Proof Let $W=v^b-v$. Then, by (3.10) and (3.11) and (6.1), W satisfies

$$\begin{cases}\dfrac{\mathrm{d}W}{\mathrm{d}t}+\alpha W-\Delta W-\Delta W_t=\mathrm{e}^{-bz(\vartheta_t(\omega))}g\left(u^b\right)-g(u)+\mathrm{e}^{-bz(\vartheta_t(\omega))}f(x)-f(x)+bz\left(\vartheta_t(\omega)\right)v^b,\\ W(x,0)=v_0^b(\omega)-v_0(x)=\mathrm{e}^{-bz(\omega)}u_0^b-v_0,\end{cases}\tag{6.3}$$

where we have used the relations $v(x,t)=u(x,t)$. Taking the inner product of (6.3) with W in $L^2(\mathbb{R}^n)$ we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|W\|^2+\|\nabla W\|^2\right)+\alpha\|W\|^2+\|\nabla W\|^2\\ &=\left(\mathrm{e}^{-bz(\vartheta_t(\omega))}g\left(u^b\right)-g(u),W\right)+\left(\mathrm{e}^{-bz(\vartheta_t(\omega))}f(x)-f(x),W\right)\\ &\quad+\left(bz\left(\vartheta_t(\omega)\right)v^b,W\right)\end{aligned}\tag{6.4}$$

By conditions (3.4) and (3.5), to yield that

$$\begin{aligned}&\left(\mathrm{e}^{-bz(\vartheta_t(\omega))}g\left(u^b\right)-g(u),W\right)\\ &=\left(\mathrm{e}^{-bz(\vartheta_t(\omega))}g\left(u^b\right)-g(u),W\right)+\left(1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right)\left(g(u),W\right)\\ &\le L\mathrm{e}^{-bz(\vartheta_t(\omega))}\left|\left(u^b-u,W\right)\right|+\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|\int_{\mathbb{R}^n}\left(-\beta_3|u|^{p-1}+\delta_3|u|\right)|W|\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&\le L\mathrm{e}^{-bz(\vartheta_t(\omega))}\left|\left(\mathrm{e}^{bz(\vartheta_t(\omega))}v^b-\mathrm{e}^{bz(\vartheta_t(\omega))}v+\mathrm{e}^{bz(\vartheta_t(\omega))}v-u,W\right)\right|\\
&\quad+\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|\left(-\beta_3\|u\|_p^{p-1}\cdot\|W\|_p+\delta_3\|u\|\cdot\|W\|\right)\\
&\le L\left|\left(W+u\left(1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right),W\right)\right|\\
&\quad+\mathcal{C}_1\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|\left(\|u\|_p^p+\|W\|_p^p+\|u\|^2+\|W\|^2\right)\\
&\le L\|W\|^2+\mathcal{C}_2\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|\left(\|v\|^2+\|u\|_p^p+\|W\|_p^p+\|u\|^2+\|W\|^2\right).
\end{aligned}
\tag{6.5}
$$

On the other hand,

$$
\begin{aligned}
&\left(\mathrm{e}^{-bz(\vartheta_t(\omega))}f(x)-f(x),W\right)+\left(bz(\vartheta_t(\omega))v^b,W\right)\\
&\le\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|\left(\|f(x)\|^2+\|W\|^2\right)+2\left|bz(\vartheta_t\omega)\right|\|W\|^2+\left|bz(\vartheta_t\omega)\right|\|v\|^2.
\end{aligned}
\tag{6.6}
$$

Then inserting (6.5) and (6.6) into (6.4) to see that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|W\|^2+\|\nabla W\|^2\right)+\alpha\|W\|^2+\|\nabla W\|^2\\
&\le L\|W\|^2+\mathcal{C}_3\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|\left(\|v\|^2+\|u\|_p^p+\|W\|_p^p+\|W\|^2+\|f(x)\|^2\right)\\
&\quad+2\left|bz(\vartheta_t\omega)\right|\|W\|^2+\left|bz(\vartheta_t\omega)\right|\|v\|^2.
\end{aligned}
\tag{6.7}
$$

Since $\|W\|_p^p\le 2^p\left(\|v^b\|_p^p+\|v\|_p^p\right)\le 2^p\left(\mathrm{e}^{-pbz(\vartheta_t(\omega))}\left(\|u^b\|_p^p+\|u\|_p^p\right)\right)$, by (5.7) we conclude that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|W\|^2+\|\nabla W\|^2\right)+2\|\nabla W\|^2\\
&\le(2L-2\alpha)\|W\|^2+\mathcal{C}_4\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|\left(\|v\|^2+\|u\|_p^p+\|W\|_p^p+\|W\|^2+1\right)\\
&\quad+4\left|bz(\vartheta_t\omega)\right|\|W\|^2+2\left|bz(\vartheta_t\omega)\right|\|v\|^2\\
&\le\left(2L-2\alpha+\mathcal{C}_4\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|+4\left|bz(\vartheta_t\omega)\right|\right)\|W\|^2\\
&\quad+\mathcal{C}_5\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|\left(\|v\|^2+\|u\|_p^p+\mathrm{e}^{-pbz(\vartheta_t(\omega))}\|u^b\|_p^p+1\right)+2\left|bz(\vartheta_t\omega)\right|\|v\|^2
\end{aligned}
\tag{6.8}
$$

Hence, we can rewrite (6.8) as

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|W\|^2+\|\nabla W\|^2\right)+m_1(t,\omega)\left(\|W\|^2+\|\nabla W\|^2\right)\le\mathcal{C}_5 m_2^b(t,\omega),
\tag{6.9}
$$

where $m_1(t,\omega):=\min\left\{-\left(2L-2\alpha+\mathcal{C}_4\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|+4\left|bz(\vartheta_t\omega)\right|\right),2\right\}$, independent of b and $\mathbb{P}$-a.s. bounded for each $t\in\mathbb{R}$;

$$
m_2^b(t,\omega):=\left(\left|1-\mathrm{e}^{-bz(\vartheta_t(\omega))}\right|+2\left|bz(\vartheta_t\omega)\right|\right)\left(\|v\|^2+\|u\|_p^p+\mathrm{e}^{-pbz(\vartheta_t(\omega))}\|u^b\|_p^p+1\right).
\tag{6.10}
$$

By applying Gronwall's lemma to (6.9), we find that

$$
\begin{aligned}
&\left\|W(t,\omega,W(0))\right\|^2+\left\|\nabla W(t,\omega,W(0))\right\|^2\\
&\le\mathrm{e}^{\int_0^t m_1(\tau)\mathrm{d}\tau}\left(\|W(0)\|^2+\|\nabla W(0)\|^2\right)+\mathcal{C}_6\int_0^t\mathrm{e}^{\int_s^t m_1(\tau)\mathrm{d}\tau}m_2^b(s)\mathrm{d}s.
\end{aligned}
\tag{6.11}
$$

Now for each fixed $t \in \mathbb{R}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, consider the last term in (6.11). First, notice that, by (6.10),

$$\int_0^t e^{\int_s^t m_1(\tau)d\tau} m_2^b(s)ds \le e^{\int_s^t |m_1(\tau)|d\tau} \sup_{0\le s\le t}\left(\left|1-e^{-bz(\vartheta_t(\omega))}\right| + 2\left|bz(\vartheta_t\omega)\right|\right) \times \int_0^t \left(e^{-pbz(\vartheta_t(\omega))}\left\|u^b\right\|_p^p + \|u\|_p^p + \|v\|^2 + 1\right)ds. \tag{6.12}$$

According to (4.9), by replacing ω with $\vartheta_{-t}\omega$, we conclude that

$$\begin{aligned} &\int_0^t e^{-pbz(\vartheta_t(\omega))}\left\|u^b\left(s,\omega,u_0^b(\omega)\right)\right\|_p^p ds \\ &\le \sup_{0\le s\le t} e^{b|(2-p)z(\vartheta_s\omega)|+\gamma(t-s)} \cdot \int_0^t e^{-2bz(\vartheta_s\omega)+\gamma(s-t)}\left\|u^b\left(s,\omega,u_0^b(\omega)\right)\right\|_p^p ds \\ &\le \mathcal{C}_7(t,\omega) \end{aligned} \tag{6.13}$$

where $\mathcal{C}_7(t,\omega)$ is independent of *b*, $\mathbb{P}$-a.s. bounded for each fixed *t*, and given by

$$\mathcal{C}_7(t,\omega) := c \sup_{0\le s\le t} e^{|(2-p)z(\vartheta_s\omega)|+\gamma(t-s)} \sup_{0\le b\le 1}\rho(\vartheta_t\omega), \tag{6.14}$$

where $\rho(\omega)$ is the tempered random variable given by (4.12).

By taking $b=0$ in (6.13) we find that

$$\int_0^t \left\|u\left(s,u_0(x)\right)\right\|_p^p ds \le ce^{\gamma t}. \tag{6.15}$$

Similarly, from (4.8) we know that, when $b=0$,

$$\begin{aligned} &\int_0^t \left\|v\left(s,v_0(x)\right)\right\|^2 ds \\ &\le \int_0^t e^{-\gamma s}\left(\left\|v_0(x)\right\|^2 + \left\|\nabla v_0(x)\right\|^2 + \frac{\|f\|^2}{\alpha}\int_0^s e^{\gamma(\tau-s)}d\tau\right)ds \\ &\le \frac{1}{\gamma}\left(1-e^{-\gamma t}\right)\left(\left\|v_0(x)\right\|^2 + \left\|\nabla v_0(x)\right\|^2\right) + \left(\frac{1}{\gamma^2}\left(\frac{1}{2} - e^{-\gamma t} + \frac{1}{2}e^{-2\gamma t}\right)\right)\frac{\|f\|^2}{\alpha}. \end{aligned} \tag{6.16}$$

Therefore, from (6.11)-(6.16), to yield that

$$\begin{aligned} &\left\|v^b\left(t,\omega,v_0^b(\omega)\right) - v\left(t,v_0(x)\right)\right\|^2 + \left\|\nabla v^b\left(t,\omega,v_0^b(\omega)\right) - \nabla v\left(t,v_0(x)\right)\right\|^2 \\ &\le e^{\int_0^t m_1(\tau)d\tau}\left(\left\|v_0^b(\omega) - v_0(x)\right\|^2 + \left\|\nabla v_0^b(\omega) - \nabla v_0(x)\right\|^2\right) \\ &\quad + l(t,\omega)\sup_{0\le s\le t}\left(\left|1-e^{-bz(\vartheta_s\omega)}\right| + b\left|2z(\vartheta_s\omega)\right|\right). \end{aligned} \tag{6.17}$$

where

$$\begin{aligned} l(t,\omega) := \mathcal{C}e^{\int_s^t |m_1(\tau)|d\tau}\Bigg(&\mathcal{C}_7(t,\omega) + e^{\gamma t} + \left(1-e^{-\gamma t}\right)\left(\left\|v_0(x)\right\|^2 + \left\|\nabla v_0(x)\right\|^2\right) \\ &+ \left(\frac{1}{2} - e^{-\gamma t} + \frac{1}{2}e^{-2\gamma t}\right)\frac{\|f\|^2}{\alpha}\Bigg) \end{aligned}$$

is $\mathbb{P}$-a.s. bounded for each $t \ge 0$ (since $z(\vartheta_t\omega)$) is pathwise continuous) and independent of *b*. This proof is completed. □

Theorem 6.3 Assume that $f \in L^2(\mathbb{R}^n), b \in (0,1]$ and (3.3)-(3.5) hold. Then,

$\mathbb{P}$-a.e. $\omega \in \Omega$, we have

$$\lim_{b \downarrow 0} \operatorname{dist}_{H^1(\mathbb{R}^n)} \left(\mathcal{A}_b(\omega), \mathcal{A}_0 \right) = 0.$$

Proof To achieve the result, it suffices to verify conditions (1), (2) and (3) in Lemma 2.10.

Notice that, condition (1) is actually proved by Lemma 6.2. For condition (2), since Lemma 4.1, has proved that random dynamical system ϕ possesses a closed random absorbing set $\{K(\omega)\} \in \mathfrak{D}$, which is given by

$$K(\omega) = \left\{ v \in H^1(\mathbb{R}^n) : \|\nabla v\|^2 \le \rho(\omega) \right\},$$

where

$$\rho(\omega) = 1 + \frac{\|f\|^2}{\alpha} \int_{-\infty}^{0} e^{-2bz(\vartheta_s \omega) + \gamma s} \mathrm{d}s,$$

it is readily to obtain that, $\mathbb{P}$-a.e.,

$$\limsup_{b \downarrow 0} \|K(\omega)\| = 1 + \frac{\|f\|^2}{\alpha},$$

which deduces condition (2) immediately. Now consider condition (3). Given $b \in (0,1]$. From Lemma 4.3 we know that $\tilde{R}(\omega) \in \mathfrak{D}$ is also closed and tempered random absorbing set ϕ_b in $H^1(\mathbb{R}^n)$, where

$$\tilde{R}(\omega)_b = \left\{ v \in H^1(\mathbb{R}^n) : \|v\|^2 + \|\nabla v\|^2 \le e^{-\tilde{\gamma}} \left(\tilde{R}_1(\omega) + \tilde{R}_2(\omega) \right) + \int_{-1}^{0} e^{-\tilde{\gamma} - 2bz(\vartheta_\tau \omega)} \|f\|^2 \mathrm{d}\tau \right\},$$

with $\tilde{R}_1(\omega)$ and $\tilde{R}_2(\omega)$ are tempered random variables in $\omega \in \Omega$ and continuous in *b*. Let

$$\tilde{R}(\omega) = \left\{ v \in H^1(\mathbb{R}^n) : \|v\|^2 + \|\nabla v\|^2 \le \sup_{0 \le b \le 1} \left[e^{-\tilde{\gamma}} \left(\tilde{R}_1(\omega) + \tilde{R}_2(\omega) \right) \right] + \int_{-1}^{0} e^{-\tilde{\gamma} + |2bz(\vartheta_\tau \omega)|} \|f\|^2 \mathrm{d}\tau \right\}.$$

Then, we know $\tilde{R}(\omega)$ is compact in $H^1(\mathbb{R}^n)$. From

$$\bigcup_{0 \le b \le 1} \mathcal{A}_b(\omega) \subseteq \bigcup_{0 \le b \le 1} \tilde{R}(\omega)_b \subseteq \tilde{R}(\omega),$$

it follows that $\bigcup_{0 \le b \le 1} \mathcal{A}_b(\omega)$ is precompact in $H^1(\mathbb{R}^n)$. Hence, condition (3) is clear and this proof is completed. □

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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